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PLATES

Some problems of Diophantine approximation: A further note on the trigonometrical series associated with the elliptic theta-functions. By Prof. G. H. HARDY and Mr J. E. LITTLEWOOD.

1. This note contains a short addition to a memoir, with a similar title, published in 1914 in the *Acta Mathematica*[*]. In that memoir we considered the sums

$$s_n{}^2 = s_n{}^2 (x, \theta) = \sum_{\nu \leq n} e^{(\nu - \frac{1}{2})^2 \pi i x} \cos (2\nu - 1) \pi\theta,$$

$$s_n{}^3 = s_n{}^3 (x, \theta) = \sum_{\nu \leq n} e^{\nu^2 \pi i x} \cos 2\nu\pi\theta,$$

$$s_n{}^4 = s_n{}^4 (x, \theta) = \sum_{\nu \leq n} (-1)^\nu e^{\nu^2 \pi i x} \cos 2\nu\pi\theta,$$

where x and θ are real and x irrational[†]. There is plainly no real loss of generality in supposing either x or θ to be positive and less than unity, if it be understood that θ may be zero.

Our main results may be stated as follows. We denote by $s_n = s_n (x, \theta)$ any one of the sums $s_n{}^2, s_n{}^3, s_n{}^4$. Then, in the first place,

$$s_n = o(n) \quad \dots\dots\dots\dots\dots\dots(1\cdot1),$$

for every irrational x, and uniformly in θ [‡]. And this equation is a best possible equation of its kind; there is no function $\phi = \phi(n)$, tending to infinity with n, such that

$$s_n = O \begin{pmatrix} n \\ \phi \end{pmatrix} \quad \dots\dots\dots\dots\dots\dots(1\cdot2),$$

for every irrational x [§].

[*] G. H. Hardy and J. E. Littlewood, 'Some problems of Diophantine Approximation', *Acta Mathematica*, vol. 37 (1914), pp. 193–238.

[†] The second and third sums reproduce one another when $\theta + \frac{1}{2}$ is written for θ. They are considered separately for the sake of formal symmetry in the analysis.

[‡] p. 213 (Theorem 2·14). It should be observed that we there use s_n in the more restricted sense of $s_n (x, 0)$.

[§] p. 225 (Theorem 2·221).

On the other hand much more than (1·1) is true for special classes of values of x. In particular, if

$$x = \frac{1}{a_1 +} \; \frac{1}{a_2 +} \; \ldots,$$

and the partial quotients a_n are bounded, then

$$s_n = O\left(\sqrt{n}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1\cdot3),$$

and again uniformly in θ*. And this result too is a best possible of its kind, for

$$s_n = o\left(\sqrt{n}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1\cdot4)$$

is false for $\theta = 0$ and *any* irrational x†.

2. There was one obvious gap in our former results. We did not give any simple criterion for distinguishing the classes of irrationals x for which

$$s_n = O\left(n^a\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2\cdot1),$$

where α is an assigned number between $\frac{1}{2}$ and 1. The theorems of this character which we proved‡ were avowedly tentative and unsatisfactory. We did not even prove that *some* equation of the type (2·1) holds for every *algebraic* x. It is this gap which we propose to fill in the present note.

We denote by p_n/q_n a typical convergent to x, taking

$$\frac{p_0}{q_0} = \frac{0}{1}, \; \frac{p_1}{q_1} = \frac{1}{a_1}, \; \frac{p_2}{q_2} = \frac{a_2}{a_1 a_2 + 1}, \; \ldots \; ;$$

and write, as in our former memoir,

$$x = \frac{1}{a_1 + x_1}, \; \; x_1 = \frac{1}{a_2 + x_2}, \; \ldots,$$

$$a_1{}' = a_1 + x_1, \; \; a_2{}' = a_2 + x_2, \; \ldots,$$

$$q_n{}' = a_n{}' \, q_{n-1} + q_{n-2}.$$

We shall say that an irrational x is *of class k* if

$$q_{n+1} < A q_n{}^k \quad \ldots\ldots\ldots\ldots\ldots\ldots(2\cdot2),$$

where $A = A(x)$ is independent of n. We shall use A generally to denote a number of this kind, not the same in different formulae. If x is of class k, and $k < k'$, then x is of class k'.

If x is of class k,

$$\mid p_n - q_n x \mid = \frac{1}{q'_{n+1}} > \frac{A}{q_{n+1}} > \frac{A}{q_n{}^k}.$$

<hr />

* p. 213 (Theorem 2·141).
† p. 225 (Theorem 2·22).
‡ p. 214 (Theorems 2·142, 2·143).

Further $$| p - qx | > \frac{A}{q^k} \quad ....(2\cdot 3)$$

for all positive integral values of p and q. Thus a number of class k might be defined as one for which $(2\cdot 3)$ is true. If a_n is bounded, $(2\cdot 2)$ is true with $k = 1$, so that x is of class 1. In particular a quadratic surd is of class 1, and every algebraic number is of finite class.

We shall now prove

THEOREM A. *If x is of class k then*

$$s_n = O\,(n^a), \quad a = \frac{k}{k+1} \quad ....(2\cdot 4),$$

and uniformly in θ.

In particular we have, as a corollary,

THEOREM B. *If x is algebraic then $s_n = O\,(n^a)$, for some value of a less than 1, and uniformly in θ.*

We shall also prove that Theorem A is in a sense the best theorem of its kind. This will follow from

THEOREM C. *It is possible to choose an x of class k and a θ so that*

$$s_n \neq o\,(n^a), \quad a = \frac{k}{k+1} \quad ....(2\cdot 5).$$

3. We require the following lemmas:

Lemma 1: $$q_{n+1} + x_{n+1} q_n = \frac{1}{x x_1 x_2 \ldots x_n}.$$

For

$$q_{n+1} + x_{n+1} q_n = (a_{n+1} + x_{n+1})\, q_n + q_{n-1} = \frac{q_n}{x_n} + q_{n-1} = \frac{q_n + x_n q_{n-1}}{x_n}.$$

As $$q_1 + x_1 q_0 = a_1 + x_1 = \frac{1}{x},$$

the lemma follows. As an obvious corollary we have

Lemma 2: $$q_{n+1} < \frac{1}{x x_1 x_2 \ldots x_n}.$$

4. We can now prove Theorem A. If we choose ν so that

$$n x x_1 \ldots x_{\nu-1}\, x_\nu < 1 \leqq n x x_1 \ldots x_{\nu-1} \quad ....(4\cdot 1),$$

we have*

$$s_n = O\,(n \sqrt{x x_1 \ldots x_{\nu-1}}) + O\left(\frac{1}{\sqrt{x x_1 \ldots x_{\nu-1}}}\right) = O\,(n \sqrt{x x_1 \ldots x_{\nu-1}}),$$

$$(4\cdot 21)$$

* *L.c.*, p. 213.

1—2

and also $\qquad s_n = O\left(\dfrac{1}{\sqrt{xx_1 \ldots x_\nu}}\right)$ ....(4·22).

We write
$$xx_1 x_2 \ldots x_{\nu-1} = n^{-j} \quad (0 < j \leqq 1).$$
Then
$$\frac{1}{x_\nu} = a_{\nu+1} + x_{\nu+1} < a_{\nu+1} + 1 < A a_{\nu+1} < A\frac{q_{\nu+1}}{q_\nu} < A q_\nu^{k-1},$$
by (2·2); and so, by Lemma 2,
$$\frac{1}{x_\nu} < A \, (xx_1 \ldots x_{\nu-1})^{-(k-1)} < A n^{(k-1)j}.$$

It follows, from (4·21) and (4·22), that
$$s_n = O\,(n^{1-\frac{1}{2}j}), \quad s_n = O\,(n^{\frac{1}{2}kj}) \quad \ldots\ldots\ldots\ldots(4\cdot3),$$

or $\qquad s_n = O\,(n^\gamma), \quad \gamma = \mathrm{Min}\,(1 - \tfrac{1}{2}j, \tfrac{1}{2}kj) \quad \ldots\ldots\ldots(4\cdot4).$

Now $\qquad 1 - \tfrac{1}{2}j \leqq \dfrac{k}{k+1} \quad \left(j \geqq \dfrac{2}{k+1}\right),$

$$\tfrac{1}{2}kj \leqq \frac{k}{k+1} \quad \left(j \leqq \frac{2}{k+1}\right).$$

Hence in any case
$$s_n = O\,(n^{\frac{k}{k+1}}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4\cdot5),$$
which proves the theorem.

Theorem B is an immediate corollary, since an algebraic number of degree m is of class $m - 1$ [*].

5. The proof of Theorem C also requires only a slight modification of our former analysis. We take $\theta = 0$, and write, as before,

$$q = e^{\pi i \tau} = e^{\pi i x - \pi y} = r e^{\pi i x} \quad (x > 0,\ y > 0,\ 0 < r < 1) \ldots(5\cdot1),$$

$$\vartheta_3 = \vartheta_3\,(0, \tau) = 1 + 2\sum_1^\infty q^{n^2} \quad \ldots\ldots\ldots\ldots(5\cdot2).$$

Suppose it were true that $s_n = o\,(n^a)$. Then the series
$$1 + 2\,\Sigma q^{n^2} = 1 + 2\Sigma e^{n^2 \pi i x}\, r^{n^2} = \Sigma u_m\, r^m$$

---

[*] By the classical theorem of Liouville: see, e.g. Borel, *Leçons sur la théorie des fonctions* (ed. 2, 1914), pp. 26–29. It has indeed been shown by A. Thue ('Über Annäherungswerte algebraischer Zahlen', *Journal für Math.*, vol. 135 (1909), pp. 284–305) that an algebraic number of degree m is of class $\frac{1}{2}m + \epsilon$ for every positive ϵ. See Borel, *Leçons sur la théorie de la croissance* (1910), pp. 164–165. More recently C. Siegel ('Approximation algebraischer Zahlen,' *Math. Zeitschrift*, vol. 10 (1921), pp. 173–213) has shown that an algebraic number of degree m is of class $2\sqrt{m} - 1$.

would satisfy the condition

$$U_m = u_0 + u_1 + \ldots + u_m = o\,(m^{\frac{1}{2}a}),$$

and we should have

$$\Im_3 = \Sigma u_m\,r^m = (1-r)\,\Sigma U_m\,r^m = o\,\{(1-r)^{-\frac{1}{2}a}\} = o\,(y^{-\frac{1}{2}a}) \ldots (5\cdot3).$$

It is therefore sufficient to show that $(5\cdot3)$ is false for an appropriate x; that is to say that

$$|\,\Im_3\,(0,\,\tau)\,| > A y^{-\frac{1}{2}a} \quad \ldots\ldots\ldots\ldots\ldots(5\cdot4)$$

for a sequence of values of y whose limit is zero.

We suppose that

$$q_{n+1} > A q_n{}^k \quad \ldots\ldots\ldots\ldots\ldots\ldots(5\cdot5)$$

for an infinity of values of x, and consider, as on p. 229 of our former memoir, the range R_n of values of y defined by

$$\frac{1}{q'_{n+1}{}^2} \leqq y \leqq \frac{1}{q_n{}^2}.$$

It is sufficient to fix our attention on a single value of y, viz.

$$y = q_n{}^{-k-1},$$

which plainly falls within R_n when $k > 1$.

We employ (as on p. 226 *et seq.*) the linear transformation

$$T = \frac{c + d\tau}{a + b\tau} = \pm\frac{p_{n-1} - q_{n-1}\tau}{p_n - q_n\tau},$$

where the sign is chosen so as to make $ad - bc = 1$; and here we make another assumption, viz. that this transformation is one of the types which (following Tannery and Molk) we denoted by 1°, 2°, 5°, or 6°, and which transform $\Im_3\,(0,\,\tau)$ into one of the functions $\Im_3\,(0,\,T)$ or $\Im_4\,(0,\,T)$. It is plain that this may be secured by an appropriate choice of x^*.

This being so we have, as on p. 230†,

$$|\,\Im_3\,| > A\left(\frac{1}{q'_{n+1}{}^2} + q_n{}^2 y^2\right)^{-\frac{1}{4}} > A\,(q_n{}^{-2k} + q_n{}^2 . q_n{}^{-2-2k})^{-\frac{1}{4}} > A q_n{}^{-\frac{1}{2}k}$$

$$> A y^{-\frac{k}{2(k+1)}} = A y^{-\frac{1}{2}a},$$

which proves the theorem.

---

* We cannot prove that $s_n = o\,(n^a)$ is *never* true for an irrational of class k; for it is possible that every n for which $(5\cdot5)$ is true should give rise to a transformation of type 3° or 4°.

† The condition $a_n = O\,(1)$, used there, is only required in connection with cases 3° and 4°, here excluded.

Sur le principe de Phragmén-Lindelöf. Par MARCEL RIESZ. (Extrait d'une lettre adressée à M. G. H. Hardy.)

[*Received* 11 July 1921.]

M. Carlson m'a fait observer que le théorème qui figure au nº 1 de ma lettre, adressée à vous et insérée à ce recueil (vol. XX. pt. i, p. 205–207) est un corollaire immédiat de son théorème (*cf.* nº 2) et que, de plus, ma démonstration est essentiellement identique à celle qu'il a donnée pour son théorème dans sa Thèse (Upsal, 1914).

Some peculiarities of the Wilson ionisation tracks and a suggested explanation. By J. L. GLASSON, M.A., Caius College, Cambridge. (Communicated by Prof. Sir E. Rutherford.)

[*Read* 31 October 1921.]

1. *The β-ray tracks.* Mr Wilson* draws attention to the fact that the β-rays show gradual bending resulting in large deviations. He gives a generally accepted explanation, namely that the scattering of β-rays is mainly or entirely of the cumulative or compound type being due to a large number of successive deflections each in itself inappreciable.
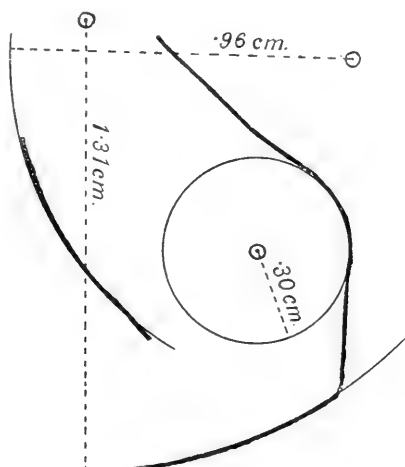
Fig. 1

I think that Mr Wilson's plate 7·2 is not fully described or explained in this way. This track is traced in fig. 1, and three circles are drawn which the tracks follow with great accuracy for long distances. The approximate radii of curvature, allowing for the magnification, are also inserted. It seems to describe the phenomenon more adequately to say that the β-ray moves in a succession of circular arcs of random radii, length and direction. In the case of the β-rays liberated by X-rays the same description seems generally applicable, except that in some cases abrupt bends seem to occur. These bends can be accounted for on the existing theory of scattering

* The photographs referred to in this paper are those published by Mr C. T. R. Wilson, F.R.S., in the following papers: (*a*) *Proc. Roy. Soc.* vol. 87 A, p. 277, 1912; (*b*) *Proc. Roy. Inst. of Gt Btn.* vol. 20, p. 668, 1913.

but it is generally admitted that the circular arcs cannot be explained on any theory without some additional hypothesis. Compton*
has made an attempt to provide this by attributing an additional
property to the electron, namely, magnetic moment. It is claimed
that this theory can account for a certain effect observable. But
as the theory has not yet been fully accepted, and as there are
other phenomena observable in the photographs for which Compton's
theory offers no ready explanation, it seems that there is still room
for another attempt upon the problem.

If we start from the conception that these arcs are a result
of compound scattering, then we must assume that the collisions
are not random but are co-ordinated in such a way as to produce
a uniform rate of bending. This would involve a regular arrangement of all the atoms with which the rays collide and does not
seem very probable. But there is a simple single agency which
can produce the circular arcs, namely, a uniform magnetic field.
If we assume the existence of such fields and also that the deviations produced by atomic collisions are extremely small under the
conditions prevailing here, then the perfection of the arcs is fully
explained. On this view the electron moves in a circle not because
of the collisions but in spite of them. The field required to give the
observed curvature must be of the order of several hundred Gauss,
it must extend over an area of a square centimetre or so in some
cases, and must last for at least 10^{-10} sec. The only explanation
which seems to explain all the facts adequately is that the fields
are produced within the gas itself. They might conceivably be produced by the existence of transient crystalline forms. This hypothesis
receives support in several directions but in this paper that derivable
from further study of the Wilson photographs is particularly considered.

2. *The parallel tracks.* If such magnetic fields exist, then two
β-rays entering the same field or group of fields should describe
parallel tracks. This prediction is immediately verifiable. A few
illustrations of this effect are traced in figs. 2 and 3. Quincke† has
drawn attention to a large number of regularities of this type, which
seem to be fairly well accounted for on the hypothesis here adopted.
The spiral forms to which Compton has drawn attention are obviously to be expected when the whole of the path of a ray lies in
the same field. The three sixes at the bottom of fig. 2 (Mr Wilson's
plate 8·4) seem to illustrate this type of effect. Quincke gives many
other examples of it. In the case of the three tracks *A*, *B* and *C* of
fig. 3, we seem to have evidence that the fields can travel some
distance without change of form or else that there are certain
peculiar configurations of the fields which are particularly probable.

3. *The periodicity of the ions.* The distribution of the ions along a single track seems inconsistent with any theory of chance distribution. In Mr Wilson's plate 7·2 there are eight groups of ions in regular sequence. Plate 8·6 shows the same effect, the average distance between the groups being about ·03 cm. in this case. This periodicity in the ionisation is presumably evidence of a similar periodicity in the arrangement of the molecules and is an obvious consequence of the hypothesis here proposed. The distance between
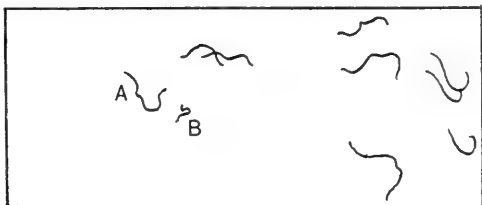
Fig. 2
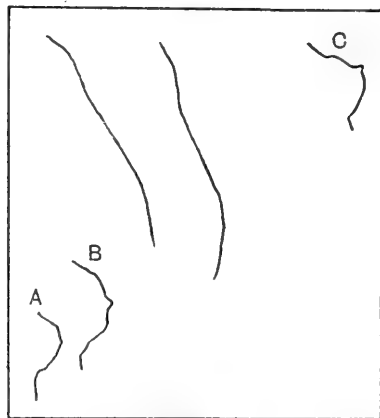
Fig. 3

the groups would then be determined in some way by the distance between successive layers of the crystalline forms.

4. *The curvature of α-ray tracks.* The α-ray tracks are not absolutely straight lines, but are in the majority of cases slightly curved. The radius of curvature in some cases is of the order of 50 cm. and from this we can calculate that the field strength is of the order of several hundred Gauss, in harmony with the value deduced from the β-ray curvatures. This agreement seems to afford strong confirmation of the hypothesis that the fields really are magnetic.

5. *The emanation α-ray curves.* The most striking feature of Mr Wilson's plate 6·4 is the existence of 3 α-rays which are within

$2°$ of parallelism. There are only 20 α-ray tracks visible on this plate and if the direction of projection of the α particles is a random one, the chance that three out of twenty should fall within $2°$ of one another, is quite small. There are three other pairs of parallel rays. Further on this plate there is evidently one direction in which very few particles are projected. By drawing all the particles as radiating from a common origin, I find that the particles tend to be projected in two directions inclined at about $60°$ to one another. Fig. 5 of the Royal Institution paper exhibits these effects much more markedly. Out of fourteen tracks and portions of tracks visible on this plate three are within $2°$ of parallelism, and there are two other parallel pairs. The tendency for all the rays to point in definite directions is also very marked. There is one pair of quadrants in which only a single ray occurs. The chance of getting this result on a random distribution is small. We can account for it at once if we assume that the atoms of radium emanation are polar and that the field is polarised. This is a simple consequence of the hypothesis adopted. Apart from any particular view as to the nature of the orienting forces, this plate seems to suggest the idea that the α particle is ejected from the nucleus in a definite direction.

6. *Discussion.* Of the five phenomena here described, the first and second seem to be the most obvious. The other three are admittedly less conspicuous and it was only by the help of the theory here proposed that I was led to their observation. But although taken separately the arguments may appear weak, they seem to form in combination a strong basis for the hypothesis that gaseous crystalline forms exist in Mr Wilson's apparatus. The water vapour present is in a state of six-fold supersaturation, and its temperature is about $-30°$ C. Even in unsaturated water vapour large aggregations are known to occur and in some cases molecules having the formula $36\,H_2O$ have been detected by vapour density measurements. This is the average size and much larger aggregates must temporarily occur. It does not seem likely that these aggregates are to be regarded as amorphous in view of the polar nature of the water molecule. In the case of the solid state the aggregation is almost universally crystalline and it is in harmony with the ideas of the continuity of state that such a conspicuous feature of the solid state should persist though in a lessened degree, in the liquid and gaseous states. Molecular aggregation in liquids is a generally recognised phenomenon, and in the case of certain liquids it has been found that this aggregation results in the exhibition of quasi-crystalline optical properties. Such liquids are said to form liquid crystals. It does not seem to be impossible therefore that in many hitherto unrecognised cases, molecular association, both in the liquid and gaseous state, means crystalline association.

Determination of the coefficient of viscosity of Mercury. By J. E. P. WAGSTAFF, M.A., St John's College. (Communicated by Prof. R. Whiddington.)

The following is a brief account of a series of experiments for determining the coefficient of viscosity of a conducting liquid carried out in the Physics Laboratories of the University of Leeds.

Poiseuille's Law states that for stream line motion of an incompressible fluid of viscosity η through a tube of length l cm. and radius a cm.

$$\frac{dQ}{dt} = \frac{\pi (p_1 - p_2) a^4}{8\eta l}$$
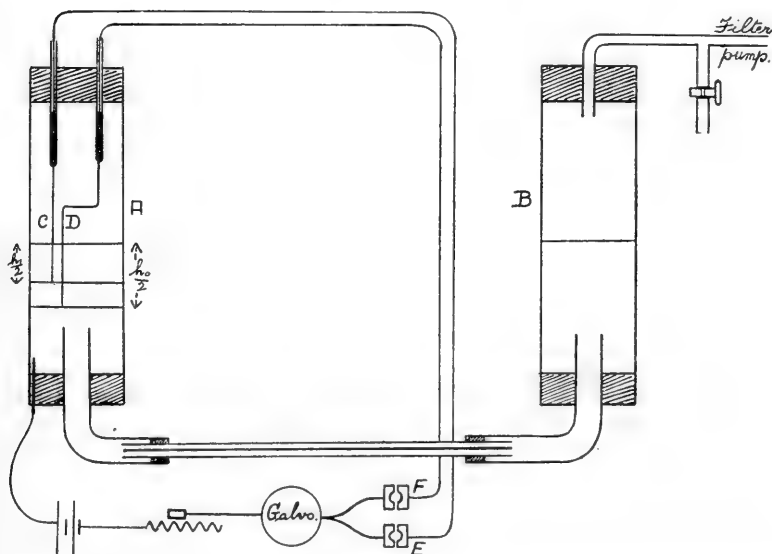
where $\dfrac{dQ}{dt}$ is the volume of liquid which flows through the tube per second corresponding to a pressure difference $p_1 - p_2$. The Fig. shows the arrangement of the apparatus. The mercury is contained in two identical cylindrical reservoirs A and B of about 1 inch diameter, which are connected at their lower extremities by a very narrow piece of capillary tube of uniform bore. Initially the mercury in A is drawn into B by connecting the latter to a filter

pump, after which B is again put into communication with the
atmosphere, and the time taken for the difference in level between
the two surfaces to change from h_0 to h_1 is measured.

If at any instant, h is the difference of level between the two
mercury surfaces, at that instant

$$p_1 - p_2 = h\rho g.$$

Let $b = $ radius of either of the two cylindrical reservoirs, then
corresponding to a rise δx in the level of the mercury in A,

$$\delta Q = \pi b^2 \delta x$$

and

$$\delta h = -2\delta x.$$

$$\therefore \frac{\delta Q}{\delta h} = -\frac{\pi b^2}{2};$$

$$\therefore -\frac{\pi b^2 \delta h}{2h} = \frac{\pi \rho g a^4}{8\eta l} \cdot \delta t;$$

$$\log \frac{h_0}{h_1} = \frac{\rho g a^4}{4\eta l b^2} t,$$

where t is the time that elapses for the difference in level of the
two surfaces to change from h_0 to h_1. This time was measured by
the following device: long platinum wires C and D, sealed into
tubes containing mercury were placed in one of the cylinders, so
that their free ends were at known distances below the common
level of the mercury when in equilibrium, say $\frac{h_0}{2}$ and $\frac{h_1}{2}$ respectively.

These tubes were connected as shown to a battery and a Weston
Students' Galvanometer so that initially when either of the plugs
E and F was inserted a suitable deflection was obtained. Mercury
was now drawn out of A into B until both the platinum wires
were out of the mercury. At this instant the plug F was inserted,
both tubes being now open to the atmosphere, and the instant was
noted when the galvanometer needle was deflected. The plug was
now removed from F and placed in E, and the instant noted when
the galvanometer was again deflected. The time between these
two successive deflections corresponded to the interval that elapsed
from the instant the two surfaces were at a distant h_0 apart until
they were at h_1 apart.

The mean radius $\overline{(a^2)}$ of the capillary tube was found by inserting
a thread of mercury and comparing its length and mass at a known
temperature. The results obtained in several experiments are
tabulated below.

<div align="center">

Series I.

Radius of tube = ·02868 cm.

$2b = 2\cdot311$ cm.

$l = 19$ cm.

θ (temperature during experiment) $= 16^\circ$ C.

</div>

$\frac{h_0}{2}$ (cm.)	$\frac{h_1}{2}$ (cm.)	t (sec.)	η
2·03	1·30	80·4	·01618
		80·4	
		80·6	

Series II (performed by Colin Barnes of Leeds University).

<div align="center">

Length of tube $= 20\cdot10$ cm.

$2b = 2\cdot56$ cm.

$\theta = 14^\circ$ C.

$a = \cdot03838$ cm.

</div>

$\frac{h_0}{2}$ (cm.)	$\frac{h_1}{2}$ (cm.)	t (sec.)	η
1·255	·635	50, 50·2,	·0162
		49·8, 50,	
		50·2, 50,	
		50, 49·8,	
		50, 50, 50,	
		50, 50	

Warburg found that η at $17\cdot2^\circ$ C. for mercury is ·016329 and a later determination by Umani gives $\eta = \cdot01577$ at 10° C. The values obtained above are seen to be in close agreement with these. The method has been applied successfully to the determination of the viscosity of liquid electrolytes.

A laboratory method of determining Young's Modulus for a microscopic cover slip. By J. E. P. WAGSTAFF, M.A., St John's College. (Communicated by Prof. R. Whiddington.)

[*Read* 31 October 1921.]

This paper is a short account of some experiments carried out in the Physics Laboratory of the University of Leeds on the elastic properties of thin glass discs.

A thin slip of glass A (diameter 2 cm.) whose modulus of elasticity is to be found is made to rest on the top of a pressure chamber B, as shown in Fig. in the upper surface of which is a circular
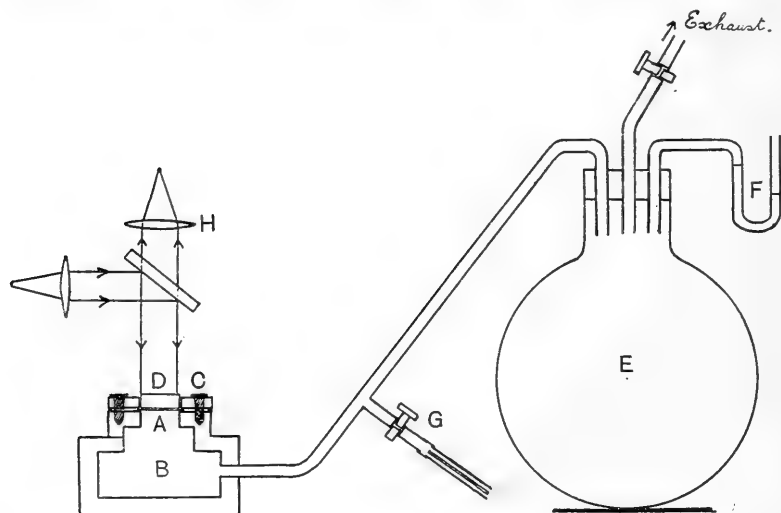
hole of diameter 1·93 cm. In order to clamp the edges of the slip, a plate C having a hole of diameter 1·97 cm. concentric with that in the upper face of the chamber, is screwed down on to the top surface of the glass slip. A small plate of glass D (diameter 1·95 cm. approximately and thickness 1 millimetre) is made to rest on the top of the slip, care being taken that the air in the space between the slip and the plate remains at atmospheric pressure. The chamber is connected to a large reservoir of air E, which can be partially exhausted by a filter pump and the pressure is measured by a Mercury manometer F as shown. The air in the reservoir can be allowed to escape very slowly through a fine tube by opening

the tap G. The film of air between A and D is illuminated with a parallel beam of monochromatic light by means of a sodium flame, a convex lens, and a semi-transparent mirror and is viewed by a telescope at H.

When a circular plate of material of radius R cm. and thickness t cm. is clamped at the edge and subjected to a uniform pressure p over one face the centre sags an amount

$$\delta = \frac{3}{16} \cdot \frac{pR^4}{Et^3} \left(1 - \frac{1}{m^2}\right)$$

where E is Young's Modulus of the material and $\frac{1}{m}$ is Poisson's Ratio.

Assuming for glass $m = 4$, corresponding to a pressure difference p

$$\delta = \frac{45pR^4}{16^2 Et^3}$$

and the change in sag $\delta_1 - \delta_2$ corresponding to a change in pressure $p_1 - p_2$ is

$$\delta_1 - \delta_2 = \frac{45}{16^2} \cdot \frac{(p_1 - p_2) R^4}{Et^3}.$$

This change in sag corresponding to a pressure change $p_1 - p_2$ is determined by observing the circular fringes when the chamber B is placed in communication with the reservoir E and counting the number of fringes which close into the centre as the pressure is changed from p_1 to p_2, by opening the tap G. If N is the number of fringes which disappear for light of wave length λ

$$\delta_1 - \delta_2 = \frac{N\lambda}{2}.$$

As the fringes get distorted due to irregularities in the surface of the glass when the pressure difference between the two faces of the disc approaches zero, the pressure in the chamber is during an experiment kept definitely less than atmospheric.

In a second series the cover slip was freely supported, and the system of fringes formed in the film between this and a second light disc supported on the edge of the former was examined. The formula giving Young's Modulus in this case becomes

$$\frac{n\lambda}{2} = \frac{189}{16^2} \frac{R^4 (p_1 - p_2)}{Et^3}.$$

It will be noticed that the two sets of observations are in satisfactory agreement.

Experiment I.

Thickness of cover slip = ·02117 cm.
Diameter of aperture = 1·93 cm.

Manometer readings in cm. of Hg		Pressure difference between two faces of cover slip	Number of fringes that have disappeared	Young's Modulus
49·75	35·95	13·8	0	
48·75	36·95	11·8	25	$5·8 \times 10^{11}$
47·7	37·9	9·8	50	$5·8 \times 10^{11}$
46·85	38·75	8·1	70	$5·9 \times 10^{11}$
49·85	35·85	14·0	0	
48·8	36·8	12·0	25	$5·8 \times 10^{11}$
47·85	37·8	10·05	50	$5·7 \times 10^{11}$
47·0	38·5	8·5	70	$5·7 \times 10^{11}$
46·65	38·9	7·75	80	$5·7 \times 10^{11}$
46·2	39·35	6·85	90	$5·8 \times 10^{11}$
45·75	39·75	6·0	100	$5·8 \times 10^{11}$

Experiment II.

The cover slip examined was that used in I.

Manometer readings in cm. of mercury		Pressure difference between two faces of cover slip	Number of fringes that have disappeared	Young's Modulus
45·65	39·75	5·9	0	
45·45	39·95	5·5	25	$4·9 \times 10^{11}$
45·15	40·20	4·95	50	$5·8 \times 10^{11}$
44·85	40·45	4·4	75	$6·1 \times 10^{11}$
44·65	40·70	3·95	100	$5·9 \times 10^{11}$
44·4	40·95	3·45	125	$5·8 \times 10^{11}$
44·2	41·15	3·05	150	$5·8 \times 10^{11}$
43·9	41·4	2·5	175	$5·9 \times 10^{11}$
43·7	41·6	2·1	200	$5·8 \times 10^{11}$
45·6	39·8	5·8	0	
45·1	40·25	4·85	50	$5·8 \times 10^{11}$
44·9	40·45	4·45	75	$5·5 \times 10^{11}$
44·65	40·7	3·95	100	$5·64 \times 10^{11}$
44·4	40·95	3·45	125	$5·7 \times 10^{11}$
44·15	41·2	2·95	150	$5·8 \times 10^{11}$
43·9	41·4	2·5	175	$5·7 \times 10^{11}$

On a problem concerning the Riemann ζ-function. By S. WIGERT, Stockholm. (Communicated by G. H. Hardy.)

1. Let us denote by a a positive real number and by $S_\mu(n)$ the arithmetic function $\sum\limits_{dd'=n} d^{-\mu}$. If the real part σ of the variable s is greater than 1, then we have*

$$\sum_{n=1}^{\infty} \frac{S_1(n) S_a(n)}{n^s} = \prod_p \sum_{\lambda=0}^{\infty} \frac{(1-p^{-\lambda-1})(1-p^{-a(\lambda+1)})}{(1-p^{-1})(1-p^{-a})} \frac{1}{p^{\lambda s}}$$

$$= \prod_p \frac{1-p^{-2s-a-1}}{(1-p^{-s})(1-p^{-s-1})(1-p^{-s-a})(1-p^{-s-a-1})}$$

$$\frac{\zeta(s)\,\zeta(s+1)\,\zeta(s+a)\,\zeta(s+a+1)}{\zeta(2s+a+1)}$$

$$= \psi(s). \qquad \qquad \ldots\ldots(1)$$

Now suppose $a > 3$ and x real and positive. It is known that the function $F(x) = \sum\limits_{n=1}^{\infty} S_1(n) S_a(n) e^{-nx}$ can be expressed by the following integral

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(s)\,\psi(s)\,ds}{x^s},$$

if $a > 1$. We now transform this integral by Cauchy's theorem and obtain

$$F(x) = Q(x) + \frac{1}{2\pi i} \int_{-\frac{a}{2}-i\infty}^{-\frac{a}{2}+i\infty} \frac{\Gamma(s)\,\psi(s)\,ds}{x^s}, \qquad \ldots\ldots(2)$$

where

$$Q(x) = \frac{\zeta(2)\,\zeta(a+1)\,\zeta(a+2)}{\zeta(a+3)} \frac{1}{x} + \frac{\zeta(a)}{2}\log x + b - \frac{\zeta(a)}{24}x, \Bigg\}$$
$$b = \frac{\zeta(a)}{2}\left\{\frac{\zeta'(a+1)}{\zeta(a+1)} - \log 2\pi\right\} - \frac{\zeta'(a)}{2}. \Bigg\} \qquad (3)$$

As for the integral in (2), it is $o(x^{\frac{a}{2}})$, for $x \to 0$†.

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* This is not a new result. It is contained in a more general formula indicated by the late Mr S. Ramanujan (*Messenger of Mathematics*, Vol. XLV, 1916, p. 15).
† See Landau, *Handbuch* I, p. 265.

But if we admit an upper limit less than 1 for the real parts of the complex zeros of the ζ-function, we can deform the path of integration and replace $o\left(x^{\frac{1}{2}a}\right)$ by $O\left(x^{\frac{1}{2}a+\lambda}\right)$, where $0 < \lambda < \frac{1}{4}$. Again, if we suppose $F\left(x\right) - Q\left(x\right) = O\left(x^{\frac{1}{2}a+\lambda}\right)$, we can prove the formula

$$\Gamma\left(s\right)\psi\left(s\right) = \int_0^\infty x^{s-1}\left\{F\left(x\right) - Q\left(x\right)\right\} dx,$$

which holds for $-1 > \sigma > -\dfrac{a}{2} - \lambda$. Hence it follows that

$$\zeta\left(s\right) \neq 0 \text{ for } 1 > \sigma > 1 - 2\lambda > \tfrac{1}{2}\,{}^*.$$

The order of magnitude (for $x \to 0$) of $F\left(x\right) - Q\left(x\right)$ being thus most intimately connected with the position of the zeros of $\zeta\left(s\right)$, the question arises whether this function can be represented in a manner which does not involve the function $1/\zeta$. For this purpose we may proceed as follows. If we write

$$f\left(x\right) = \sum_{n=1}^\infty S_1\left(n\right) e^{-nx}, \qquad \ldots\ldots(4)$$

it is easy to show that

$$\sum_{n=1}^\infty S_1\left(mn\right) e^{-mnx} = \frac{1}{2m} \sum_{\nu=1}^m \left\{ f\left(x + \frac{2\pi\nu i}{m}\right) + f\left(x - \frac{2\pi\nu i}{m}\right) \right\}$$
$$= \frac{1}{m} \sum_{\nu=1}^m Rf\left(x + \frac{2\pi\nu i}{m}\right),$$

and furthermore [†]

$$F\left(x\right) = \sum_{n=1}^\infty S_1\left(n\right) S_a\left(n\right) e^{-nx} = \sum_{m=1}^\infty \frac{1}{m^a} \sum_{n=1}^\infty S_1\left(mn\right) e^{-mnx}$$
$$= \sum_{m=1}^\infty \frac{1}{m^{a+1}} \sum_{\nu=1}^m Rf\left(x + \frac{2\pi\nu i}{m}\right). \quad (5)$$

2. The series $f\left(z\right) = \sum_{n=1}^\infty S_1\left(n\right) e^{-nz} = -\log \prod_{n=1}^\infty \left(1 - e^{-nz}\right)$ is well known from the theory of the elliptic modular functions. If $R\left(\omega/i\right) > 0$, and if a, b, c, d are four integers, such that $ad - bc = 1$, the function f satisfies [‡] the transformation formula

$$f\left(\frac{2\pi\omega}{i}\right) + \frac{\pi\omega}{12i} = f\left(\frac{2\pi\omega_1}{i}\right) + \frac{\pi\omega_1}{12i} + \tfrac{1}{4}\log\left\{-\left(c\omega + d\right)^2\right\}$$
$$+ K\left(a, b, c, d\right)\frac{\pi i}{12}, \quad (6)$$

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* Compare Gram, *Acta Math.* 27, p. 290.

† Wigert, *Acta Math.* 37, pp. 134 *et seq.*

‡ Tannery et Molk, *Fonctions elliptiques*, Vol. 2. See also Dedekind's commentary on Riemann, *Fragmente über die Grenzfälle der elliptischen Modulfunctionen (Gesammelte Werke, pp. 466 et seq.).*

where $\omega_1 = \dfrac{a\omega + b}{c\omega + d}$ and $K\,(a, b, c, d)$ is an integer, whose form is of no importance. Suppose first ν and m without common factor and let us write

$$z = -2\pi i\omega = x + \frac{2\pi\nu i}{m}; \quad c = m,\ d = \nu,\ a\nu - bm = 1.$$

Then

$$\omega = -\frac{\nu}{m} - \frac{x}{2\pi i}, \quad m\omega + \nu = -\frac{mx}{2\pi i},$$

and

$$\omega_1 = \frac{a}{m} - \frac{1}{m\,(m\omega + \nu)} = \frac{a}{m} + \frac{2\pi i}{m^2 x}, \quad z_1 = -2\pi i\omega_1 = \frac{4\pi^2}{m^2 x} - \frac{2\pi a i}{m},$$

so that, from (6)

$$Rf\left(x + \frac{2\pi\nu i}{m}\right) = \frac{\pi^2}{6m^2 x} + \tfrac{1}{2}\log x + \tfrac{1}{2}\log m$$

$$- \tfrac{1}{2}\log 2\pi - \frac{x}{24} + Rf\left(\frac{4\pi^2}{m^2 x} - \frac{2\pi a i}{m}\right). \quad (7)$$

We have again to establish the corresponding formula in the general case, where ν and m are no longer relative primes. Denoting then by $D\,(\nu, m)$ the greatest common divisor of ν and m, it is plain that we must have

$$Rf\left(x + \frac{2\pi\nu i}{m}\right) = \frac{\pi^2 D^2\,(\nu, m)}{6m^2 x} + \tfrac{1}{2}\log x + \tfrac{1}{2}\log \frac{m}{D\,(\nu, m)} - \tfrac{1}{2}\log 2\pi$$

$$- \frac{x}{24} + Rf\left(\frac{4\pi^2 D^2(\nu, m)}{m^2 x} - \frac{2\pi a'\,(\nu, m)\,D\,(\nu, m)\,i}{m}\right), \quad (7\ bis)$$

where

$$a'\,(\nu, m)\,\frac{\nu}{D\,(\nu, m)} \equiv 1,\ \text{mod.}\ \frac{m}{D\,(\nu, m)}.$$

If we refer back to (5), we see that the function $F\,(x)$ will be represented in the form

$$F\,(x) = \frac{\pi^2}{6x} \sum_{m=1}^{\infty} \frac{1}{m^{a+3}} \sum_{\nu=1}^{m} D^2\,(\nu, m) + \frac{\zeta\,(\alpha)}{2}\log x - \frac{\zeta'\,(\alpha)}{2}$$

$$- \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m^{a+1}} \sum_{\nu=1}^{m} \log D\,(\nu, m) - \frac{\zeta\,(\alpha)}{2}\log 2\pi - \frac{\zeta\,(\alpha)}{24}\,x$$

$$+ \sum_{m=1}^{\infty} \frac{1}{m^{a+1}} \sum_{\nu=1}^{m} Rf\left(\frac{4\pi^2 D^2(\nu, m)}{m^2 x} - \frac{2\pi a'\,(\nu, m)\,D\,(\nu, m)\,i}{m}\right). \quad (8)$$

2—2

From (2), (3) and (8) we obtain the curious formulae

$$\left. \begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m^{a+3}} \sum_{\nu=1}^{m} D^2\left(\nu, m\right) &= \frac{\zeta\left(a+1\right)\zeta\left(a+2\right)}{\zeta\left(a+3\right)}, \\ \sum_{m=1}^{\infty} \frac{1}{m^{a+1}} \sum_{\nu=1}^{m} \log D\left(\nu, m\right) &= -\zeta\left(a\right)\frac{\zeta'\left(a+1\right)}{\zeta\left(a+1\right)}, \end{aligned} \right\} \quad \dots(9)$$

which can also be proved directly*.

In this manner we have really formed a new expression for the integral in (2), but its further discussion seems to be as difficult as would be expected from its bearing upon the zeros of $\zeta\left(s\right)$.

3. In this section we add a few remarks on the last term of (8). First, it may be observed that the series

$$\sum_{m=1}^{\infty} \frac{1}{m^{a+1}} \sum_{\nu=1}^{m} Rf\left(\frac{4\pi^2 D^2\left(\nu, m\right)}{m^2 x} - \frac{2\pi a'\left(\nu, m\right) D\left(\nu, m\right) i}{m}\right) \quad (10)$$

is absolutely and uniformly convergent for $0 \le x \le h$. Since the function $yf'(y)$ is bounded for $y \ge 0$, we have

$$\left| \sum_{m=q+1}^{\infty} \frac{1}{m^{a+1}} \sum_{\nu=1}^{m} Rf \right| \le \sum_{m=q+1}^{\infty} \frac{1}{m^{a+1}} \sum_{\nu=1}^{m} f\left(\frac{4\pi^2 D^2\left(\nu, m\right)}{m^2 x}\right)$$
$$< \frac{Cx}{4\pi^2} \sum_{m=q+1}^{\infty} \frac{1}{m^{a-2}} < \frac{Cx}{4\pi^2\left(a-3\right) q^{a-3}},$$

which proves the assertion.

On the other hand, the *majorant*

$$\sum_{m=1}^{\infty} \frac{1}{m^{a+1}} \sum_{\nu=1}^{m} f\left(\frac{4\pi^2 D^2\left(\nu, m\right)}{m^2 x}\right)$$

can give us no information about the order of magnitude (for $x \to 0$) of the series (10). We know it, in fact, to be $o\left(x^{\frac{1}{2}a}\right)$, but the majorant cannot be $O\left(x^{\frac{1}{2}\left(a-1\right)+\epsilon}\right)$, for any positive value of ϵ. To show this we may write†

---

* It is easily verified that we have, more generally,

$$\sum_{\nu=1}^{p\lambda} D^k\left(\nu, p^\lambda\right) = \left(p^\lambda - p^{\lambda-1}\right)\frac{p^{\left(k-1\right)\lambda} - 1}{p^{k-1} - 1} + p^{k\lambda},$$

and this leads to

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{\nu=1}^{m} D^k\left(\nu, m\right) = \frac{\zeta\left(s-k\right)\zeta\left(s-1\right)}{\zeta\left(s\right)},$$

where σ must be greater than $k+1$ if $k > 1$. Otherwise $\sigma > 2$ is sufficient. The second formula in (9) follows (for $s = a+1$) by differentiating with respect to k and putting $k = 0$.

† $\phi\left(m\right)$ is the well-known Eulerian function. It was shown by Landau that $\lim_{m \to \infty} \frac{\phi\left(m\right)\log\log m}{m} = e^{-\gamma}$, where γ is Euler's constant.

$$\sum_{m=1}^{\infty} \frac{1}{m^{\alpha+1}} \sum_{\nu=1}^{m} f\left(\frac{4\pi^2 D^2\,(\nu,\,m)}{m^2 x}\right) > \sum_{m=1}^{\infty} \frac{1}{m^{\alpha+1}} \sum_{\nu=1}^{m} \exp\left\{-\frac{4\pi^2 D^2\,(\nu,\,m)}{m^2 x}\right\}$$

$$> \sum_{m=1}^{\infty} \frac{\phi\,(m)}{m^{\alpha+1}}\, e^{-4\pi^2/m^2 x} > \sum_{m=p}^{\infty} \frac{1}{m^{\alpha+\epsilon}}\, e^{-4\pi^2/m^2 x} > \int_{p}^{\infty} \frac{e^{-4\pi^2/xt^2}\,dt}{t^{\alpha+\epsilon}}$$

$$= \frac{1}{2^{\alpha+\epsilon}\,\pi^{\alpha-1+\epsilon}}\, x^{\frac{1}{2}(\alpha-1+\epsilon)} \int_{0}^{4\pi^2/p^2 x} \omega^{\frac{1}{2}(\alpha-3+\epsilon)}\, e^{-\omega}\,d\omega,$$

for any positive value of ϵ. If x is sufficiently small, we can choose

$$p = 1 + \left[\sqrt{\frac{8\pi^2}{\alpha x}}\right].$$

The function $t^{-\alpha-\epsilon}\, e^{-4\pi^2/xt^2}$ then decreases for $t > p$, and

$$\frac{4\pi^2}{p^2 x} \geqq \left(1 - \frac{1}{p}\right)^2 \frac{\alpha}{2} > \frac{\alpha}{2} - \epsilon.$$

It is obvious, then, that the order of magnitude of (10) is due entirely to the variations of sign.

Note on Prof. T. H. Morgan's Theory of Hen Feathering in Cocks.
By M. S. PEASE. (Communicated by Professor Punnett.)

[*Read* 28 November 1921.]

It has been shown* that in the ovary of the hen there are cells whose appearance and chemical reactions are very similar, if not identical, with the luteal cells in the corpus luteum of mammals. In three recent publications†, Prof. T. H. Morgan has stated that such luteal cells can be found in the interstitial tissue of the testes of henny feathered cocks. "The abundance of these clear cells," says Prof. Morgan‡, "supposedly gland cells with endocrine influences, in the testes of hen feathered birds is in sharp contrast to their absence in normal adult cock birds. It seems to follow therefore, that the hen feathering in the Sebright is due to the presence of these cells whose function is the same as of similar cells in the female, *i.e.* the suppression in both of cock feathering."

With a view to confirming this very attractive theory, Prof. Punnett suggested that I should look through material collected from birds in his experiments on the inheritance of hen feathering§. The material has been taken from 28 birds at various ages, at every season of the year, and over a number of years. Several methods of staining were used, but for the most part Mallory's technique (cited by Boring and Pearl‖) was adopted. No difficulty was found in identifying the luteal cells in the ovaries and in some of the testes. But in material where spermatogenesis was in full swing with plenty of mature spermatozoa in the tubules, no indubitable luteal cells were found, either in the hen feathered or in the normal control material. Conversely, wherever material was taken from immature birds, or from birds where spermatogenesis was inactive or only beginning, luteal cells were always found not only in the hen feathered material, but also in the control material from normal birds. The complete results are given in the following table:

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* Boring and Pearl, *Anat. Record*, XIII, 1917.
† (*a*) Boring and Morgan, *Journal Gen. Phys.* 1918.
   (*b*) Morgan, *Endocrinology*, Vol. IV, No. 3.
   (*c*) Morgan. The genetic and the operative evidence relating to secondary sexual characters. Carnegie Institute, 1919.
‡ *Op. cit.* (note † (*c*)), p. 34.
§ *Journal of Genetics*, Vol. XI, No. 1.
‖ *Anat. Record*, XIII, 1917.

No. of Bird	Hatched	Killed	Age (months)	State of Spermatogenesis	Luteal Cells	Condition of Plumage
Ex.-27/21	iv. 21	17. ix. 21	5	Just beginning	A few visible	Normal cock feathers just coming up
Ex.-27/21	2. v. 21	2. x. 21	5	Not yet begun	Plenty	Normal cock feathers coming up
252/18	10. vi. 19	5. x. 21	28	Active	Very few, if any	Normal cock feathered, just moulted
Ex.-27/21	2. v. 21	17. ix. 21	4	Not active	Many visible	Normal cock feathers just coming up
Ex.-27/21	2. v. 21	10. ix. 21	4	Not active	Many visible	Male feathers coming up
Ex.-6W/20	—	—		Not yet begun	Plenty	Male feathered
40/20	14. iii. 20	2. viii. 20	5	Active	None seen	Male feathered
292/18	10. vi. 19	16. xii. 20	18	Inactive	Many, but not very clear	Male feathered
357/15	20. v. 15	22. ii. 16	9	Active	A very few	Male feathered
3/16	28. i. 16	17. iii. 16	2	Not active	Many	Male feathered
29/20	6. iii. 20	16. viii. 20	5	Active	No indubitable l.c. seen	Intermediate*
101/20	8. iv. 20	6. v. 21	13	Active	Very few visible	Intermediate
115/20	9. iv. 20	4. v. 21	13	Active	Very few, if any	Intermediate
61/20	14. iii. 20	10. viii. 20	5	Active	Only a very few rather dubious	Intermediate
52/20	11. iii. 20	10. viii. 21	5	Just beginning	A good smattering	Intermediate
7/15	1. iii. 15	26. x. 15	7	Active	None	Intermediate
Ex.-23/21	iv. 21	6. x. 21	6	Not active	Many	Intermediate feathers just coming up
267/15	5. v. 15	22. ii. 16	9	Active	A few very dubious	Intermediate
434/15	28. v. 15	22. ii. 16	10	Active	None	Intermediate
70/20	30. iii. 20	27. x. 20	7	Active	None	Hen feathered
70/20	30. iii. 20	8. iv. 21	13	Active	None	Hen feathered
Ex.-18/21	iv. 21	12. ix. 21	5	Beginning, but no full grown sperms yet formed	Some visible	Full hen feathered
Ex.-18/21	iv. 21	12. ix. 21	5	Not yet begun	Visible	Full hen feathered
3/20	21. ii. 20	10. viii. 20	6	Active	Only a very few, but quite distinct	Hen feathered
265/15	3. v. 15	22. ii. 16	9	Active	None	Hen feathered, very slightly tipped
Ex.-18/21	v. 21	4. x. 21	5	Not active	Many	Hen feathered
63/11	6. iii. 11	7. v. 15	14	Spermatogenic tissue quite degenerate. This bird was sterile†	Many	Hen feathered
Ex.-18/20	16 day old embryo			Inactive	Testis consists very largely of l.c.	Not determinable

* For the genetic nature of the intermediate birds, see Punnett, *Journal of Genetics*, Vol. xi, No. 1.
† This bird is referred to by Cutler and Doncaster, *Journal of Genetics*, Vol. v, No. 2.

These results point to the conclusion that the quantity of luteal cells in the testes depends on the state of spermatogenesis and is not, as Prof. Morgan has on several occasions asserted, associated with the type of plumage of the bird. It would seem more likely that the luteal cells constitute a supply of food for the actively forming
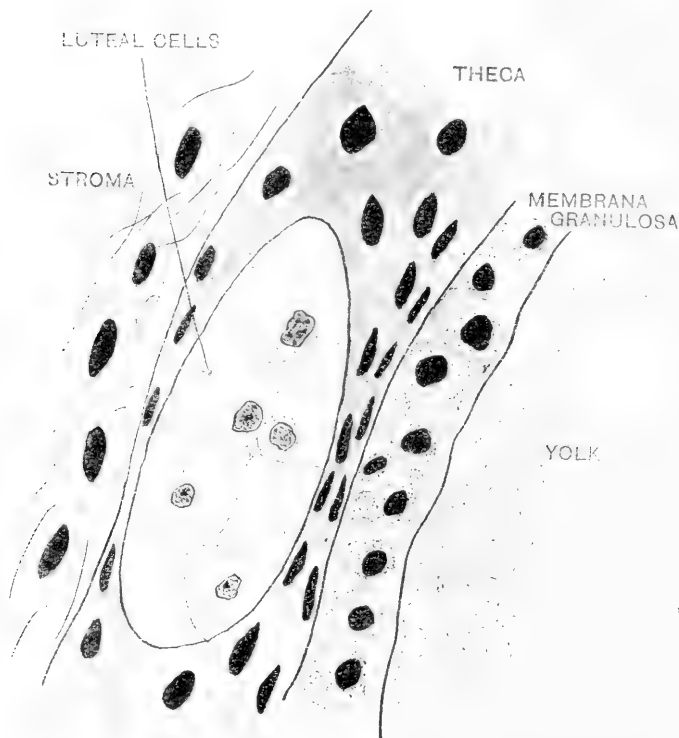
Fig. 1. Ovary.

sperms, which supply becomes used up as spermatogenesis proceeds. There is, let it be said at once, no direct evidence for this conjecture, but at any rate it does not so far conflict with observation.

One cannot escape the suspicion that Prof. Morgan has based his theory on material from two birds only, and that in these two birds spermatogenesis happened to be inactive. For in the first

place, he says* " ...in the summer of 1918 I had some new material
derived from a castrated Sebright male that had partly regenerated
its testes and was again going back to hen feathering and pieces
from one of the old testes of a castrated bird." The slides described
are apparently from this "new material" and nothing is subse-
quently said about any previous material. In any case, it should
be noted, some of the material was admittedly obtained from

LUTEAL CELLS   INTERSTITIAL TISSUE
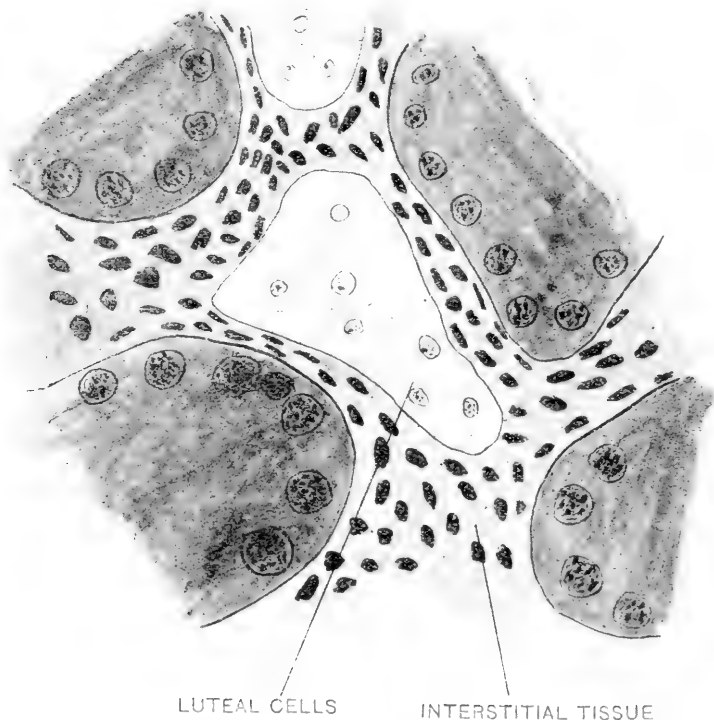
Fig. 2. Testis of Normal Cock. Spermatogenesis inactive.

regenerated testes, and it cannot be assumed without further in-
vestigation that this would be normal†. In the second place, Prof.
Morgan's own figures bear out the suggestion put forward in this
paper; for in the drawings‡ where he shows luteal cells in the testes,

* *Op. cit.* p. 1 note † (c), p. 34.
† My own observations on this point are confined to one henny feathered bird.
The regenerated testicular tissue on the peritoneum and on the wall of the body
cavity showed active spermatogenesis and no luteal cells.
‡ *Op. cit.* p. 1, note † (a) and (b).

no divisions at all are taking place in the sperm mother cells: moreover no control material from normal cocks for comparison is shown; in no figure does he show spermatozoa and luteal cells on the same slide.
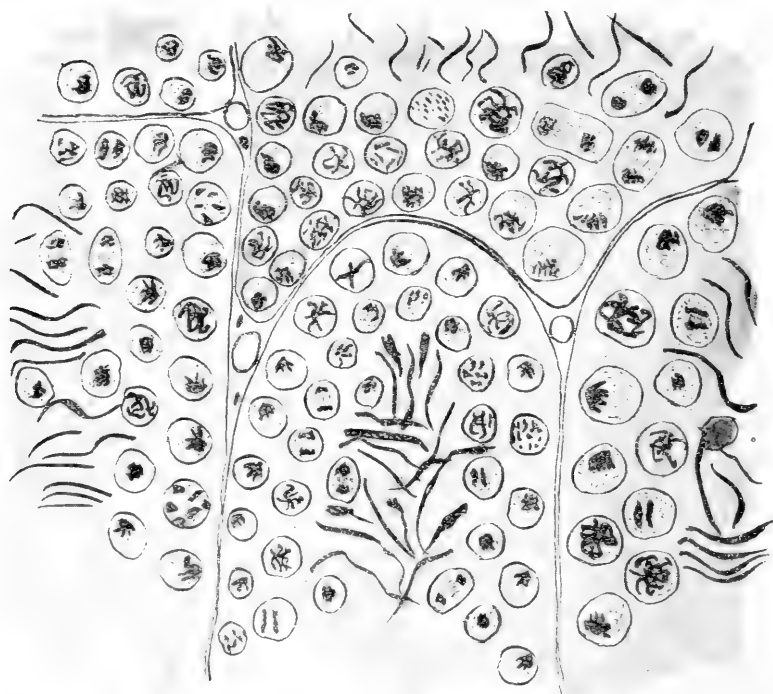
Fig. 3. Testis of Henny Feathered Cock. Spermatogenesis active: no luteal cells.

The Bionomics of certain parasitic Hymenoptera. By MAUD D.
HAVILAND, Research Fellow of Newnham College. (Communicated
by Mr H. H. Brindley.)

The relations of an animal with its enemies, predatory or
parasitic, its commensals, or its symbionts, form what may for
convenience be termed a bionomical complex.

Aphides, with their parasites and hyperparasites, form a well
defined complex of considerable intricacy. Numerous species of
Aphidiidae (Braconidae) are obligative internal parasites of plant-
lice, and their larvae, during development, are liable to infestation
by certain Cynipids, Chalcids, and Proctotrypids, which are there-
fore hyperparasites of the aphid.

The Cynipids (*Charips*) are endoparasites, and oviposit in the
Aphidius larva before development is complete, and while the
aphid host is still living (1). The Chalcids (*Asaphes*, *Pachycrepis*,
etc.) and the Proctotrypids (*Lygocerus*) are ectoparasites, and
infest the *Aphidius* after the latter has devoured the aphid, and
has woven a cocoon within the empty skin (2). The Cynipid is
always a parasite of the *Aphidius* and therefore a hyperparasite
of the aphid. The status of the Chalcids and Proctotrypids is more
difficult to determine, for, although usually parasites of *Aphidius*,
and therefore standing in the same relation to the aphid as does
Charips, they may on occasion be parasitic on each other. An
Aphidius cocoon sometimes contains two hyperparasites of either,
or of both, these forms, the result of successive ovipositions. This
phase of parasitism is sometimes called "superparasitism"; but
as the word means neither more nor less than hyperparasitism, a
term already employed to define cases where the parasite is itself
attacked by another parasite, I would suggest replacing this
etymological hybrid by "epiparasitism." In such cases in the
aphid complex, a single hyperparasite emerges from the cocoon,
having caused the death of its rival, not by direct attack, but by
obtaining the major share of the food.

Epiparasitism then may be defined as successive infestations of
a single host by two or more species, or by two or more individuals
of the same species, of parasite.

But occasionally a Chalcid hyperparasite larva may be found
with the larva of another Chalcid or Proctotrypid on its body.
The explanation seems to be that the intention of the second
hyperparasite was to oviposit upon the *Aphidius*, but when by
chance her ovipositor encountered the larva of the first, she was

unable to distinguish between it and the proper host, and placed her egg upon it. The mature larva of *Lygocerus* was never found thus infested. In this form, the last instar is capable of active movement in the cocoon when irritated, and thus possibly evades the ovipositor of another hyperparasite.

It is clear that this phase of parasitism differs somewhat from ordinary epiparasitism. It has been called "accidental super-parasitism," but might better be termed "metaparasitism," and defined as the direct attack of one epiparasite upon another.

The terms suggested here may be illustrated with examples from the aphid complex as follows:

Parasitism	Aphid + *Aphidius*.
Epiparasitism	Aphid + *Aphidius* + *Lygocerus* and *Asaphes*.
Metaparasitism	Aphid + *Aphidius* + *Asaphes* + *Lygocerus*.
Hyperparasitism	Aphid + *Aphidius* + *Lygocerus* or *Asaphes* or *Charips*.

Objection may be taken that the distinctions are too subtle for new nomenclature; but, of late years, the practice of introducing parasites to control pests in a new area has been much extended, and, before importing a parasite, it is very important to ascertain to what extent it is potentially metaparasitic.

In Hawaii (3) for instance, the efficiency of *Opius humilis*, Sil. in the control of the fruit fly (*Ceratitis capitata*) has been reduced by the introduction of two other forms, *Diachasma tryoni* and *Tetastichus giffordianus*, which, although primary parasites of the fly, proved to be metaparasitic on *Opius* when epiparasitic with it, and therefore tend to supplant it.

Cases such as these are of interest as a possible indication of the origin of parasitism in the Hymenoptera Parasitica. Thus epiparasitism, which is brought about by a high proportion of parasites to the host population, may lead to metaparasitism, and thence to obligative hyperparasitism. The origin of primary parasitism still remains to be accounted for. The suggestion that the parasitic habit arose in a common ancestor of the existing Parasitica, and was perpetuated by natural selection, involves the assumption of a considerable initial mutation. It seems more probable that in this group, the parasitic developed from the inquiline habit. In other words, the proto-Hymenoptera were phytophagous, and oviposited on plants. Later, for protection of the eggs and larvae, they resorted to galls and other vegetable deformities produced by members of their own, or other tribes of insects, and from inquilinism to parasitism is possibly not a great step.

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(2) 1921. *Ibid.* vol. 65, pt. III.
(3) 1918. Pemberton, C. E. and Willard, H. F., *Journ. Agric. Research*, Washington, vol. 12, no. 5.

A Method of testing Microscope Objectives. By Dr HARTRIDGE.

[*Read* 31 October 1921.]

In order that any substantial improvement may be introduced in the immediate future in the design of microscope objectives, it is essential that better methods of testing should be evolved than those worked out by Abbe and the older microscopists.

Methods previously in use had four disadvantages: (1) they tested the lens as a whole, (2) they did not give quantitative results, (3) they depended on the judgment and memory of the observer, (4) they tested an objective when combined with a certain eye-piece, not the objective only.

Recently two methods of testing microscope objectives have been proposed. (1) is a modification of the Hartmann test used for camera lenses and the like*, (2) is a proposed modification of an interference method of testing lenses†.

The first method has certain disadvantages. In order that the rays through the separate small apertures (placed at the back of the objective) should not interfere with one another, the photographic plate, used to record the positions of the rays, has to be moved a long way inside and outside the focus. Thus in the tests described by the author the displacement of the plate was 10·15 cms. (the distance of objective from its focus being 16·5 cms.). This big displacement causes the distances between the marks left by the rays on the plate to be large compared with the displacements of the rays by aberrations. Since also the centres of the marks on the plate are indefinite the accuracy of the determinations of aberration is small.

The second method (Twyman's) has in its application to camera lenses been recently described in some detail. The determination of aberrations depends on the recognition of a certain interference pattern and on its interpretation in terms of aberration. No data have so far been given of any tests on microscopic objectives carried out with it. And it would seem that with high power objectives at all events such a method of testing would prove very difficult.

The methods to be described by the author do not suffer from these disadvantages. In these methods either the focussing points

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* L. C. Martin, The physical meaning of spherical aberration, *Trans. Optical Soc.* Nov. 10th, 1921.

† Twyman, *Phil. Mag.* [6], XLII, 1921, p. 777.

of rays from different zones, or the lateral displacements of an image formed by the rays from different zones are determined. For the latter method either direct visual observation may be used or the displacements as recorded in a photographic plate can be subsequently measured. The principles on which these methods are based may be briefly described as follows:

According to geometrical optics, with a perfect lens system, points in the object are represented as corresponding points in the image. The image produced by an imperfect lens system differs from the above in that points in the object are represented as patterns, whose size, position and light distribution depend on diffraction and on the aberrations present. When as is generally the case all zones of an imperfect objective operate together, and white light is being employed, the image pattern is of a complicated nature, but if the aperture of the objective be sub-divided and examined piece by piece with approximately monochromatic light, then the pattern can be measured in detail by suitable micrometric apparatus. It is clear however that the dimensions of the image pattern will vary not only with the amounts of the different aberrations present in the objective, but also on the scale of the image (*i.e.* the magnification) and this will depend on the focal length of the objective. Therefore if objectives of high power are not to be penalized to the advantage of those of low, it is necessary either that the images should be on the same scale before the dimensions of the image patterns are measured, or that after the measurements have been made, the values should be reduced to unit magnification. In this way the values obtained should be comparable between one objective and another, and be independent of focal length, tube length and magnifying power.

Further the part of the aperture of the objective forming a given image pattern should be expressed in terms of N.A. so that the values at unit magnification may be plotted on squared paper against the N.A. values of the apertures.

This method of lens testing consists of two processes; firstly isolation of portions of the lens aperture the boundaries being determined in terms of N.A., and secondly the measurement of the sizes and positions of the image patterns thus produced.

With regard to the isolation of the required portion of the lens aperture, it was found that there were two alternative methods: (*a*) to illuminate the object on the stage by a full cone of light, and to limit the rays proceeding to the image plane by post objective stops. This method has the advantage that the isolation of a given objective zone is carried out very efficiently; it has the disadvantage however that the image pattern is to a considerable extent disturbed by diffraction effects which appear to be directly traceable to the post objective stop: (*b*) to limit the rays illum-

inating the object on the stage to narrow bundles corresponding to the particular objective aperture to be tested, by means of suitable stops placed in the lower focal plane of a well corrected condenser system. This method has the disadvantage that isolation of the required aperture is not perfect, because the diffraction caused by the object causes a wider zone to be filled than would otherwise be the case ; on the other hand it is found that the definition of the image pattern is particularly good, being practically unspoiled by diffraction effects, and therefore greatly facilitating the micrometric measurements which form the basis of the method. For these reasons I adopted the latter method as being the more advantageous, depending on the selection of a special test object to reduce diffraction to a minimum.

DESCRIPTION OF APPARATUS.

Test object.

With regard to test objects, since isolation is performed as I have just mentioned before the rays reach the stage, a test object producing diffraction effects to the smallest possible extent should be selected. Such an object may be readily obtained by mounting any fine metal filings, *e.g.* silver, between a slide and coverglass with Canada Balsam. One metal particle of small size is selected as the test object.

Microscope.

The illumination of the test slide was effected by an oil immersion Conrady condenser of 1·35 N.A. which was fitted with an auxiliary lens as previously described*, so that an iris diaphragm attached to the tail piece of the microscope should form the radiant at the correct "tube length" from the condenser. On this iris was focussed the image of a small automatic arc lamp (or later a pointolite lamp). A movable carrier actuated by a graduated micrometer screw, bearing the slit-shaped aperture, was placed between the auxiliary lens and the Conrady condenser, so that it admitted light to the required part of the aperture of the condenser, and so past the test object to the corresponding part of the aperture of the objective. The micrometer scale was calibrated in terms of N.A. by the method that I have previously described†. The objective being attached to the nose piece, an image of the test slide was caused to fall on the plane of an index placed at the upper end of the draw tube on the special stage shown in the diagram. This index was prepared by mounting a sable hair between two glass slips

* *Journ. Quekett Micro. Club*, XIV, Nov. 1919.
† *Journ. R. Micro. Soc.* 1918, p. 337.

with balsam. The end of the index, with the magnified image of
the test slide in coincidence with it, were together magnified and
presented to the eye of the observer by means of a high power
compound eyepiece, consisting of a 1-inch objective and a × 8 eye-
piece, the whole magnifying about 50 diameters. Experiment
showed that an eyepiece of this magnification was required.

The Micrometer.

For the measurements of the image pattern a glass plate micro-
meter was found to meet requirements because of its precision and
freedom from back lash. The glass plate was 1 mm. thick and was
mounted below the index already described, on a stand separate
from the rest of the instrument, so that rays from the objective
going to form the magnified image of the test slide passed through
it, as shown in Fig. 1.

The following is a brief description of the method of measure-
ment.

The test slide having been so adjusted on the stage that the
edge of a speck of silver coincided with the point of the index,
with the micrometer at its zero, alteration was now made in the
part of the aperture of the objective under illumination by shifting
the slit. If then the image pattern was seen to move, the glass
plate micrometer was tilted until the image of the silver edge once
more coincided with the point of the index. The tilt of the glass
plate could then be ascertained from a scale which was so arranged
that the deflections produced by the plate should be proportional
to the scale readings of the pointer by which it was tilted, and
from the tilt the actual movement of the image could be ascer-
tained.

Calibration of the Micrometer.

Theory shows that the deflection obtained for a given angle of
tilt varies not only with the thickness of the glass plate, but also
with the optical tube length. But the latter is not always readily
ascertainable, since it does not correspond in any way with the
length of the mechanical tube. Further the deflection when found
must be divided by the magnification of the objective in order, as
pointed out above, to state the size and position of the image
pattern at unit magnification. But the magnification of the ob-
jective also varies with the tube length. A method was therefore
sought for that would avoid the measurements of magnification
and optical tube length, and would calibrate the tangent scale of
the glass plate micrometer directly in terms of unit magnification.
I found that this could be done by substituting a ruled glass stage

micrometer for the test slide, keeping the tube the same length as that used for the tests. By placing on the stage the ruled glass micrometer so that two of its standard lines lay on either side of the index, then, by rotating the glass plate, the index could be

High power microscopic eyepiece

Index mounted on slide

Glass plate micrometer mounted on separate stand

Tube length adjustment

Objective under test

Test object on slide

Substage condenser

Micrometer for moving slit

Auxiliary lens

Iris diaphragm of radiant
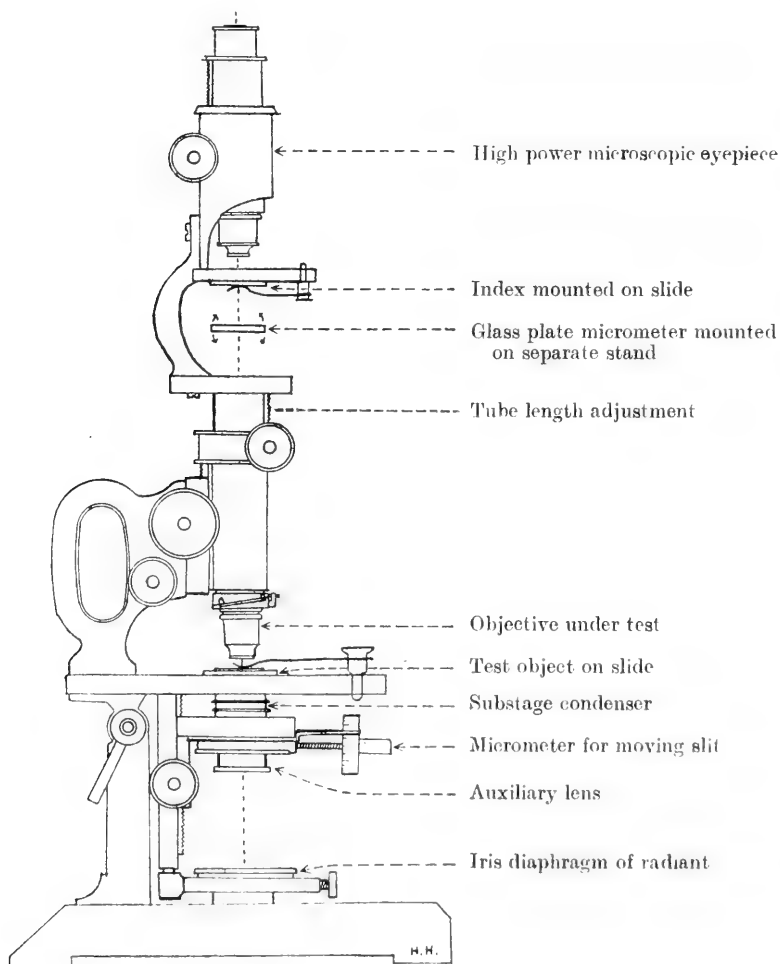
Fig. 1

made to coincide first with one and then with the other, and in this way the glass plate scale could be calibrated with each objective to be tested in terms of the same stage micrometer, that is at unit magnification.

Performance Curves and their Interpretation.

The form of the curves found under certain typical conditions according to theory will now be briefly described.

Centre of Field.

Perfect lens and correct focus give, when the deflections are plotted against N.A., a straight line parallel with the aperture axis as shown at **A**, Fig. 2. If the focus is imperfect the curve is still a straight line, but it is inclined to the axis, as shown at B, Fig. 2. When chromatic aberration is present the inclination of the line differs with the wave-length of the light.

Spherical aberration gives an **S**-shaped graph. For example C in Fig. 2. When the spherical aberration is regular (*i.e.* can be corrected by a change in tube length) the curve is found to follow the equation

$$x = 4 \left(\frac{y}{N} - \frac{y^2}{N^2} \right),$$

where N = total N.A. of lens and y = N.A. of aperture considered. If the experimental curve deviates from one given by this equation, then irregular (zonal) spherical aberration is present which cannot be corrected by any change in tube length. For example D, Fig. 2 which shows under correction of spherical aberration in the zones of large N.A. At E, Fig. 2 is shown effect of attempting to correct this under correction of the outer zones by over-correcting the whole lens by increasing the optical tube length. At F is shown the graph of a lens in which the attempt has been made to correct the outer zones by over-correcting the intermediate ones. If the shape of the curve differs with the wave-length of the light, then tube-length varies with wave-length and the elimination of spherical aberration for one colour leaves other colours uncorrected. Thirdly zonal spherical aberration absent for some colours may be present for others.

Periphery of Field.

A straight horizontal line denotes a perfect lens in correct focus, as for centre of field. If inclined the focus is imperfect. The difference in the inclinations of performance curves for centre and periphery shows the amount of curvature of field present. See Fig. 2, G, H, I.

A bent line is due to the presence of aberrations. If **S**-shaped spherical aberration is present. If **L**-shaped disobedience of the sine condition is indicated. See Fig. 2, J.

Chromatic difference of magnification is shown by inclination of line when wave-length is plotted against deflection. See Fig. 2, K.
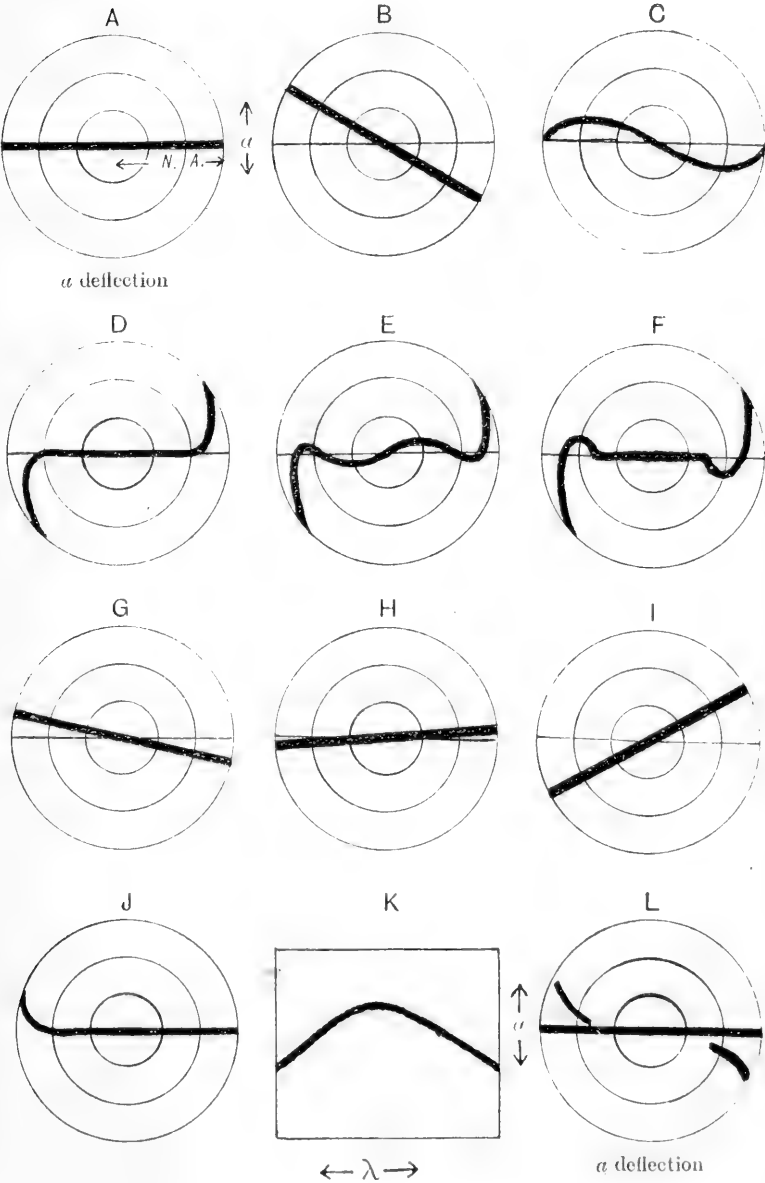
Fig. 2

True flare is shown by presence of a second image distinct from that of the ordinary image pattern, but the effects of flare are closely imitated by any uncorrected aberration when of large amount. See Fig. 2, L.
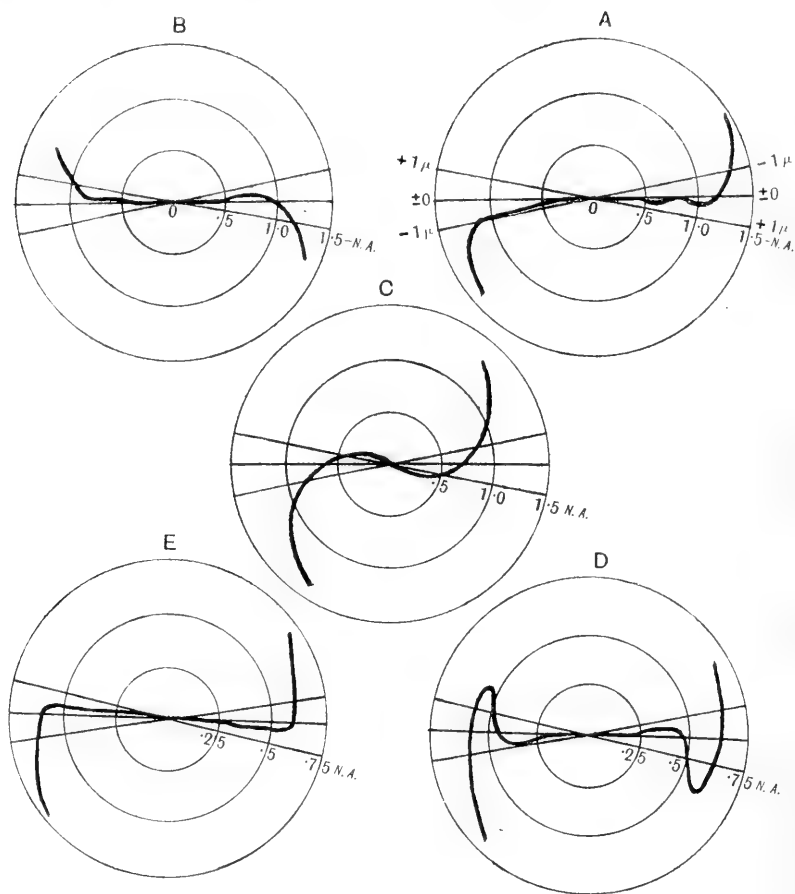
Fig. 3

The actual performance curves of a few objectives are given in Fig. 3.

B is that of a Zeiss apo. 2 mm. oil immersion 1·4 N.A., the tube length being a little too long, *i.e.* spherical over correction.

A is that of a Leitz 2 mm. oil immersion 1·3 N.A. tube length slightly too short.

C is same lens and adjustment as for A, but using glycerine as the immersion fluid instead of cedar oil. The curve shows nearly regular spherical aberration to be present. Cf. Fig. 2, C.

D is that of a Watson 4 mm. Holos ·95 N.A. dry, the under correction in the outer zones being partially corrected by using a long tube length. Cf. Fig. 2, E.

E is that of a Beck 4 mm. ·82 N.A. dry correct tube length and focus, showing under correction of spherical aberration in the outer zones.

The Insects and Arachnids of Jan Mayen. By W. S. BRISTOWE.
(Communicated by Mr H. H. Brindley.)

[*Read* 28 November 1921.]

The faunas of isolated islands are always interesting and of an
arctic one like Jan Mayen doubly so on account of the severe
hardships through which its inhabitants have to pass during the
long winter.

A brief analysis of my captures during August 1921 (exclusive
of water material to be mentioned later) shows that the following
arthropods, 65 per cent. of which are new to the island, were
obtained:

> Arachnida: 5 spiders, 6 mites, 1 tick.
> Hymenoptera: 2 Ichneumons.
> Diptera: about 12 species.
> Apterygota: 8 Collembola.

An examination of the known distribution of the Jan Mayen
species shows that, whereas all the spiders and ichneumons and
most of the Collembola are British, the flies, with one exception,
are not. There is reason to believe that at one time an insect fauna
existed in the south of the United States, among other regions,
similar to that found in the arctic. A period of cooling and of the
establishment of an arctic or circumpolar fauna followed. Then
came a glacial epoch during which all animal life was driven south.
Later on the temperature began to rise and the survivors, which
had now become adapted to a cold climate, began to pass north,
some settling on the high mountains and forming the alpine faunas
while stragglers reached the home of their ancestors. How does
this theory fit our facts? In the first place it should be noted that
arctic spiders and insects differ very little from those found in
temperate regions and in many cases identical species are found
in both. The same may be said of the arctic faunas of the east and
west hemispheres, several similar species being found in both. These
facts hold for the Jan Mayen species, for many are also found in
Continental Europe and a few in arctic North America, while the
remainder have been recorded from other arctic localities and
belong to genera represented in temperate regions. The theory is
further borne out by there being species in Jan Mayen identical
with those found in the Swiss and French Alps and Scottish moun-
tains. An interesting fact is that as they get further north the

altitude at which they are found decreases. Taking the spiders, we find that in Switzerland they occur above 7000 feet and in Scotland none have been taken below 3000 feet, yet in Jan Mayen they were all to be found wherever there was vegetation, sometimes almost at sea level. The case of the flies cannot be dealt with satisfactorily till they have been identified, but it is conceivable that isolation coupled with evolution should account for new species.

This brings us to the difficult question of how these arthropods ever reached an island so isolated as Jan Mayen, for it cannot be argued that it ever had any land connection. They must have crossed more than two hundred miles of open sea. It will be noticed that a large proportion of the species have also been recorded from Britain, Greenland and Siberia, while practically none occurs in Iceland. It may be that birds are largely responsible for the transport of insects, mites, or their eggs, from one place to another by these taking shelter amongst their feathers or being attached to their legs. They might even swallow and not digest them. A friend recently extracted a living weevil from a bird's dropping, but these beetles have a very tough integument. Birds may therefore transport insects to Jan Mayen from Greenland and possibly Britain during migrations. The driftwood so plentiful in all the bays of the island is supposed to come from the mouths of Siberian rivers. The eggs of spiders laid in a crack in the wood and protected by a waterproof covering of silk might easily travel across the sea, and it is conceivable that the eggs or even mature forms of other arthropods could do so also. The above appear to be the most likely means whereby the fauna of Jan Mayen arose, but in a subject like this where nothing definite can be proved unlikely explanations also should be considered. It is known that spiders can float enormous distances by their gossamer streamers. The flight of a fly is supposed to be a matter of a mile or two, but, in sailing up the fiords of Norway, I have seen a large number of flies and a few ichneumons flying round the ship even where the water way was nearly five miles wide. I watched a bluebottle following the ship for some way, then I saw it descend and settle on the surface of the water just where it was churned up most by the screw. It rested here for perhaps ten seconds before it rose and flew away. Long flight certainly seems a possible means whereby flies can travel long distances over water. Some mites can walk over water and Collembola do not get wet when submerged, though it is highly improbable that they can travel far across the sea. Another suggestion which has been advanced seems unlikely to be correct, *viz.*, that some of the insects, particularly Collembola, may have passed north actually on the ice, or with occasional rests on it, during an ice connection between the arctic and subarctic zones.

The shortness of the summer necessitates growth in great haste in order that the life cycle may be completed, and it has been suggested that more than one year is necessary for this, but my observations led me to suppose that in Jan Mayen at any rate this is not the case. It must be borne in mind that during summer there is no night, and in all probability the activities of Arthropods are fairly continuous. My grounds for holding that in the case of the spiders maturity is reached in one year were the comparative scarcity of immature specimens and the number of dead mature ones found towards the end of our visit, *i.e.*, early in September. An interesting modification due to the necessary haste was that all the eggs were beginning to hatch, whereas in temperate climates the winter is passed in the egg stage. The young would remain inside their protective silken cocoons during the winter and emerge directly the thaw arrived. Flies probably pass the winter in the pupal stage, for it is known that pupae can be frozen until they are as brittle as twigs and yet suffer no harm, and I found several pupae and larvae in advanced stages towards the end of our stay. The tick which is parasitic on sea birds and which has a very wide distribution was found in large numbers under stones on the cliffs where birds were accustomed to sit. It afforded one point of special interest, *viz.*, that it was found in all stages from the egg to maturity, and, as far as I am aware, the breeding of this form in the arctic has not previously been recorded.

As is usually the case in the arctic, species were few but specimens numerous. Under nearly every stone suitably situated amongst vegetation one or more spiders were found, and under one quite small stone I counted twenty-eight spider cocoons. Amongst the moss Collembola were numerous, and from two heads of one dandelion plant I shook eighteen ichneumons. These and the flies must be very useful in the fertilization of plants in these regions. Dandelions and *Saxifraga caespitosa* were the favourite haunts and in the latter small flies often seemed to be caught owing to the stickiness of the flower. The only arthropods to be found by turning over the driftwood on the bare stretches of sand were red mites. They acted as scavengers and were often very numerous under dead birds. They could walk over water and in this way devoured drowned insects. Flies also acted as scavengers by laying eggs in dead creatures and in bird droppings. I watched some of these and saw that the female laid her eggs whilst the male was on her back. She walked along, occasionally lowering the tip of her abdomen, while at the same time the male gave a jab with the end of his body. A fine bluebottle with a yellow head was the most "showy" member of the arthropod fauna.

Our sudden departure prevented me from doing several things in mind especially as regards the aquatic fauna. A bottle which

was let down in the North Lagoon to a depth of about 35 metres
and left for a fortnight contained Ostracods, Daphnids and Nauplii.
It also contained diatoms of various straight types, particularly
Navicula. On the last day of our stay some small trout were found
at the edge close to the entry of a stream. The water of this lagoon
tastes quite fresh.

In a small freshwater pool at the base of the Säule I picked out
three Cladocerans of the genus *Macrothrix*, and some diatoms of
the genus *Synedra* were taken in bottles left there for several days.

Fishing from the ship anchored on sand in about four fathoms
I secured swarming hordes of pink-eyed sand-hoppers. These and
also *Clione* were washed up by the sea and appeared to form the
chief food of the birds, among which ringed Plovers, Sanderlings
and Turnstones strutted up and down looking for them all day.

The following is a list of the Arachnids and Insects collected on
the island in August 1921, followed by a few fungi and mosses
which I happened to pick up and which have been identified for me:

Arachnida

Araneida

Coryphoeus mendicus L. Koch. Everywhere. Specimens eating
Collembola.

Sib., Nov. Zem., Spits., Scotland.

Hilaira frigida Thor. The only spider previously recorded.

Arc. N. Amer., E. and W. Green., Scotland above 3000 ft.

Micryphantes nigripes Sim. Not previously recorded from the
arctic.

French and Swiss Alps. Scottish mountains above 3500 ft.

Erigone tirolensis L. Koch. No males.

Sib., Nov. Zem., Spits., Tyrol, Swiss Alps (2700 metres),
Scotland.

Microneta sp.? One immature specimen.

Acarina

Bdella littoralis Linn. Scavenger.

Sib., Nov. Zem., Spits., Iceland, W. and E. Green., Nor.,
Eng.

Cyta brevirostris L. Koch.

Sib., Nov. Zem., Green., Eur.

Rhagidia gelida Thor.

Sib., Nov. Zem., Spits., Green., Ice., Lap.

Ceratoppia bipilis Herm.

Sib., Nov. Zem., Green.

Scutovertex lineatus Thor.

Sib., Nov. Zem., Spits., Eng., Eur.

Acarina
> *Cyrtolaelops Kochii* Trag. New to Jan Mayen.
>> Sib., Nov. Zem., Green., Eng.
> *Ixodes putus* O. P. C. New to Jan Mayen.
>> Eng., Green., Alaska, Cape Horn, Brit. Columbia, etc.

Hymenoptera
> *Stenomacrus intermedius* Hlgr. ⎫ New to the arctic. Probably
>>   ,,   *cubiceps* Thor. ⎭ parasitic on fungus-gnat larvae.

Aptera
> Collembola
>> *Xenylla humicola* Fab.
>>> Green., Nov. Zem., Eur., N. America, Eng.
>> *Onychiurus armatus* Tullb.
>>> Sib., Green., Eur., Brit.
>> *Onychiurus duodecimpunctatus* Fol.
>>> Arctic Canada.
>> *Isotoma sensibilis* Tullb.
>>> Nov. Zem., N. Amer., Eur., Brit.
>> *Isotoma viridis* Bour. Only species previously recorded.
>>> Sib., Green., Spits., N. Amer., Eur., Brit.
>> *Isotoma grisesiens* Schaffer.
>>> Eur., Brit.
>> *Sminthurides aquaticus* Bour.
>>> Eur., Brit.
>> *Sminthurinus niger* Lubb.
>>> Sib., Eur., Brit.

Diptera
> *Cynomyia mortuorum.* The only British species.
>> Sib., Nov. Zem., Green., Ice., Brit.
> *Limnophora.* Three species not yet identified.
> *Scotophaga.* Two species.
> *Acroptera.* One species.
> *Nemocera.* About five species.

Fungi
> *Omphalia* ⎫
> *Entoloma* ⎬ Agarics.
> *Hebeloma* ⎭
> *Pleurotus.*
> *Cordyceps.* Growing on spiders.
> *Rhytisma.* Growing on the leaves of *Salix herbacea* L.

Mosses
> *Tetraplodon mniodes* Br. and Sch. Previously recorded.
> *Bartramia subulata* Br. and Sch.
> *Dicranoweisia crispula.*

PROCEEDINGS

An Attempt to separate the Isotopes of Chlorine. By E. B. LUDLAM. (Communicated by Mr F. W. Aston.)

[*Read* 27 February 1922.]

PART I.

The work described in this paper was commenced immediately following the discussion on the subject of Isotopes at the Royal Society meeting on March 3, 1921 (*P.R.S.* May 2, 1921, p. 87). Since that time details of work by Harkins have been published (*J. Amer. Chem. Soc.* August, 1921) and a letter in *Nature*, July 14, 1921, from Brönsted and Hevesey announcing the success of their methods of separation. Although the experiments described below do not lead to a satisfactory positive result they possess some interest and value at the present juncture and will be briefly described.

The method adopted is the one suggested by Sir J. J. Thomson at the Royal Society discussion. As the masses of the two isotopes are different their average velocities will be different, and consequently the number of impacts per second on a surface exposed to the gas will be different, in inverse ratio to the square root of the mass. If, on striking the surface, they are removed by solution or chemical action, the composition of the residual gas should steadily alter, the process resembling diffusion but differing experimentally in technique.

Hydrogen chloride is superior to chlorine for a number of reasons, chief of which is that it only contains two types of molecule, HCl_{35} and HCl_{37}, whereas chlorine contains the three types $Cl_{35}Cl_{35}$, $Cl_{37}Cl_{37}$, $Cl_{35}Cl_{37}$.

A great variety of absorbents were tried but finally pure water was preferred. The HCl from a Kipps apparatus containing conc. sulphuric acid and large lumps of ammonium chloride, was passed through two wash bottles containing saturated HCl solution and issued from a jet over the surface of the water, producing a slight depression. The heavy solution immediately sank, setting up the circulation of about two litres of water and maintaining the water surface in a constant condition throughout the experiment. The vacuum above the surface was about 2 cm. of mercury and the rate of flow of HCl was so regulated that only a small fraction was able to escape absorption and pass forward into another vessel where it

was all absorbed. The solution of this residual gas was added to a known weight of silver dissolved in nitric acid, the whole slowly and carefully evaporated to dryness and the silver chloride fused and weighed.

Identical operations were carried out with the gas which had not been passed over a water surface first. Absolute values for the ratio of silver to chlorine were not being sought: the method was extremely simple and accurate.

In order to arrive at the most convenient rate for passing the gas a large number of preliminary experiments were performed and the solution of HCl obtained in the circulation water and the final absorption water was titrated, in suitable quantity, with standard sodium hydroxide solution. In some experiments a little methyl orange was added to the water in the end vessel so that it was possible to note the very gradual nature of the change in colour from yellow through orange to pink.

All water used for the quantitative work was distilled water which had been re-distilled, using a silica condenser, and was kept in a silica flask; similarly the nitric acid used for dissolving the silver was the middle portion of the distillate obtained by distilling ordinary pure nitric acid twice in the same silica apparatus and it was used as distilled without being kept at all.

Glass had to be used for the flask in which the residual HCl was absorbed, but the solution was dilute and was poured into a silica vessel at the end of each experiment, so that it was only in contact with glass for about two hours and the amount of alkali dissolved should be very small indeed.

The silver was obtained from Messrs Johnson and Mathey, guaranteed 99·9 % pure: such traces of impurity as might be present would probably be copper. For the purpose in hand absolute purity was not important; it was necessary that the material should be absolutely homogeneous.

The vessels used for the precipitation, evaporation and weighing were beakers of silica, which is superior to glass in that it can be obtained perfectly clean by a final heating in the blowpipe flame without any fear of cracking, being hard there is no loss by abrasion, and it is less likely to condense films of moisture on its surface. The beakers were tested for loss of weight on heating and cleaning, and at first there was a slight loss, but, afterwards, experiment showed that there was no loss when the vessels were taken round the same cycle of operations as in the actual determination, culminating in gentle heating to 460° C. and cooling in a vacuum desiccator.

The balance was a beautiful modern instrument very kindly placed at my disposal by Mr C. T. Heycock. With pans unloaded a deflection of one scale division was produced by ·095 milligram. To Mr Heycock also I am indebted for the use of a set of weights

with National Physical Laboratory certificate giving the correct weight in vacuo, and with this set I calibrated the set I used.

All weighings were performed against a counterpoise beaker of the same size and shape as the one containing the silver chloride, small pieces of silica having been added to bring the weights still nearer together. In this way atmospheric effects were eliminated and the actual metal weights were very few and small in bulk. The final weighings were always made in the early morning after the beakers had been left in the balance room over night.

With these precautions great confidence can be placed in the weighings.

Some trial experiments were made to obtain experience in the evaporating and drying of the silver chloride. The method finally adopted was to use a form of air bath consisting of silica plates and a glass bell jar. It was heated by a Bunsen burner and the acid fumes could be drawn off by a water pump. The beaker was always kept covered by a watch glass, separated from it by small glass rests. The silver chloride was kept screened from the light by large sheets of brown paper and the colour remained perfectly white. When dry it was gently heated until the silver chloride had all melted, then placed in a vacuum desiccator containing calcium chloride and sodium hydroxide until cold. It was then transferred to another desiccator, taken to the balance room and weighed several hours later. The fusion was repeated until there was no loss greater than a tenth of a milligram, the final weighing then being made after the beaker had been left in the balance room over night.

The following are the actual weights obtained:

Ordinary HCl (*a*)

wt. of silver $= 2\cdot9671$ wt. of AgCl $= 3\cdot9429$

corrected for air displaced $= 2\cdot96744$,, $= 3\cdot943705$

Ag : Cl $= 1 : \cdot3290$ (0).

'*Residual*' HCl (*b*) Reduction in volume 1000 : 15

wt. of silver $= 2\cdot7401$ wt. of AgCl $= 3\cdot6413$

corrected   ,, $= 2\cdot74041$,, $= 3\cdot64208$

Ag : Cl $= 1 : \cdot3290$ (3).

(*c*) Reduction in volume 1000 : 28

wt. of silver $= 5\cdot9053$ wt. of AgCl $= 7\cdot8477$

corrected   ,, $= 5\cdot90589$,, $= 7\cdot84938$

Ag : Cl $= 1 : \cdot3290$ (8).

The specific gravity of silver was taken as $10\cdot5$ and of silver chloride as $5\cdot6$.

Richards' value by the same method was

Ag : Cl $= 1 : \cdot3287$.

The difference between this result and (*a*) may be attributed to the extra purity of his silver.

It is clear that the differences obtained all lie in the region of experimental error, and this would not have been the case had the method been providing an efficient means of separation.

The increase of (*b*) over (*a*) for 1 gm. of silver is ·00003 gm. For 3 gms. of silver (about the actual wt. taken) this would be ·00009 gm.: a perfect separation* should have given an increase of ·0012 gm.

The increase of (*c*) over (*a*) for 1 gm. of silver is ·00008. For the 6 gms. of silver this would be ·00048: a perfect separation should have given ·0021 gm.

Hence, if the separation had been effected, the weighings would have shown it with certainty.

PART II.

Ammonia Gas the Absorbing Surface.

As no separation of the isotopes of chlorine had been effected by passing HCl over water, another method was adopted, exactly the same in principle but free from the objection that the 'mixing' at the surface of the water might be very imperfect owing to the extreme rapidity with which the absorption takes place even under greatly reduced pressure.

The method chosen was to make use of the fact that ammonium chloride can be made to sublime and dissociate by a convenient rise of temperature, and association precedes condensation.

By adding a slight excess of HCl before each sublimation and removing the excess remaining after condensation it was hoped that the latter would show an increase in density; the chief assumption being that recombination would take place as the result of molecular impact and that the lighter HCl would make a greater number of impacts per second than the heavier in virtue of its greater average velocity. It was also assumed that mere volatilisation and condensation alone, as distinct from dissociation, would effect practically no separation at all under the conditions of the experiment.

Twenty grams of ammonium chloride were employed. This quantity would give rise to about 9 litres of HCl at ordinary temperature and pressure, if dissociation were complete, but as the degree of dissociation is not much above 66 % the volume of HCl would only be about 6 litres. The presence of excess HCl would tend to reduce this to an extent dependent on its partial pressure. As about 20 c.c. at ordinary temperature and pressure were added, this would produce a pressure of about one-hundredth of an atmosphere at the ordinary temperature in the two litre flask employed,

* Lord Rayleigh, *Phil. Mag.* 1896, II. p. 493.

and double that at the temperature employed, about 290° C., say 1·5 cm. of mercury. The observed pressure during sublimation from the bulb into the neck of the flask was about 14 cm. but when the neck was heated and condensation took place in the bulb, the manometer only showed a pressure of about 4 cm. In the former case the effect may be neglected, in the latter case it might have been of importance if our final result had shown that any appreciable separation was being attained. Making rough allowance for this effect we may set the volume of HCl produced at 5 litres.

The operation is, then, analogous to passing 5 litres of HCl over a large surface of ammonia with which it is perfectly mixed and allowing combination to proceed until only 20 c.c. of HCl remain. The reduction in volume is 1000 : 4 and the increase in the density of the hydrochloric acid should amount to ·058 (say, from 36·5 to 36·558). One gram of silver should combine with ·32922 gm. of this heavier chlorine as compared with ·3287 obtained by Richards for ordinary chlorine, difference ·00052.

In our experiments about 3 gms. of silver were employed, so that the actual increase in weight to be sought was

$$\cdot 00052 \times 3 = \cdot 00156,$$

which is fifteen times the experimental error and should be determinable with certainty.

It is, perhaps, well to point out that even had the reduction in volume been only half that claimed, there would still have been an increase in weight on 3 gms. of silver of ·0013, and the success of the method does not depend in any important degree on being able to calculate with great accuracy the relation between the volumes of the total and residual HCl.

As before, the HCl was prepared by the action of conc. sulphuric acid on ammonium chloride in a Kipps apparatus, washed twice with saturated HCl solution and in this case dried by P_2O_5. A tube, 20 c.c. in volume, was filled with the gas which could then be allowed to enter the two litre flask by opening a tap. The flask contained 20 gms. of pure ammonium chloride and was placed in an electrically heated oven. The temperature was raised to 290° C. and kept there for about an hour and a half, after which time all the ammonium chloride had sublimed into the neck. It was then allowed to cool and when quite cool the residual HCl was pumped off through a U tube containing pure water, a P_2O_5 tube being interposed to prevent any moisture finding its way into the tubes leading to the flask.

Another 20 c.c. of HCl were allowed to enter the flask and the ammonium chloride sublimed back into the bulb of the flask by means of an electrical heater placed round the neck. On cooling, the residual HCl was removed as before and this cycle of operations

was repeated until a sufficient quantity of HCl solution had been obtained to precipitate all the silver which had been weighed out and dissolved in nitric acid in advance. The same water was not used for the whole series, but was changed three times, consequently the acid was always very dilute and the effect on the glass would only be very slight.

Before commencing the final series of operations tests were made by titration to see that of the 20 c.c. admitted practically the whole was obtained in the water after pumping out, and this was found to be the case.

With the apparatus in its final form 29 sublimations were carried out, and the solutions of HCl were added as they were obtained to the silica crucible containing the silver nitrate. For the twenty-ninth, fresh water was used and this was added cautiously to the clear liquid above the precipitate. It produced no turbidity, showing that all the silver had been precipitated.

The silver chloride was dealt with as described in Part I, the only difference being that a smaller and lighter silica crucible and counterpoise were used and the evaporation was conducted in a copper steam oven at 95° C. The crucible was covered by a watch glass resting on glass supports and the oven was lined with clean sheet lead. As before, the final weighings of the fused silver chloride were made in the morning after the crucible and counterpoise had been standing all night in the balance room.

Results.

(1) *Ordinary* HCl (*i.e.* HCl passed through the apparatus and absorbed in the water in the U tube but the ammonium chloride not heated):

> 3·1311 gms. Ag gave 4·1607 gms. AgCl uncorrected.

> 3·13145 gms. Ag gave 4·1616 gms. AgCl corrected.

> 1 gm. Ag combined with ·3289 (7) gm. Cl.

(2) '*Heavy*' HCl:

> 2·5990 gms. Ag gave 3·4540 gms. AgCl uncorrected.

> 2·5993 gms. Ag gave 3·45474 gms. AgCl corrected.

> 1 gm. Ag combined with ·3290 (3) gm. Cl.

This is a gain of 6 in 32,000.

This is equivalent to an actual gain on 3 gms. of silver of 1·9 tenths of a milligram. This can be weighed with certainty, but, nevertheless, must be considered as being of the order of experimental error when, in addition to weighing, all the manipulation is contributory.

DISCUSSION OF RESULT.

There is a positive increase in weight, but very minute. The only differences in treatment in the two cases were that the 'heavy' HCl had been obtained as residue after the sublimation of the ammonium chloride and that it had been kept for several days in a glass U tube. This tube was not of new glass and had been used in preliminary experiments for a month before quantitative work was begun. Nevertheless it might contribute so slight an amount.

To account for the failure to obtain a separation we must suppose that the determining factor in the re-combination of the ammonia with the HCl is not mere collision. It is known that a trace of moisture is necessary before the combination takes place; the relatively few water molecules may be combined with HCl molecules and these may be the nuclei round which all the condensation occurs. Even so, one would have expected the lighter molecules to get there most frequently.

If, however, there is some attractive force between the molecules of ammonia and HCl, due to polarisation in the molecules themselves and exerted when they approach in favourable positions, the chance of such a position will not depend at all on the mass of the two HCl molecules and no separation would be expected.

RESULT OF PART I AND PART II.

$$Ag : Cl$$

Ordinary HCl (August)	$1 : \cdot 3290$ (0)	
,, (December)	$1 : \cdot 3289$ (7)	
Heavy HCl (August) (water surface)	$\begin{cases} 1 : \cdot 3290 \ (3) \\ 1 : \cdot 3290 \ (8) \end{cases}$	Mean $1 : \cdot 3290$ (6)
,, (December) (ammonia)	$1 : \cdot 3290$ (3)	

Richards' result by same method, using specially pure silver,
$$1 : \cdot 3287.$$

Increase August (average) $\cdot 6$ in 3290 — about 1 in 5000.
December ,, $\cdot 6$,, $=$,, 1 in 5000.

SUMMARY.

Hydrogen chloride at a pressure of a few centimetres of mercury was passed over (*a*) a water surface, (*b*) ammonia gas, and a small fraction allowed to remain uncombined. In neither case was there an increase in density greater than 1 in 5000 and this could be attributed to experimental error.

My thanks are due to Sir J. J. Thomson, at whose suggestion the work was undertaken, for his continued interest and advice.

The Measurement of Magnetic Susceptibilities at High Frequencies. By MAURICE H. BELZ, B.Sc., Barker Graduate Scholar of the University of Sydney. (Communicated by Professor Sir E. Rutherford, F.R.S.)

The heterodyne beat method, employing thermionic valves, has been successfully employed by Herweg[*], Whiddington[†], and others in the measurement of physical quantities. In the present paper, an account is given of its application to the measurement of susceptibilities of fairly low order at frequencies ranging from 3×10^5 to 4×10^5 per second.

If in a circuit containing inductance L and capacity C, oscillations are maintained which have a frequency very different from the natural frequency of the coil L alone, then the frequency is given very approximately by

$$n = \frac{1}{2\pi \sqrt{LC}}.$$

If the inductance is altered to $L + dL$, the capacity remaining unchanged, the new frequency will be $n + dn$, where

$$\frac{dn}{n} = -\frac{1}{2} \cdot \frac{dL}{L}.$$

The change in inductance can, under proper conditions, be produced by inserting the specimen inside the coil, hence since a change in n of 1 per second can easily be measured, with a frequency of 3×10^5 per second, it is possible to measure the ratio dL/L to 1 part in 150,000.
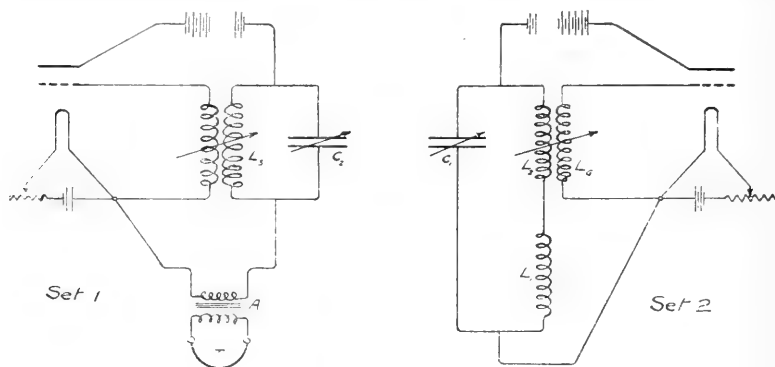
Fig. 1

* Herweg, *Verh. d. D. Phys. Ges.* 21, 572, 1919; *Zeit. f. Phys.* 3, 36, 1920.
† Whiddington, *Phil. Mag.* 40, 634, 1920.

The experimental arrangements are shown in Fig. 1. Set 1 is an ordinary oscillating circuit having an amplifier A and telephone T in the plate circuit. Set 2 is the oscillating circuit in which the changes in inductance were produced. The plate inductance was divided into two parts L_1 and L_2 between which there was no mutual inductance, the part L_2 being coupled to the grid inductance L_G to maintain the oscillations, the part L_1 being uncoupled. The two sets were loosely coupled to produce the heterodyne note in the telephone T.

The specimen was inserted inside the coil L_1, the whole change in the inductance $L_1 + L_2$ causing the note in T to alter. In practice this note was adjusted so as to be slightly out of tune with a third oscillating system producing an audible note, thus giving a certain number of beats per second. The insertion of the specimen caused this number to change, and by observing this change it is possible to calculate the susceptibility of the specimen.

Let A be the area of cross section of the coil, A' that of the specimen. Then if l is the length of the coil, l' the length of the specimen, the volume susceptibility of which is K_v, the new inductance of the circuit is

$$L_1' = L_1 \left[A - A' + (1 + 4\pi K_v)\, A' \right] l'/(Al)$$
$$= L_1 \left[1 + 4\pi K_v A'/A \right] l'/l.$$

Therefore $dL = L_1 \,.\, 4\pi K_v \,.\, A'l'/(Al),$

$$\frac{dL}{L} = \frac{dL}{L_1 + L_2} = 4\pi K_v \,.\, \frac{A'}{A} \,.\, \frac{l'}{l} \,.\, \frac{L_1}{L_1 + L_2},$$

whence $\quad K_v = \dfrac{1}{2\pi} \,.\, \dfrac{dn}{n} \,.\, \dfrac{\text{volume of coil}}{\text{volume of specimen}} \,.\, \dfrac{L_1 + L_2}{L_1}.$

On account of the very high frequency oscillations employed, it was necessary to shield everything from outside electrostatic disturbances. The whole of the apparatus was enclosed in earthed metal lined boxes, and the capacity C_1, by means of which final small adjustments were made, was provided with a long spindle which projected beyond the box. The coil L_1 was outside the box, but was specially shielded from all disturbances, both from outside and inside. The whole success of the experiment resulted from the precaution to shield the coil from the electrostatic effects of the specimen. The importance of this precaution was realised in some work similar to that described here, in which an electric discharge took the place of the specimen. Without it, the dielectric constant of the substance produces effects which completely mask the magnetic effect. The coil was wound on a long glass tube on which a very thin layer of platinum, 10^{-5} cm. thick, had been deposited

and which was earthed. The coil was protected on the outside by a metal cylinder.

The inductances of the coils L_1, L_2, and L_3 were designed to give approximately the same frequency of oscillation, but it was found that the fundamentals of the two circuits could not, on account of the limitations in the size of the box, be made to produce a sufficiently low beat note for comparison with the audible note from the third circuit. This synchronisation effect has been examined by Mr E. V. Appleton. In practice, the fundamental of Set 1 was made to beat with the first overtone of Set 2 to give the required note.

To determine the frequency of the fundamental of Set 2, the capacity C_2 was altered until the mid-point of the silent space was reached. The wave length of Set 1 was then observed with a calibrated Townsend wave-meter, and the frequency calculated. Let this be N, then if n is the frequency of the fundamental of Set 2, $N = 2n$.

Let the frequency of the third circuit be m. Then when Set 1 is altered so as to give q beats per second between the heterodyne and the audible note, the frequency of Set 1 is $2n \pm m \pm q$. Now if the frequency of the fundamental of Set 2 is altered by dn, the frequency of the first overtone is altered by $2dn$, so that the frequency of the heterodyne beat note is now $m \pm q \pm 2dn$, whence if a change of p beats per second is counted when the specimen is inserted, $p = 2dn$.

Therefore $$p/N = dn/n.$$

Hence $$K_v = \frac{1}{2\pi} \cdot \frac{p}{N} \cdot \frac{\text{volume of coil}}{\text{volume of specimen}} \cdot \frac{L_1 + L_2}{L_1} \quad \ldots\ldots(\text{i}).$$

Before the formula can be applied several corrections have to be investigated. Eddy currents are produced in the substance, the effect of which is to diminish the inductance and produce a change of frequency in a direction opposite to that due to the susceptibility. This effect should depend on the conductivity of the substance. In order to test the point, a solution of sulphuric acid, which had a susceptibility below the limit of reading and a conductivity far greater than that of any of the substances examined, was inserted inside the coil. No change in the beat note was observed over 10 seconds. It was thus certain that this correction is negligible.

Further corrections are required for the demagnetising effect on the field due to the magnetism induced on the specimen; for the absorption in the platinum shield; and for the fact that the specimen had a length small compared with the length of the coil. However, the effect of all these corrections can be shown to produce a change in K_v comparable with the experimental error, and so can be included therein.

Employing formula (i), the susceptibilities of certain iron salts have been calculated. In each case, over the range of frequency employed it was found that K_v was constant. The specimens were contained in short lengths of glass tube, and were inserted by means of a silk thread passing over a pulley. The change of beat note was observed on insertion and removal. The effect of the glass was previously found to be zero.

The results with iron salts are included in Table I.

TABLE I.

Substance	Mass susceptibility of water-free salt, K_m	Temperature
$FeCl_3$ (solution)	$90 \cdot 7 \times 10^{-6}$	15° C.
$FeSO_4$, $7H_2O$ (crystals and powder) ...	$74 \cdot 0 \times 10^{-6}$	16° C.
$FeSO_4$ $(NH_4)_2SO_4$, $6H_2O$ (crystals) ...	$41 \cdot 1 \times 10^{-6}$	16° C.

These results compare well with the values found by balance and low frequency methods, the mean of these values, taken from Landolt and Börnstein's Tables (1912), being 91×10^{-6}, 78×10^{-6}, and 44×10^{-6} respectively. This is probably to be expected, for it is only at very high frequencies when the period of oscillation is comparable with the 'time of relaxation' of the molecule that a change in the permeability would occur. Such changes have been observed in the case of iron and nickel by Arkadief* who found that for a frequency of 2×10^{10} per sec., the permeability of iron was about 8.

I wish to record my best thanks to Sir Ernest Rutherford for suggesting the problem and for his helpful advice and encouragement during its progress; also to Mr E. V. Appleton for initiation into the technique of the experiment and for many useful suggestions.

* Arkadief, *Ann. d. Phys.* 58, 2, 105. 1919.

CAVENDISH LABORATORY,
    CAMBRIDGE.
    27 *February*, 1922.

Note on an attempt to influence the random direction of α particle emission. By G. H. HENDERSON. (Communicated by Prof. Sir E. Rutherford, F.R.S.)

[*Read* 27 February 1922.]

§ 1. *Purpose of Experiment.*

In a recent communication to the Cambridge Philosophical Society Mr J. L. Glasson* drew attention to some peculiarities in the ionization photographs of Mr C. T. R. Wilson, which depicted the α ray tracks due to radium emanation which was diffused throughout the expansion chamber. It was pointed out that there seems to be a tendency for all the α ray tracks to point in definite directions. It was suggested ·by Mr Glasson that such an effect was unlikely to be due to chance but might be explained by assuming that the α particles were emitted from the emanation nucleus in a definite direction, the emanation atoms themselves being oriented by some orienting force, possibly a magnetic field.

It occurred to the writer that such a suggestion could be readily tested by an ionization method which gives an average effect, for which a statistical study of a large number of photographs would be necessary. The verification of Mr Glasson's suggestion that the α ray tracks may be oriented in definite directions in some conditions, is evidently a matter of some importance. For such an effect must have the following consequences, (1) that in some manner the emanation atoms have been oriented in a definite direction, and (2) that the α particle itself is emitted from the emanation nucleus in a definite direction. The experimental verification of (2) would be of assistance in furthering our views of nuclear structure. A negative result (which, to anticipate, is the result of the present attempt) would not necessarily lead to a negation of (2) but may only mean that the forces applied were unable to orient the atoms as in (1).

§ 2. *Apparatus.*

The experimental arrangement used is indicated in the accompanying diagram. At one end of the glass vessel *A* is fixed the ionization chamber *B*, formed of an insulated brass disk connected to a quadrant electrometer, and a gauze 4 mm. in front of the disk and maintained at a potential of 80 volts.; 4 mm. in front of the gauze was a second gauze connected to earth, thus forming

* Glasson, *Proc. Camb. Phil. Soc.* XXI, p. 7. 1921.

the usual protective antechamber to the ionization chamber proper. At the other end of A was fitted a narrower tube closed by a ground glass joint carrying the emanation chamber C. This chamber was formed by widening the end of a glass capillary tube and covering the end with a thin sheet of mica of about 1 cm. air equivalent. A fraction of a millicurie of radium emanation was used.

The distance from C to the ionization chamber was about 22 cm. and accordingly the air pressure in the chamber A was reduced until the α particles could enter the ionization chamber. To allow for the effect of β and γ ray ionization, a thin aluminium sheet D of 8 cm. air equivalent could be moved to intercept the α rays. Thus the ionization current due to β and γ rays, which was always small compared with that of the α rays, could be measured and allowed for. The sheet D was connected by means of a thread to a piece of brass rod in the rubber tube forming the connection to the pump. By moving the rod along the rubber tube the sheet D could be moved from outside the vessel into any position desired.
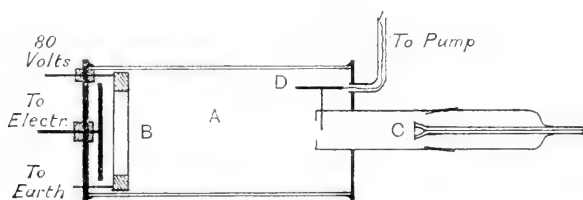
§ 3. *Results.*

By means of an electromagnet with split pole pieces a magnetic field of 860 gauss could be applied either perpendicular or parallel to the axis of the α ray beam. The effect of the β and γ rays was allowed for and as the α ray beam was limited by diaphragms so that the particles entered only the central portion of the ionization chamber, no appreciable error could arise due to the bending of the α rays by the field. The air pressure was adjusted so that the ionization chamber was at the maximum of the ionization curve due to emanation α particles.

The results obtained were entirely negative. Single ionization measurements with parallel and perpendicular fields agreed within 1·5 per cent., the experimental error, while the averages agreed well within 1 per cent. As a little less than half of the ionization measured was due to emanation α particles (the rest being due to the active deposit) this result means that if any effect at all is produced by the magnetic field, it is certainly less than 2 per cent.

The air pressure in the vessel was also adjusted until the maxima due to the α particles from radium A and radium C fell within the

ionization chamber and an orienting effect of the field on the solid active deposit was looked for with similar negative results. Equally unsuccessful was an attempt with a deposit of thorium C on a nickel plate.

Thus a magnetic field of the order of magnitude which Mr Glasson has suggested might be present in the Wilson apparatus is quite unable to produce any appreciable influence on the distribution of α ray tracks, and the cause of such distributions as those called attention to, if not due to chance, must be looked for elsewhere.

On theoretical grounds no appreciable effect could be expected. Taking Langevin's[*] theory of paramagnetism as a basis it may be shown simply that, unless the emanation atom possessed an extraordinarily high magnetic moment, an appreciable orienting of the emanation atoms could only take place at extremely low temperatures.

The possible effect of an electric field was also looked for. Small electrodes were sealed into the emanation chamber and ionization currents measured (1) when the electrodes were at the same potential, and (2) when they were charged so as to give a potential gradient of about 400 volts per cm. perpendicular to the α ray beam. Again no appreciable effect was observed.

[*] Langevin, *Ann. de Chim. et de Phys.* 5, p. 70. 1905.

Determination of the Coefficient of Rigidity of a glass plate. By J. E. P. WAGSTAFF, M.A., Lecturer in Physics at the University of Leeds. (Communicated by Prof. R. Whiddington.)

[*Read* 27 February 1922.]

The following is a short account of an interference method of determining the rigidity of a glass plate, which was originally devised as a laboratory experiment. Since no experiments made along similar lines have been found, it was thought an account of the apparatus and the results which have been obtained might be of interest. The work was suggested by Cornu's experiments (*Comptes Rendus*, 1869) on the measurement by an interference method of the bending of glass plates, and was carried out in the Physics Laboratories of Leeds University.

Description of apparatus. The beams, whose rigidities have so far been determined, have been 10 to 15 cm. long, of rectangular section, having a width of 1·8 cm. and a depth of ·18 cm. approximately. The surfaces of the beams are reasonably flat and give broad straight fringes when placed in contact with an optically flat surface. One end of the beam *D* is clamped tightly at *A* (see Fig. 1) to a fixed vertical pillar *B*, and the other end is fastened
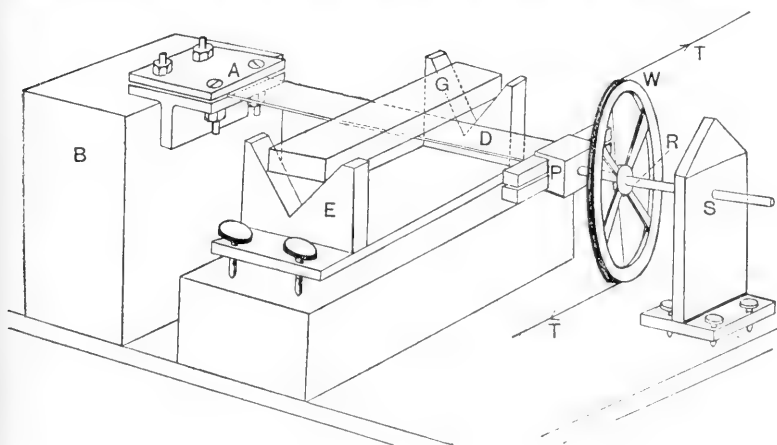
Fig. 1

securely to a cross-piece *P* attached to a circular rod *R* which rests in a socket *S*; the height of *S* can be accurately regulated by three levelling screws. In this way the plate can be subjected to known torsional stresses by means of suitable couples applied to the wheel *W*. A piece of optically worked glass *G* rests on a small

table *E* fitted with three screws for accurate adjustment of the height, and is arranged so that a very thin film of air separates the two plates, the two V's in which *G* rests acting as a geometrical slide. Before torsion is applied to the lower plate *D*, the fringe system formed in the thin air film is made as broad as possible by adjusting the screws attached to the table *E* and also those attached to the frame carrying the socket. When these adjustments have been made and broad fringes are obtained, the plate can be assumed
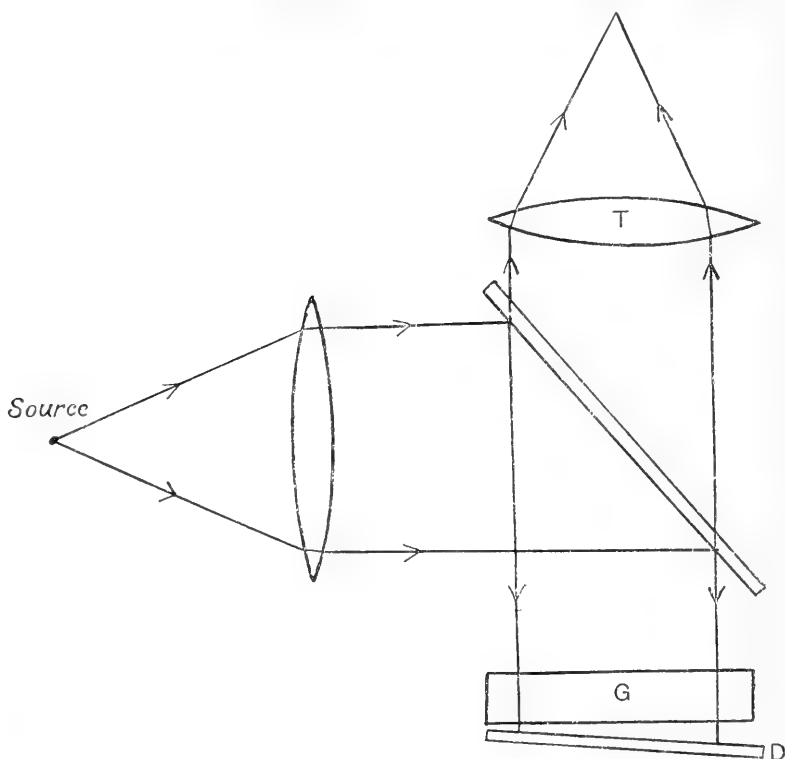
Fig. 2

to be free from all bending stresses. The film is illuminated, as illustrated in Fig. 2, with monochromatic light and the fringe system is observed and measured by means of a travelling microscope *T*. When a pure couple is applied to the beam, the fringe system, consisting of a number of rectangular hyperbolas, can be brought into view by adjusting the position of the glass *G* very slightly. By the measurement of these curves, the rigidity of the plate may be determined. A photograph of the fringes accompanies the paper.
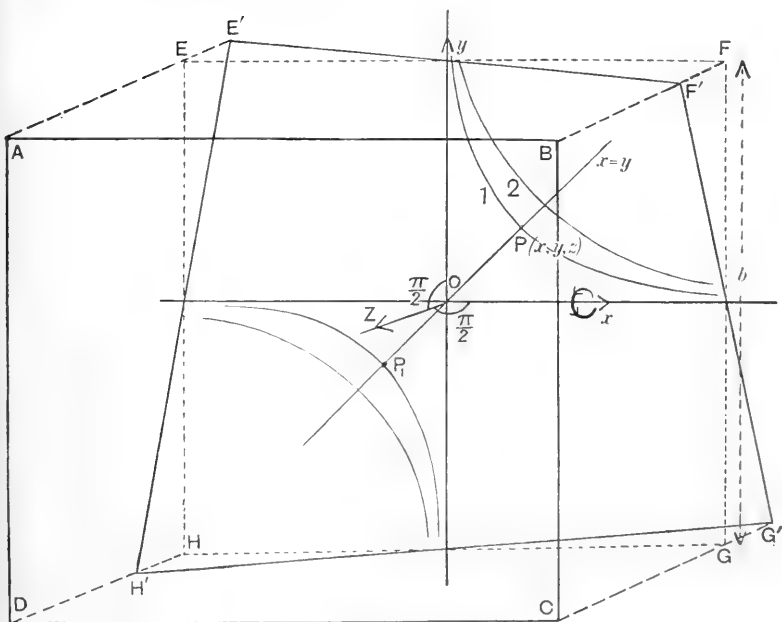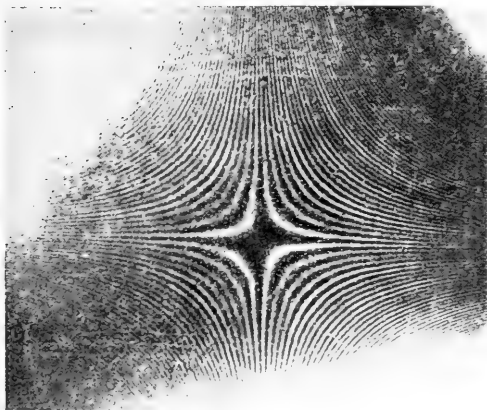
Fig. 3

Theory. Let the surface $ABCD$ (Fig. 3) represent the portion of the under-surface of the plate G above the plate D. Let $EFGH$, parallel to $ABCD$, represent the adjacent surface of the plate D originally, and $E'F'G'H'$ the surface when the couple is applied. The new surface is called a helicoid and the lines $E'F'$ and $H'G'$ are portions of uniform right-handed helices. Take a system of rectangular axes at a particular point O on the upper surface of the plate D, such that Ox is parallel to the length of the plate and Oy is parallel to the surface $ABCD$.

Let μ = Rigidity of the material of the plate.

C = Couple applied to the plate.

θ = Angle turned through about Ox by a point Q whose original coordinates are x, y, 0.

b = Width of plate.

d = Thickness of plate.

Then* $$C = \frac{\mu b d^3 \theta}{3x} (1 - \cdot 63 d/b).$$

Now $\theta = z/y$, where z is the new third coordinate of Q, and hence

$$C = \frac{\mu b d^3 z}{3xy} (1 - \cdot 63 d/b).$$

Let t be the thickness of the air film between the two plates originally. If, after the lower plate has been twisted, the thickness at any point x, y becomes t', then

$$t' = t - z = t - \frac{3xyC}{\mu b d^3 (1 - \cdot 63 d/b)} .$$

Hence those points of the film for which the thickness $t - z$ is constant lie on the rectangular hyperbolas given by the above equation. Corresponding to $t - z = \frac{1}{2}N\lambda$, where N is an integer and λ is the wave-length of the light, a dark fringe is formed. The dark lines in the fringe system are given by

$$\tfrac{1}{2}N\lambda = t - \frac{3xyC}{\mu b d^3 (1 - \cdot 63 d/b)} .$$

If $t = \frac{1}{2}n\lambda$, where n is a particular value of N, the asymptotes of the hyperbolic fringe system will also be dark lines. The equations of the hyperbolas in this case become

$$\frac{3xyC}{\mu b d^3 (1 - \cdot 63 d/b)} = \frac{(n - N)\lambda}{2} ,$$

where for the hyperbola marked 1 in Fig. 3, $n - N = 1$, and for the hyperbola marked 2 in Fig. 3, $n - N = 2$, etc.

* Love, *Mathematical Theory of Elasticity*, 2nd edit. p. 311.

In order to determine μ, two different methods have been adopted. The values of x corresponding to a given value of y for the different hyperbolas have been observed. In this case

$$x = \frac{\lambda \mu b d^3}{6 C y} (n - N)(1 - \cdot 63 d/b),$$

from which μ can be calculated.

Again, if the microscope be arranged to move along the line $x = y$, which bisects the asymptotes, and if the semi-diameters, OP, etc., for the different hyperbolas be measured, then for the hyperbola $n - N = s$,

$$xy = \tfrac{1}{2} OP^2,$$

and

$$\frac{3 OP^2 . C}{2 \mu b d^3 (1 - \cdot 63 d/b)} = \frac{s\lambda}{2}.$$

Thus, by measuring the semi-diameters of the different hyperbolas the rigidity can be found.

It has been found more convenient to measure the distance $PP_1 = 2OP$, where P_1 is the point of intersection of $x = y$ with the second branch of the same hyperbola, as the origin O is rather indistinct, whereas the points P and P_1 are perfectly well defined.

Observations have been made with widely varying loads; the results are tabulated below.

Series I.

Width of glass plate $= b = 1 \cdot 835$ cm.
Mean thickness of plate $= d = \cdot 1793$ cm.
Diameter of wheel $= 7 \cdot 42$ cm.
Tensions in wires attached to wheel $= 227 \cdot 5 \times 981$ dynes.
Wave length $= \lambda = 5 \cdot 89 \times 10^{-5}$ cm.

y	$n-N$	x (measured by microscope)	μ. Dynes per cm.^{2} per unit shear	y	$n-N$	x	μ
$\cdot 3867$ cm.	8	$\cdot 3185$ cm.	$2 \cdot 59 \times 10^{11}$	$\cdot 3688$ cm.	4	$\cdot 169$ cm.	$2 \cdot 64 \times 10^{11}$
	12	$\cdot 478$,,	$2 \cdot 61 \times 10^{11}$		8	$\cdot 330$,,	$2 \cdot 58 \times 10^{11}$
	16	$\cdot 6502$,,	$2 \cdot 66 \times 10^{11}$		12	$\cdot 4963$,,	$2 \cdot 59 \times 10^{11}$
	19	$\cdot 7691$,,	$2 \cdot 65 \times 10^{11}$		16	$\cdot 6714$,,	$2 \cdot 63 \times 10^{11}$
					19	$\cdot 8035$,,	$2 \cdot 64 \times 10^{11}$

y	$n-N$	x (measured by microscope)	μ	y	$n-N$	x	μ
$\cdot 3688$ cm.	8	$\cdot 3294$ cm.	$2 \cdot 58 \times 10^{11}$	$\cdot 6187$ cm.	4	$\cdot 103$ cm.	$2 \cdot 54 \times 10^{11}$
	12	$\cdot 497$,,	$2 \cdot 59 \times 10^{11}$		12	$\cdot 308$,,	$2 \cdot 53 \times 10^{11}$
	16	$\cdot 675$,,	$2 \cdot 64 \times 10^{11}$		16	$\cdot 4065$,,	$2 \cdot 50 \times 10^{11}$

The following two sets of observations, taken at different times, show the results when the distances from the origin of the points of intersection of the various hyperbolas with the line $x = y$ were measured.

$n - N$	Distance along $x = y$	μ	$n - N$	Distance along $x = y$	μ
4	·3469 cm.	$2{\cdot}56 \times 10^{11}$	4	·3525 cm.	$2{\cdot}63 \times 10^{11}$
8	·49125 ,,	$2{\cdot}56 \times 10^{11}$	8	·493 ,,	$2{\cdot}58 \times 10^{11}$
12	·6082 ,,	$2{\cdot}61 \times 10^{11}$	12	·6075 ,,	$2{\cdot}61 \times 10^{11}$
16	·6987 ,,	$2{\cdot}59 \times 10^{11}$	16	·6995 ,,	$2{\cdot}60 \times 10^{11}$
20	·78125 ,,	$2{\cdot}59 \times 10^{11}$	20	·783 ,,	$2{\cdot}60 \times 10^{11}$
24	·8600 ,,	$2{\cdot}61 \times 10^{11}$			

Series I gives the mean value $\mu = 2{\cdot}59 \times 10^{11}$ dynes per cm.[2] per unit shear.

SERIES II.

In a short series of observations, in which the distance PP_1 was measured, the following results were obtained:

Width of glass plate $= b = 1{\cdot}825$ cm.

Mean thickness of plate $= d = {\cdot}17532$ cm.

Diameter of wheel $= 7{\cdot}42$ cm.

Tensions in wires attached to wheel $= 367 \times 981$ dynes.

Wave length $= \lambda = 5{\cdot}89 \times 10^{-5}$ cm.

$n - N$	Distance along $x = y$	μ
4	·52835 cm.	$2{\cdot}57 \times 10^{11}$
8	·74815 ,,	$2{\cdot}58 \times 10^{11}$
12	·91705 ,,	$2{\cdot}58 \times 10^{11}$

Series II gives the mean value $\mu = 2{\cdot}576 \times 10^{11}$ dynes per cm.[2] per unit shear.

SERIES III.

These observations were taken by Mr L. Fouracre of Leeds University.

Width of glass plate $= b = 1{\cdot}83$ cm.

Mean thickness of plate $= d = {\cdot}1740$ cm.

Diameter of wheel $= 7{\cdot}42$ cm.

Tensions in wires attached to wheel $= 265 \times 981$ dynes.

Wave length $= \lambda = 5{\cdot}89 \times 10^{-5}$ cm.

x	$n-N$	y (measured by microscope)	μ	x	$n-N$	y	μ
·3876 cm.	4	·1254 cm.	2.63×10^{11}	·5472 cm.	4	·0912 cm.	2.70×10^{11}
	8	·2394 ,,	2.52×10^{11}		8	·1824 ,,	2.70×10^{11}
	12	·3648 ,,	2.56×10^{11}		12	·2736 ,,	2.70×10^{11}
	16	·4788 ,,	2.51×10^{11}		16	·3420 ,,	2.54×10^{11}

In the following observations the tensions in the wires $= 465 \times 981$ dynes.

x	$n-N$	y	μ	$n-N$	Distance along $x = y$	μ
·4788 cm.	4	·057 cm.	2.48×10^{11}			
	8	·114 ,,	2.60×10^{11}	8	·3306 cm.	2.60×10^{11}
	12	·171 ,,	2.60×10^{11}	12	·4104 ,,	2.67×10^{11}
	16	·2166 ,,	2.47×10^{11}	16	·4674 ,,	2.60×10^{11}

Series III gives the mean value $\mu = 2.59 \times 10^{11}$ dynes per cm.^{2} per unit shear.

In conclusion, the coefficient of rigidity thus obtained may be compared with that deduced from determinations of Young's Modulus E, and Poisson's Ratio σ, by direct observation for the same glass beam. The values obtained as a result of several determinations were $E = 6.07 \times 10^{11}$ dynes per cm.^{2} per unit elongation, $\sigma = ·2$. Hence

$$\mu = \frac{E}{2(1+\sigma)} = 2.53 \times 10^{11} \text{ dynes per cm.}^2 \text{ per unit shear.}$$

It gives me great pleasure to thank Dr G. F. C. Searle, F.R.S., for revising the manuscript before publication.

Low Voltage Glows in Mercury Vapour. By G. STEAD, M.A., and
E. C. STONER, B.A., Honorary Research Student of Emmanuel
College, Cambridge.

When the anode voltage in a soft thermionic tube exceeds a
certain critical value, a general glow fills the tube, and there is an
apparently discontinuous change in current. On decreasing the
voltage, there is a marked lag, and the glow disappears at a potential
considerably lower than that at which it appeared. Although the
phenomenon is familiar*, a completely satisfactory explanation
does not seem to have been given.

In the present work mercury vapour was used, and the effect
of varying the filament temperature and the pressure was investi-
gated in some detail.

Apparatus. The main observations were made using a valve of
the form shown in Fig. 1. The grid was a flat spiral, about 1·5 mm.
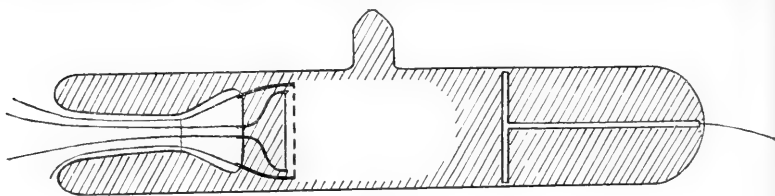
Fig. 1

from the tungsten filament. Clean, freshly distilled mercury was
placed in the valve which was evacuated and baked in the usual
way. (The mercury which distilled over during the baking was
caught in a suitable trap so that it could be reintroduced into the
tube.) The pressure on sealing off, as measured by a McLeod gauge,
was less than ·00005 mm. Different pressures of mercury could
be obtained by heating up the tube in an electric oven. It may here
be noted that change in the anode voltage produced little change
in the effects observed, and the anode current was always small
compared with the grid current. The current measured is the total
current from filament to grid and anode, the two being connected.

Experiments have also been made on a valve similar to that
shown in the figure but without an anode. The general nature of
the effects is exactly the same.

* Cf. Stead and Gosling, *Phil. Mag.* XL. p. 424 (1920), and G.E.C. Research Staff,
Phil. Mag. XL. p. 585 (1920); XLI. p. 685 (1921); XLII. p. 227 (1921).

OBSERVATIONS.

15° *C.* *Pressure* ·0012 *mm. mercury.*

A typical example of the type of characteristic obtained is shown in Fig. 2. With the low pressure, and the arrangement of grid and anode, the ordinary ionisation kink corresponding to 10·4 volts is not observed. The emission increases gradually as the voltage is raised through A to B. At B under definite conditions of pressure and filament current an increase in voltage of less than $\frac{1}{20}$—probably less than $\frac{1}{100}$—of a volt is sufficient to produce a bright blue glow which fills the whole tube in which before no glow whatever was visible. At the same time there is an enormous increase in the current passing between the filament and the anode and grid. On further increase of voltage, there is up to 40 volts only a very slight increase of current (C-D). On decreasing the voltage the change is very slight until E is reached. Here there is a sudden
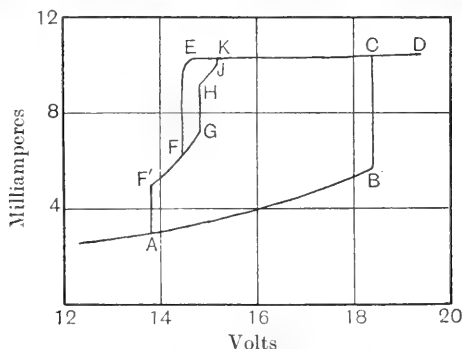
Fig. 2. Typical characteristics at 15° C.

discontinuous decrease of current, and a sudden marked decrease in the brightness of the glow. Further, the glow disappears entirely underneath the filament. The space between the grid and anode is now filled with a blue glow fainter than before, but quite easily seen in a bright room. Gradual diminution of the voltage results in a decrease in the length of the glow. The limit is sharply marked, as a dome, as suggested in Fig. 1. By varying the voltage between F and F' the dome can be made to rise and fall in a beautifully regular manner; it could not, however, be lowered indefinitely, as the glow invariably disappeared suddenly, with decrease of current (F'-A) when its length became slightly less than about two centimetres. (For brevity, the faint glow will be referred to as the first glow, and the bright glow as the second; and the glow beneath the filament as the under glow.) If the glow is obtained in the dome state (the path $ABCEF$ must be traversed) and the voltage in-

creased, the faint glow completely fills the grid anode space. With further voltage increase the second glow suddenly appears at G, the current increasing from G to H. There is then a gradual increase from H to J. The JK jump corresponds to the appearance of the under glow. Decrease is by the EF path, under glow and second glow disappearing almost simultaneously.

The spectrum of the glows was examined by a Hilger Wave-Length Spectroscope. As photographs were not taken, some of the fainter lines may have been missed. Those observed indicated that the spectrum (as far as the visible region was concerned) showed the ordinary mercury arc lines:

6234	$2S - 4P$	5676	$2s - 5P$
6123	—	5461	$2p_1 - 2s$
6073	$2s - 4P$	4916	$2P - 3S$
5790	$2P - 2D$	4358	$2s - 2p_2$
5770	$2P - 2d_2$	4348	$2P - 3D$

The most prominent lines were those underlined. No change was observed in the spectrum when the second glow appeared, but all the lines became brighter.

With a hotter filament the current increase corresponding to the second glow is relatively much greater.
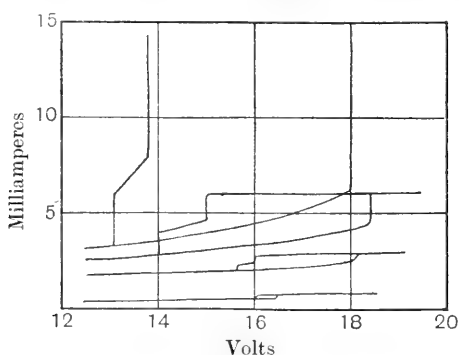
Fig. 3. Series of characteristics at 15° C.

Fig. 3 shows the result of a series of observations with different filament currents. The appearance voltage at first increases with increase of filament current, and then decreases. The disappearance voltage decreases regularly.

Although in general at constant voltage, the current remains constant with time after the glow point is passed, with smaller filament temperatures, and so smaller currents, there is a gradual decrease of current with time towards an asymptotic limit. Such

time effects seem to be associated with phenomena of decrease of
current with increase of voltage and *vice versa*.

60° C. *Pressure* ·025 *mm.*

The glow appearance voltages were lower than at 15° C., but
at this pressure there were more marked changes in current with
time and voltage after the glow point was reached. At the glow
appearance point the sudden increase in current was followed by
a decrease with time at first rapid, but slowing down to a limit.
On increasing the voltage the current decreased; on decreasing
the voltage again the current increased. The final current after
disappearance of the glow was usually slightly less than the original,
though there seemed to be no large permanent effect on the filament.

77° C. *Pressure* ·075 *mm.*

At this pressure, though time decrease effects were noticed for
saturation currents less than 4 m.a., the phenomena became fairly
regular again. The critical voltages were lower. With different
filament currents, the appearance voltage varied regularly between
12·4 and 13 volts, the disappearance between 11 and 12 volts. Only
one glow was produced, as with the smaller currents at 15° C.
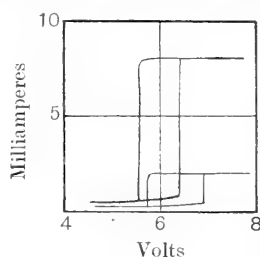
Fig. 4. Characteristics at 130° C.

130° C. *Pressure* 1·15 *mm.*

The lowest voltages for appearance and disappearance were
obtained in this region. Two curves are shown. The disappearing
voltage was as low as 5·6; the appearing 6·4—both well below the
ionisation potential of mercury at 10·4 volts. The glow at this
pressure was still visible though it was confined to the filament
grid region. There was no sharp line of demarcation, the glow
gradually shading off. Observations of the glow are obviously
difficult under the conditions; but with a small direct vision spectro-
scope the mercury lines were readily seen on the bright continuous
spectrum background with applied voltages well under 8 volts.

145° C. *Pressure* 2·3 *mm.*   160° C. *Pressure* 4·17 *mm.*

The curves for these pressures show very well the gradual change
in the characteristics with change in filament current. With small

filament currents little or no hysteresis is shown. (See lower curve
in Fig. 5.) Also, what is apparently the usual ionisation kink
becomes marked. At 160° with smaller currents there seems to be
hysteresis for this ionisation potential, but with higher currents a
kink of the usual kind is obtained. Similar effects were noticed at
165° C.

At higher pressures the glow voltages increase, and the kinks
become less and less marked, eventually disappearing. At 230° C.
(43 mm.) the currents were very small, rising only to ·1 m.a. at
40 volts, and no kinks were noticed.
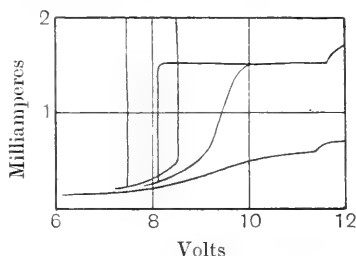
Fig. 5. Characteristics at 145° C.
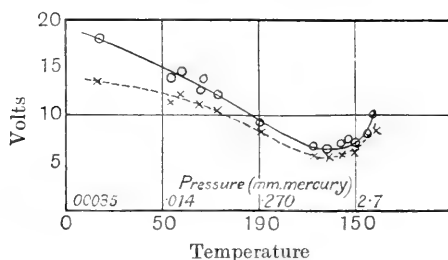
Fig. 6. Curves showing appearance voltages (full line) and disappearance
voltages (dotted line) at different pressures.

Summary of observations.

At a definite pressure the appearing voltage at first increases
and then decreases with the filament current, and seems to approach
a definite lower limit. (As increasing the filament temperature
cannot be carried on indefinitely, it is impossible to be quite certain
of this.) Taking as values the lowest appearing and disappearing
voltages observed at the various pressures, these can be plotted
against the pressure, with the result shown in Fig. 6. The full line
represents the appearing voltage, and the dotted line the dis-
appearing voltage.

At the conclusion of the above experiments an induction coil discharge was passed through the tube to see if any appreciable amount of hydrogen had accumulated. The amount was estimated at $\frac{1}{100}$ mm., the effect of which should be negligibly small.

With other types of valves the characteristics obtained were somewhat different. With a guard ring cylindrical type triode, which may be regarded as having a filament with an almost closed grid round it, the current gradually *decreased* with the appearance of the glow. The appearance and disappearance voltages were similar to those above.

A diode, which had as anode a filament parallel to the cathode filament, gave no glow below 27·5 volts.

DISCUSSION.

In a soft valve ionisation by collisions between electrons and gas molecules may cause an increase in the current directly owing to the production of fresh ions, and indirectly through the effect of the positive ions in neutralising the electronic space charge*. The neutralising effect will be of greater importance the nearer the positives approach the hot filament.

Let us consider for simplicity a gas with an ionisation potential V (no resonance potentials). Neglecting initial velocities, no ionisation will be produced unless the applied voltage is greater than V, and at a given voltage above this the number of ions produced will at first increase and then decrease with increase of the gas pressure.

Now consider the indirect effect. The positives will travel towards the filament, but if they are few in number, and their velocity is small, they will recombine before reaching it. As the applied voltage is increased, some will eventually reach the neighbourhood of the filament. The emission of electrons from the filament will increase owing to the partial neutralisation of the space charge. The state is essentially unstable. The increased emission current will increase the ionisation which in turn affects the emission. The measured current will increase discontinuously to a value depending on the saturation current at the temperature. On decreasing the voltage the current will not immediately return to its former value, but will remain practically constant, until the re-combination, owing to the lower velocities, is again sufficient to prevent the positives reaching the filament. The state is again unstable and a discontinuous current decrease occurs.

Such considerations explain the possibility of discontinuous current changes, and of a hysteresis effect. They also suggest an explanation of the changes in the voltages at which the discontinuous current increases and decreases occur with change of

* See Richardson, *Emission of Electricity from Hot Bodies* (1921), p. 67 *et seq.* for discussion.

filament temperature at a definite gas pressure. A rise of filament temperature will cause an increased space charge, and the appearing voltage will at first be higher. The disappearing voltage will be lower owing to the greater number of positives formed.

Omitting the 'second glow' for the present, the phenomena observed with mercury are in general agreement with the explanation proposed. It must however be concluded that mercury can be ionised below its accepted I.P. of 10·4 volts. Production of ionisation under 6 volts cannot be ascribed to electrons having initial velocities greater than 4 volts for several reasons. The number would be negligibly small (less than ·0001 % at 2400° K have an energy greater than 3 volts) and on the above views ionisation occurs before the discontinuous current increase. Moreover, the ordinary ionisation potential, in certain circumstances, is indicated by a kink in the characteristic above and distinct from the glow potential (see Fig. 5).

To explain such effects various suggestions have been made*. Although impact and radiation may be jointly responsible, it seems simpler to suppose that the outer mercury electron may be ejected from its normal 'orbit' $(1, S)$ to an outer stationary orbit, and ejected from this in turn by a second impact. There seems no reason to suppose that the electron cannot remain in an outer orbit for an appreciable time. Indeed, unless the outer 'orbits' could in some cases form stable 'resting places,' no visible glow would ever be observed, for the visible glow lines are due not to complete re-combination (return to $1, S$) but to return to outer orbits such as $(2, P)$ after complete or partial ionisation.

Although the many-lined spectrum seems only to be produced as a result of complete ionisation, a glow is not observed associated with the ordinary I.P. kink in a valve containing mercury at low pressures. This may be because it is very faint. It may be, however, that ionisation, though necessary, is not in itself sufficient to bring out the many-lined spectrum. The appearance of the mercury glow is possibly due not only directly to ionisation, but to the changed electrical environment accompanying vigorous ionisation, which in some way renders stable outer orbits, to which the electron returns in giving rise to the visible glow lines.

We have now to discuss various effects which seem to depend on the form of valve used—the shape of the first glow, and the second glow. Reverting to the gas with simply one I.P., it is obvious that ionisation will occur only within a space round the anode bounded by a definite equipotential surface. With a small anode—as in the bifilament valve—the space bounded by a given equipotential will be smaller than with a large anode. On the view

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* See Hughes, *Report on Photo-Electricity* (National Research Council Bulletin, No. 10) for summary and references.

taken above, the glow (current increase) potential should be larger, owing to a larger voltage being necessary before sufficient positives are produced for any to reach the filament without re-combination. This suggests why the glow potential observed with the bifilament valve was larger than with the grid valve (27 as compared with 17 volts). For a low glow potential it is necessary that the space through which ionisation can take place should be large, and that the positives should be able to reach the neighbourhood of the filament. The anode (or grid) must be as close as possible to the filament. The fact that in a valve with a closed cylindrical anode* no glow (or corresponding current increase) was observed supports this view.

In the guard ring cylindrical valve the positives (formed mainly beyond the anode) could not readily return to the filament. They would be repelled to the glass walls and there neutralised by an electron current from the filament. The glow should in this case be accompanied by a decrease in the filament-anode current, as was actually observed.

The boundary of the first glow (see Fig. 1) may be considered as the equipotential within which ionisation occurs. As would be expected, it varies in length with varying applied voltages. The shape of the equipotential is modified by the glass walls of the valve, with the result that the glow is dome-shaped.

At higher pressures the first glow is confined to the neighbourhood of the filament and grid. There is no second glow unless the first glow can fill the valve, reaching the walls. This suggests that the second glow is associated with a sudden change in the electric condition of the dielectric boundary. As mentioned in connection with the guard ring valve, the glass can become positively charged under first glow conditions. The electron current from the filament is partly used up in neutralising this charge. When the first glow reaches the walls the charge is neutralised by the electrons in the glow region, with the result that the electron current from the filament to the glass is diverted to the grid. The result is an increase in the filament grid current, and so in ionisation and in the brightness of the glow throughout the tube. The effect will be more marked at the higher filament temperatures when there are more electrons in the glow region. This seems to account for the facts.

Under some conditions the current decreases with increasing voltage and there are peculiar time effects (see account of observations at 60°). Such effects are usually ascribed to chemical action†. It is noteworthy that they are most pronounced when the production of positives is a maximum compared with the number of electrons.

* See Stead, *Phil. Mag.* XLI. p. 474 (1921).
† See Richardson, *Emission of Electricity from Hot Bodies* (1921), p. 138.

When this is the case the possibility arises of the positives being able to pass right through the electron atmosphere without re-combination, and to reach the filament itself. Positive charges reaching the filament will decrease the emission, which will, after the first sudden increase, decrease to an equilibrium limit. Increased voltages will increase the number of positives reaching the filament itself (owing to greater velocities) with the result that the current will decrease. The effects can thus be explained independently of any chemical action, though it is obvious that if chemical action could take place, the conditions might be favourable.

It remains to mention the bearing of these experiments on the question of the mean free path of the electron. If it is taken as four times that of the mercury atom, calculated in the usual way, it has approximately the following values at the given temperatures of the mercury vapour:

$15°$ 8·5 cm. $60°$ 3·6 mm. $100°$ ·36 mm. $140°$ ·05 mm.

With these figures may be compared the facts that even above $60°$ the glow can be obtained several centimetres long, and that ionisation occurs above $140°$, with the stated voltages applied, although the distance (1·5 mm.) between the filament and grid is many times the calculated M.F.P. of the electron.

Summary.

Experiments are described on the effect of varying conditions on the glow potentials of mercury in a thermionic tube. A special type of tube was used with a grid close to the filament, and a large space in which ionisation could take place. The glow phenomena are accompanied by current changes which may be very large. Curves are given showing the nature of the i, V characteristics obtained with different filament temperatures and different pressures.

The length of the glow, in the tube used, could be varied by varying the voltage, the edge being sharp and dome-shaped. At low pressures, and fairly high filament temperatures, a 'second glow' is obtained. The experiments show that ionisation of mercury can occur well below 10 volts (as low as 5·6 volts).

The main features of the current phenomena are explained by considering the effect of positive ions on the space charge, and the effects of re-combination. Ionisation below the I.P. is ascribed to successive impacts.

An explanation is suggested as to why in some circumstances an increase of voltage causes a decrease in current. Several minor points of interest are noticed.

An experiment illustrating the conservation of angular momentum. By G. F. C. SEARLE, Sc.D., F.R.S., University Lecturer in Experimental Physics.

§ 1. *Angular momentum of a system.* For the experiment we need only consider the case in which every particle of the system moves parallel to a fixed plane, which we take as the plane of Oxy. In the experiment this plane is horizontal. Let P (Fig. 1) be the projection on Oxy of a particle of mass m.

Let the coordinates of P relative to the fixed axes Ox, Oy be x, y, and let the components of the velocity of P be u, v. Then the components of the momentum of the particle are mu, mv. Hence, counting counter-clockwise rotation as positive, the sum of the moments about Oz of the momenta mu, mv, or the angular momentum of the particle about Oz, is $m(vx - uy)$. If H be the angular momentum about Oz of the whole system,
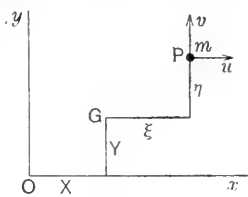
Fig. 1

$$H = \Sigma m (vx - uy) \quad \dots\dots\dots\dots\dots\dots(1).$$

Let G be the projection of the centre of gravity of the system. Let its coordinates relative to the fixed axes Ox, Oy be X, Y, and let the components of its velocity be U, V. Let $x = X + \xi$, $y = Y + \eta$, so that ξ, η are the coordinates of P relative to axes through G parallel to Ox, Oy. Then, since G is the centre of gravity,

$$\Sigma m\xi = 0, \qquad \Sigma m\eta = 0 \quad \dots\dots\dots\dots\dots(2).$$

Let $u = U + \alpha$, $v = V + \beta$, so that α, β are the components of the velocity of P relative to G. Then, since $u = dx/dt$, $v = dy/dt$, $U = dX/dt$, $V = dY/dt$, we have $\alpha = d\xi/dt$, $\beta = d\eta/dt$. But, by (2),

$$\Sigma m\, d\xi/dt = 0, \qquad \Sigma m\, d\eta/dt = 0,$$

and hence $\qquad\qquad \Sigma m\alpha = 0, \qquad \Sigma m\beta = 0 \quad \dots\dots\dots\dots\dots(3).$

Thus, by (1),

$$H = \Sigma m \{(V + \beta)(X + \xi) - (U + \alpha)(Y + \eta)\},$$
$$= \Sigma m \{VX - UY + V\xi - U\eta + X\beta - Y\alpha + \beta\xi - \alpha\eta\}.$$

Since U, V do not change from particle to particle, we may bring them outside the sign of summation. Denoting Σm by M and using (2) and (3) we have

$$H = M(VX - UY) + \Sigma m(\beta\xi - \alpha\eta) \quad \dots\dots\dots(4).$$

The first term in (4) is the angular momentum about Oz of a particle of mass M placed at G and moving with G. The second term is the angular momentum of the system about an axis through G parallel to Oz.

§ 2. *Method.* A board D (Fig. 2) is suspended by a silk thread supposed to exert no torsional control. The plane of the board is horizontal and the axis of suspension cuts the board in O. The vertical through O is the axis Oz of § 1. The inertia bar AB turns about a vertical shaft fixed to the board, the axis of the shaft passing through G, the centre of gravity of AB. A second bar C, suitably fixed to the board, acts as a counterpoise to AB. By adjusting C, the plane of D is made horizontal. By a light spring E attached to the board and operating by a string wound round a drum carried by AB, this bar can be set into motion about G relative to the board, when a thread attached to the board and holding AB in its initial position is burned. Before the thread is

burned, the system is at rest. At any later time let the axis OG of the board make an angle θ with OF, its initial direction, and let the axis AGB of the bar make an angle $\phi + \epsilon$ with OG, where ϵ is the angle between GA and GO before the thread is

Fig. 2

burned. Let ϕ be measured in the opposite direction to θ.
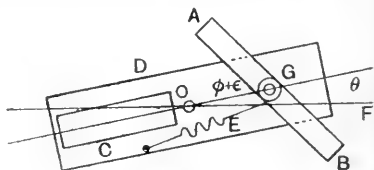
With the exception of gravity and the tension of the suspension, no external forces act on the system after the thread is burned, and hence the angular momentum of the system about the vertical axis Oz is zero. If the moment of inertia about Oz of the board, the counterpoise and all the other fittings *except* the bar AB be K_1, the angular momentum about Oz of this part of the system is $K_1 d\theta/dt$.

Let M be the mass of AB and let $OG = a$. Since the linear velocity of G is $a d\theta/dt$, the momentum of M at G is $Ma d\theta/dt$, and the moment of this momentum about Oz is $Ma^2 d\theta/dt$. This angular momentum is in the same direction as that of the board.

The angular velocity of AB in space, in the same direction as that of the board, is $d\theta/dt - d\phi/dt$, and hence, if the moment of inertia of AB about a vertical axis through G be K_2, its angular momentum about that axis is $K_2 (d\theta/dt - d\phi/dt)$. Hence, by (4), the angular momentum of AB about Oz is

$$Ma^2 d\theta/dt + K_2 (d\theta/dt - d\phi/dt).$$

The angular momentum of the whole system is zero, and hence

$$K_1 d\theta/dt + Ma^2 d\theta/dt + K_2 (d\theta/dt - d\phi/dt) = 0 \ \ldots\ldots(5).$$

Since
$$\frac{d\theta}{dt}\Big/\frac{d\phi}{dt} = \frac{d\theta}{d\phi},$$

we have, by (5), $\dfrac{d\theta}{d\phi} = \dfrac{K_2}{K_1 + K_2 + Ma^2} = \dfrac{K_2}{K_3}$ ....(6).

The quantity $K_1 + K_2 + Ma^2$, or K_3, is the moment of inertia about Oz of the whole system, when the bar AB is fixed relative to the board.

Since (6) holds at every instant, we have

$$\theta/\phi = K_2/K_3, ...(7)$$

where θ and ϕ are, respectively, the angles turned through by the board relative to the earth and by the bar relative to the board in any time, each measured from the corresponding zero.

Let the bar AB be removed from the board and be attached to a vertical torsion wire and be made to execute torsional vibrations,
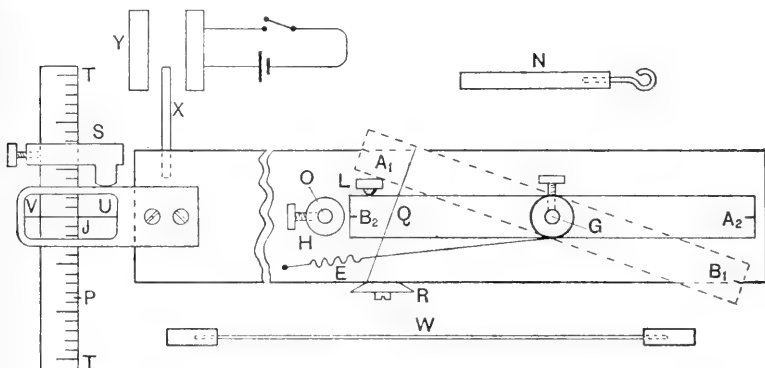
Fig. 3

the axis of suspension agreeing with that about which AB turns when on the board. Let T_2 be the periodic time. Let T_3 be the periodic time when the *complete* system is suspended from the same wire. Then $K_2/K_3 = T_2^2/T_3^2$ and thus

$$\theta/\phi = T_2^2/T_3^2   ....(8).$$

In the experiment θ/ϕ is compared with T_2^2/T_3^2.

§ 3. *Experimental details.* Fig. 3 shows some details of the apparatus. The counterpoise does not appear, as it is fixed to the part of the board which is omitted. The counterpoise and other fittings should be designed so that the axis of suspension is as nearly as possible a "principal axis." If this condition be not secured, the motion of the board will be unsteady. The ends of the torsion wire W, used in comparing K_2 with K_3, are soldered into

two short cylinders 0·5 cm. in diameter. At the centre of the board is a socket H, provided with a set-screw. Into this socket fits either the rod N, by which the system is attached to the thread, or the cylinder at the end of the torsion wire. The hole in the bar at G is of the same diameter as that in H. By a set-screw, the bar can be secured to the torsion wire.

The bar is held in its initial position $A_1 B_1$ against the pull of the spring E by a thread Q secured under the button R; this position is defined by the stop L. When the thread is burned, the bar turns round until it hits the other side of L and is arrested in the position $A_2 B_2$. To prevent rebound, a small pellet of plasticene is placed on L as shown. Unless the plasticene be re-moulded into pellet form after each impact, the bar will rebound. Though the rebound does not vitiate the experiment, it makes the observations more difficult. The angle ϕ turned through by AB relative to the board is $\pi - A_1 G B_2$. If $A_1 G = r$, we have $\sin \frac{1}{2} A_1 G B_2 = \frac{1}{2} A_1 B_2 / r$. Then

$$\phi = \pi - A_1 G B_2 = \pi - 2 \sin^{-1} (\tfrac{1}{2} A_1 B_2 / r) \text{ radians } ...(9).$$

The angle θ turned through by the board is measured by the fine wire UV, which is stretched in a metal frame carried by the board and has the same projection on Oxy as the line GO. The wire moves over a horizontal scale whose edge is TT. A zero J on TT is chosen, and TT is set perpendicular to OJ. If UV cut the edge in J initially and in P finally,

$$\tan \theta = PJ/JO \quad ....(10).$$

To determine JO, a "set square" is held so that one edge is vertical and one horizontal, as tested by a level. The vertical edge is adjusted to touch TT at J, and the horizontal distance of this edge from the axis of the suspending thread is measured.

For success, the system must be *at rest* when the thread Q is burned. The silk thread supporting the system is attached to a torsion head, which is adjusted so that UV cuts TT in some point very near J when the system is at rest. A stop S is fixed to the scale so that, when the frame touches S, UV cuts TT in J. A current sent through the coil Y attracts the small magnet X attached to the board and holds the pointer against the stop. The attraction should be only just great enough to keep the frame against the stop. The centre O of the board is then reduced to rest. A flame or a small gas jet is prepared and, when the system is at rest, the thread Q is burned. The current is stopped just before the flame is applied. The board moves round, and the reading of UV in its new position of rest at P is taken. The wire may subsequently drift very slowly from P on account of slight torsion of the silk thread or on account of draughts, and thus the reading should be taken without delay.

The supporting thread should be of *plaited* silk; *fine* fishing line may be used. If a *twisted* thread be used, it will be difficult to obtain anything like a steady zero position, since a twisted thread exerts a couple approximately proportional to the load, unless the thread be practically "unwound."

In the apparatus as finally constructed, the board is 56 cm. long, 8 cm. wide and 1·5 cm. thick. The inertia bar is 26 cm. long, 1·6 cm. wide and 1·6 cm. thick. The mass of the whole system is 1770 grms.

§ 4. *Practical example.* The following results were obtained by Mr J. A. Pattern:

Determination of θ. The values found for JP were 7·50, 7·80, 7·50, 7·70. Mean 7·625 cm. The distance JO was 38·5 cm. Hence $\tan \theta = 7·625/38·5 = ·19805$, and hence

$$\theta = 11° \ 12' \ 9'' = ·19552 \text{ radians.}$$

Determination of ϕ. The distances were, $A_1G = 15·0$ cm., $A_1B_2 = 2·42$ cm. Hence $\phi = \pi - 2 \sin^{-1} (1·21/15) = 180° - 9° \ 15' \ 14'' = \pi - ·1615$ radians. Thus

$$\phi = 2·9801 \text{ radians.}$$

Determination of T_2 *and* T_3. The inertia bar was suspended by the torsion wire and the transits were observed as follows:

Transit	Time		Transit	Time		$50T_2$	
	min.	sec.		min.	sec.	min.	sec.
0		59	50	3	43	2	44
10	1	32	60	4	16	2	44
20	2	5	70	4	48	2	43
30	2	37	80	5	21	2	44
40	3	10	90	5	53	2	43

The mean value of $50T_2$ is 163·6 secs. Hence $T_2 = 3·272$ secs.

When the whole system, including the inertia bar, was suspended by the same wire, similar observations gave $50T_3 = 638·9$ secs. Hence $T_3 = 12·778$ secs.

Comparison of results. For the times, we have

$$T_2^2/T_3^2 = 3·272^2/12·778^2 = ·06557.$$

For the angles, we have

$$\theta/\phi = ·19552/2·9801 = ·06561.$$

Hence equation (8) is closely verified.

A general condition for the quantisation of the conditionally periodic motions with an application for the Bohr atom. By VICTOR TRKAL, Ph.D., Lecturer in Theoretical Physics in the Czech University, Prague. (Communicated by Mr F. W. Aston.)

In a conservative dynamical system of k degrees of freedom let $q_1, q_2, \ldots, q_k$ be the generalized Lagrangian coordinates and let L be the kinetic potential: we shall suppose that the constraints are independent of the time, so that L is a given function of the co-ordinates $q_1, q_2, \ldots, q_k$ and of the velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k$ only, not involving the time t explicitly. If we further introduce momenta, defined in the usual manner as

$$p_r = \frac{\partial L}{\partial \dot{q}_r},$$

the total energy of the system, W, is given as †

$$\sum_{r=1}^{k} p_r \dot{q}_r - L = W = \text{Const.}; \quad L = E_{\mathbf{kin}} - E_{\mathbf{pot}}, \quad W = E_{\mathbf{kin}} + E_{\mathbf{pot}},$$

where $E_{\mathbf{kin}}$ and $E_{\mathbf{pot}}$ denote the kinetic energy and the potential energy respectively.

Multiply each side of this equation by dt and integrate from the time 0 to T and divide the equation by T. Letting now T increase beyond all limits, we have

$$\lim_{T \to \infty} \left\{ \sum_{r=1}^{k} \frac{1}{T} \int_0^T p_r \dot{q}_r dt - \frac{1}{T} \int_0^T L \, dt \right\} = \lim_{T \to \infty} \frac{1}{T} \int_0^T W \, dt = W.$$

In the case of a conditionally periodic motion, it follows that

$$\sum_{r=1}^{k} \frac{1}{T_r} \int_0^{T_r} p_r \dot{q}_r dt - \frac{1}{T^*} \int_0^{T^*} L \, dt = W,$$

where T_r and T^* denote the period of the function $p_r \dot{q}_r$ and of the kinetic potential L, respectively. Denoting further the time mean of the kinetic potential

$$\overline{L} = \frac{1}{T^*} \int_0^{T^*} L \, dt,$$

and writing frequencies ν_r instead of periods $\left(\nu_r = \dfrac{1}{T_r} \right)$, we obtain

$$\sum_{r=1}^{k} \nu_r \int p_r dq_r - \overline{L} = W \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

† Cf. E. T. Whittaker, *A treatise on Analytical Dynamics*, 2nd ed., Cambridge, 1917, p. 62.

where $\int p_r dq_r = \int_0^{T_r} p_r \dot{q}_r dt$ denotes the phase integral. Denote

$$\int p_r dq_r = I_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

the equation (1) then becomes

$$\sum_{r=1}^{k} I_r \nu_r - \bar{L} = W \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Imagine now I_r, ν_r, $\bar{L}$, W expressed as functions of 'structural' constants (e.g. of matter, charges, field) and 'kinematical' constants. These latter usually specify the shape of the path of the motion considered (e.g. the semi-axis and the numerical excentricity of an elliptical orbit of an electron rotating round a positive nucleus), in other cases they may specify for example the velocity of rotation of a spinning sphere; in the classical theory these 'kinematical' constants can generally acquire any value.

Obviously I_r, ν_r as well as $\bar{L}$ and W are functions of the 'kinematical' quantities $(a, \epsilon, \ldots)$; W, however, can be regarded as a function of I_1, I_2, ..., I_k, these latter again being functions of $(a, \epsilon, \ldots)$.

Thus we can write

$$\frac{\partial W}{\partial a} = \sum_{r=1}^{k} \frac{\partial W}{\partial I_r} \frac{\partial I_r}{\partial a} = \sum_{r=1}^{k} \nu_r \frac{\partial I_r}{\partial a}, \quad \frac{\partial W}{\partial \epsilon} = \sum_{r=1}^{k} \frac{\partial W}{\partial I_r} \frac{\partial I_r}{\partial \epsilon} = \sum_{r=1}^{k} \nu_r \frac{\partial I_r}{\partial \epsilon}, \text{ etc.}$$
$$\ldots\ldots(4),$$

since* $\quad\quad \dfrac{\partial W}{\partial I_r} = \nu_r, \quad (r = 1, 2, \ldots, k) \ldots\ldots\ldots\ldots\ldots(5).$

Now let us quantise the motion of this dynamical system; then we must use Sommerfeld's condition

$$I_r = \int p_r dq_r = n_r h, \quad (r = 1, 2, \ldots, k) \quad \ldots\ldots\ldots(6),$$

where n_r is a positive integer and h Planck's constant. Then I_1, I_2, ..., I_k are constants independent of $(a, \epsilon, \ldots)$, so that formula (3) becomes

$$\sum_{r=1}^{k} n_r h \nu_r - \bar{L} = W \quad \ldots\ldots\ldots\ldots\ldots(7).$$

Further, the formula (4) is transformed into

$$\frac{\partial W}{\partial a} = 0, \quad \frac{\partial W}{\partial \epsilon} = 0, \ldots \text{ etc.} \quad \ldots\ldots\ldots\ldots(8),$$

i.e. $\quad\quad\quad\quad\quad\quad \delta W = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$

I_r being constant and equal to $n_r h$.

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* Cf. J. M. Burgers, *Het atoommodel van Rutherford-Bohr.* (Proefschrift.) Haarlem, 1918, p. 43, § 10, equation (5). N. Bohr, "On the Quantum Theory of Line-Spectra. Part I." (*D. Kgl. Danske Vidensk. Selsk. Skrifter, Naturvidensk. og Mathem. Afd.*, 8 Raekke, IV. 1). Kobenhavn, 1918. Separate copy, p. 29, equation (5*).

Substituting into (9) the value W from equation (7), we obtain for the total quantised energy the following condition

$$\delta \left\{ \sum_{r=1}^{k} n_r h \nu_r - \bar{L} \right\} = 0 \quad \text{.....................(10)},$$

in which the variation extends to the 'kinematical' constants only. Of course, before variating, we must express the term within the brackets as a function of these 'kinematical' constants.

We notice, that the stationary states of conditionally periodic systems are determined by the condition that the difference between $\sum_{r=1}^{k} n_r h \nu_r$ and the mean kinetic potential $\bar{L}$ should be an extremum (as we shall see from examples, a minimum).

In special relativity-mechanics all the above suppositions remain valid; only the kinetic potential L must be substituted by the modified Lagrangian function

$$L = F - E_{\text{pot}}; \ F = -m_0 c^2 (\sqrt{1 - \beta^2} - 1), \ \beta = \frac{v}{c}, \quad \text{...(11)},$$

and for the kinetic energy the expression

$$E_{\text{kin}} = m_0 c^2 \left(\frac{1}{\sqrt{1 - \beta^2}} - 1 \right) \quad \text{...............(12)},$$

where v denotes the actual velocity, c the velocity of light and m_0 the mass when at rest.

According to special relativity-mechanics

$$L = F - E_{\text{pot}} = E_{\text{kin}} + F - W, \text{ since } E_{\text{kin}} + E_{\text{pot}} = W \text{...(13).}$$

Multiplying each side by dt and integrating from 0 to T, we obtain

$$\int_0^T L dt = \int_0^T (E_{\text{kin}} + F) \, dt - WT \quad \text{............(14).}$$

Introducing the principal function ('Wirkungsfunktion')

$$S = \int_0^T (E_{\text{kin}} + F) \, dt \quad \text{.....................(15)},$$

we have
$$\frac{1}{T} \left(S - \int_0^T L dt \right) = W \quad \text{.................(16)};$$

if the motion is periodic, we see that the following relation must hold

$$\lim_{T \to \infty} \frac{1}{T} \left(S - \int_0^T L dt \right) = \bar{S} - \bar{L} = W \text{............(17),}$$

where $\bar{S}$ and $\bar{L}$ denote the time means of the functions S and L respectively. Comparison with the relations (3) and (7) gives

$$\bar{S} = \sum_{r=1}^{k} I_r \nu_r \text{............................(18),}$$

by the classical theory, and

$$S = \sum_{r=1}^{k} n_r h \nu_r \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(19),$$

by the quantum theory, where

$$\bar{S} = \frac{2}{T^*} \int_0^{T^*} E_{\mathrm{kin}} dt \ \dots\dots\dots\dots\dots\dots(20),$$

in ordinary mechanics (since $F = E_{\mathrm{kin}}$) and

$$\bar{S} = \frac{1}{T^*} \int_0^{T^*} (E_{\mathrm{kin}} + F)\, dt \ \dots\dots\dots\dots(21),$$

in special relativity mechanics, T^* being the period of the functions behind the integral sign.

Hence we can summarize the chief results as follows:

(1) *The total* (classical) *energy* can be expressed as

$$W = \sum_{r=1}^{k} \nu_r \int p_r dq_r - \bar{L} = \sum_{r=1}^{k} I_r \nu_r - \bar{L}.$$

(2) *Its quantisation results from the following variation principle:*

$$\delta \left\{ \sum_{r=1}^{k} n_r h \nu_r - \bar{L} \right\} = 0.$$

EXAMPLES.

EXAMPLE 1. An oscillator vibrating linearly about a fixed equilibrium position.

The motion of this (Planck's) oscillator is given by the following known equation

$$m\ddot{\xi} = -k\xi, \quad k > 0,$$

or

$$\ddot{\xi} + 4\pi^2\nu^2\xi = 0,$$

where the constant ν denotes the frequency of this harmonic motion; integrating we obtain

$$\xi = a \cos (2\pi\nu + \vartheta).$$

The kinetic energy is

$$E_{\mathrm{kin}} = \frac{m}{2} \dot{\xi}^2 = 2\pi^2 m\nu^2 a^2 \sin^2 (2\pi\nu t + \vartheta),$$

and the potential energy

$$E_{\mathrm{pot}} = 2\pi^2 m\nu^2\xi^2 = 2\pi^2 m\nu^2 a^2 \cos^2 (2\pi\nu t + \vartheta).$$

Hence the kinetic potential

$$L = E_{\mathrm{kin}} - E_{\mathrm{pot}} = 2\pi^2 m\nu^2 a^2 \cos 2 (2\pi\nu t + \vartheta)$$

and its time-mean

$$\bar{L} = \nu \int_0^{\frac{1}{\nu}} 2\pi^2 m\nu^2 a^2 \cos 2 (2\pi\nu t + \vartheta)\, dt = 0.$$

Thus we obtain $\qquad W = \nu \int p\,dq = \bar{L} = \nu \int p\,dq,$

and the quantisation gives $\quad W = nh\nu.$

The only kinematical parameter is ν; the variation of the last expression would have here of course no meaning. Thus the last expression is the definite form for the quantised energy and agrees with the Planck expression [*].

EXAMPLE 2. A rotator spinning round its fixed axis.

The kinetic energy of such a rotator is $E_{\text{kin}} = \frac{1}{2}J\omega^2 = \frac{1}{2}J\,(2\pi\nu)^2$, where J, ω and ν have their usual significance; this is also the total energy W of the rotator as well as the kinetic potential

$$L = L = W = E_{\text{kin}}.$$

Our general condition takes the form

$$\delta\,\{nh\nu - L\} = \delta\,\{nh\nu - \tfrac{1}{2}J\,(2\pi\nu)^2\} = 0.$$

The only kinematic variable is of course ν. Hence

$$\nu = \frac{nh}{4\pi^2 J}, \quad W = \tfrac{1}{2}J\,(2\pi\nu)^2 = \frac{n^2h^2}{8\pi^2 J},$$

as it is well known from other communications [*].

EXAMPLE 3. An electron rotates round a nucleus in a circular orbit.

If we denote by m_0 the mass of the electron, v its velocity, a the radius of its circular orbit, T the period, ν the frequency, $-e$ its charge and E the nuclear charge, we have

$$\nu = \frac{1}{2\pi}\sqrt{\frac{eE}{m_0}}\,a^{-\frac{3}{2}}; \quad W = -\frac{eE}{2a}, \quad E_{\text{kin}} = \frac{eE}{2a}, \quad E_{\text{pot}} = -\frac{eE}{a}.$$

The kinetic potential $\qquad L = \bar{L} = \dfrac{3eE}{2a};$

applying our general condition

$$\delta\,\{nh\nu - \bar{L}\} = \delta\,\left\{nh\cdot\frac{1}{2\pi}\sqrt{\frac{e\bar{E}}{m_0}}\,a^{-\frac{3}{2}} - \frac{3eE}{2a}\right\} = 0,$$

we obtain, varying in a, $\qquad a = \dfrac{n^2h^2}{4\pi^2 eEm_0};$

hence the total energy

$$W = -\frac{eE}{2a} = -\frac{2\pi^2 e^2 E^2 m_0}{n^2h^2},$$

which coincides with the Bohr value[†].

* M. Planck, *Vorlesungen über die Theorie der Wärmestrahlung*, 4 Aufl. Leipzig (J. A. Barth), 1921, p. 139, form. (223 a), p. 140, form. (231).
† A. Sommerfeld, *Atombau und Spektrallinien*, 2 Aufl. Braunschweig (Fr. Vieweg & Sohn), 1921, p. 243, form. (13).

EXAMPLE 4. An electron rotates round a nucleus in an elliptical orbit.

The system involves two degrees of freedom and two equal periods (azimuthal and radial)

$$\nu = \nu' = \frac{1}{2\pi} \sqrt{\frac{eE}{m_0}} a^{-\frac{3}{2}}.$$

The total energy is $$W = -\frac{eE}{2a},$$

and the time mean of kinetic potential

$$\bar{L} = \frac{3eE}{2a}.$$

In this case the general condition

$$\delta \{n h \nu + n' h \nu' - \bar{L}\} = \delta \{(n + n') h \nu - \bar{L}\}$$

$$= \delta \left\{ (n + n') h \cdot \frac{1}{2\pi} \sqrt{\frac{eE}{m_0}} a^{-\frac{3}{2}} - \frac{3eE}{2a} \right\} = 0$$

gives the semi-axis $$a = \frac{(n + n')^2 h^2}{4\pi^2 e E m_0},$$

and the total energy

$$W = -\frac{eE}{2a} = -\frac{2\pi^2 e^2 E^2 m_0}{(n + n')^2 h^2},$$

coincident with Sommerfeld's calculations*.

EXAMPLE 5. An electron rotates round a nucleus in a 'relativistic circle.'

In this case the Coulomb's attraction balances the centrifugal force, hence

$$\frac{eE}{a^2} = \frac{mv^2}{a}, \quad \frac{eE}{a} = m_0 c^2 \frac{\dfrac{v^2}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}}.$$

Further, the total energy

$$E_{\text{kin}} + E_{\text{pot}} = W = m_0 c^2 \left(\frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} - 1 \right) - \frac{eE}{a}$$

$$= m_0 c^2 \left\{ \sqrt{1 - \frac{v^2}{c^2}} - 1 \right\}.$$

But $$\left(\frac{v^2}{c^2} \right)^2 = \left(\frac{eE}{a m_0 c^2} \right)^2 \left(1 - \frac{v^2}{c^2} \right),$$

or $$\frac{v^2}{c^2} = -\frac{1}{2} \left(\frac{eE}{a m_0 c^2} \right)^2 + \sqrt{\frac{1}{4} \left(\frac{eE}{a m_0 c^2} \right)^4 + \left(\frac{eE}{a m_0 c^2} \right)^2},$$

* A. Sommerfeld, *l.c.*, p. 267, form. (20).

and $Z = 1 + \dfrac{W}{m_0 c^2} = \sqrt{1 - \dfrac{v^2}{c^2}} = -\dfrac{eE}{2am_0 c^2} + \sqrt{1 + \left(\dfrac{eE}{2am_0 c^2}\right)^2}$.

But $\qquad v = 2\pi a \nu, \quad \nu = \dfrac{c}{2\pi a}\sqrt{1 - Z^2}$,

and $\qquad\qquad 1 - Z^2 = \dfrac{eE}{am_0 c^2}\,Z$.

Hence $\qquad a = \dfrac{eE}{m_0 c^2}\dfrac{Z}{1 - Z^2}, \quad \nu = \dfrac{c}{2\pi}\cdot\dfrac{c^2}{eE}\dfrac{(1 - Z^2)^{\frac{3}{2}}}{Z}$.

The kinetic potential

$$L = F - E_{\text{pot}} = -m_0 c^2\left(\sqrt{1 - \dfrac{v^2}{c^2}} - 1\right) + \dfrac{eE}{a}$$

$$= -m_0 c^2\,(Z - 1) + m_0 c^2\dfrac{1 - Z^2}{Z},$$

and $\qquad\qquad \bar{L} = L = m_0 c^2\left(1 + \dfrac{1}{Z} - 2Z\right)$ ....(22).

Our general condition gives

$$\delta\left\{nh\nu - \bar{L}\right\} = \delta\left\{nh\cdot\dfrac{m_0 c^3}{2\pi eE}\cdot\dfrac{(1 - Z^2)^{\frac{3}{2}}}{Z} - m_0 c^2\left(1 + \dfrac{1}{Z} - 2Z\right)\right\} = 0.$$

Evidently we can vary this expression in Z instead of in a.

Finally we have

$$\sqrt{1 - Z^2} = \dfrac{2\pi eE}{nhc}; \quad 1 + \dfrac{W}{m_0 c^2} = Z = \sqrt{1 - \left(\dfrac{2\pi eE}{nhc}\right)^2},$$

which again is the Bohr expression[*].

EXAMPLE 6. An electron rotates round a nucleus in a 're-lativistic ellipse.'

(1) Sommerfeld calculated the total energy in such a case as

$$W = -m_0 c^2\,(Z - 1), \quad Z = \sqrt{\dfrac{p^2 - p_0{}^2}{p^2 - \epsilon^2 p_0{}^2}},$$

when the equation of the 'relativistic ellipse' is

$$r = \dfrac{a\,(1 - \epsilon^2)}{1 + \epsilon\cos\gamma\phi},$$

and $p_0 = \dfrac{eE}{c}$, c denoting the velocity of light, $p^2 = \dfrac{p_0{}^2}{1 - \gamma^2}$, the semi-axis being

$$a = \dfrac{\sqrt{p^2 - p_0{}^2}\,\sqrt{p^2 - \epsilon^2 p_0{}^2}}{mc p_0\,(1 - \epsilon^2)}.$$

Hence we obtain a quadratic equation for p^2

$$p^4 - p_0{}^2\,(1 + \epsilon^2)\,p^2 + \epsilon^2 p_0{}^4 - a^2 m_0{}^2 c^2 p_0{}^2\,(1 - \epsilon^2)^2 = 0,$$

[*] A. Sommerfeld, *l.c.*, p. 330, form. (22), where $a = \dfrac{2\pi e^2}{hc}$.

then

$$p^2 = \tfrac{1}{2} \{ p_0{}^2 (1 + \epsilon^2) + p_0 \sqrt{p_0{}^2 (1 - \epsilon^2)^2 + 4a^2 m_0{}^2 c^2 (1 - \epsilon^2)^2} \};$$

further we obtain

$$Z = \sqrt{\frac{p^2 - p_0{}^2}{p^2 - \epsilon^2 p_0{}^2}} = - \frac{p_0}{2am_0 c} + \sqrt{1 + \left(\frac{p_0}{2am_0 c} \right)^2}.$$

According to the binomial theorem we have

$$1 + \frac{W}{m_0 c^2} = 1 - \frac{eE}{2am_0 c^2} + \frac{1}{2} \frac{e^2 E^2}{4a^2 m_0{}^2 c^4} + \cdots.$$

Putting $c = \infty$, we obtain $W = - \dfrac{eE}{2a}$, which is the energy in the non-relativistic case.

(2) The calculation of the two frequencies (radial and azimuthal). The areal constant

$$p = mr^2 \dot{\phi} = \frac{m_0}{\sqrt{1 - \beta^2}} r^2 \dot{\phi}, \quad \beta^2 = \frac{v^2}{c^2},$$

according to Sommerfeld* is

$$\frac{1}{\sqrt{1 - \beta^2}} = Z + \frac{eE}{m_0 c^2} \cdot \frac{1}{r}, \quad Z = 1 + \frac{W}{m_0 c^2}.$$

Hence $\quad p = m_0 \left(Z + \dfrac{p_0}{m_0 c} \dfrac{1}{r} \right) r^2 \dot{\phi}, \quad \dfrac{1}{\dot{\phi}} = \dfrac{m_0}{p} \left(Z + \dfrac{p_0}{m_0 c} \dfrac{1}{r} \right) r^2 \quad (23).$

But $\quad r = \dfrac{a (1 - \epsilon^2)}{1 + \epsilon \cos \psi}, \quad \psi = \gamma \phi, \quad \gamma = \sqrt{1 - \dfrac{p_0{}^2}{p^2}}, \quad p_0 = \dfrac{eE}{c} \quad (24).$

If T' denotes the time of revolution counted from a perihelion to the next one, we obtain from the relation

$$dt = \frac{d\phi}{\dot{\phi}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(25),$$

the period $T' = \displaystyle\int_0^{\frac{2\pi}{\gamma}} \frac{d\phi}{\dot{\phi}} = \int_0^{2\pi} \frac{d\psi}{\dot{\psi}} = \int_0^{2\pi} \frac{m_0}{\gamma p} \left(Z + \frac{p_0}{m_0 c} \frac{1}{r} \right) r^2 d\psi.$

Substituting the above values we have

$$T' = \left\{ a (1 - \epsilon^2) m_0 Z \int_0^{2\pi} \frac{d\psi}{(1 + \epsilon \cos \psi)^2} + \frac{p_0}{c} \int_0^{2\pi} \frac{d\psi}{1 + \epsilon \cos \psi} \right\} \frac{a (1 - \epsilon^2)}{\sqrt{p^2 - p_0{}^2}}.$$

But $\quad a = \dfrac{p_0}{m_0 c} \dfrac{Z}{1 - Z^2}, \quad \sqrt{p^2 - p_0{}^2} = \dfrac{p_0 Z \sqrt{1 - \epsilon^2}}{\sqrt{1 - Z^2}};$

$$\int_0^{2\pi} \frac{d\psi}{(1 + \epsilon \cos \psi)^2} = \frac{2\pi}{(1 - \epsilon^2)^{\frac{3}{2}}}, \quad \int_0^{2\pi} \frac{d\psi}{1 + \epsilon \cos \psi} = \frac{2\pi}{\sqrt{1 - \epsilon^2}}.$$

* *Ann. d. Phys.* 51 (1916), p. 48, form. (B).

Thus
$$T' = \frac{2\pi p_0}{m_0 c^2 (1 - Z^2)^{\frac{3}{2}}}.$$

Replacing the period T' by the frequency ν', we obtain
$$\nu' = \frac{m_0 c^2}{2\pi p_0} (1 - Z^2)^{\frac{3}{2}},$$

where
$$Z = \sqrt{1 + \left(\frac{p_0}{2am_0 c}\right)^2} - \frac{p_0}{2am_0 c}.$$

Hence ν' is a function of the single variable a. But we can also regard ν' as a function of the only variable Z. The second frequency is the reciprocal of the second (azimuthal) period. In increasing the angle $\psi = \gamma \phi$ from 0 to 2π in the time T', the angle $\phi = \dfrac{\psi}{\gamma}$ increases from 0 to 2π in the time $T = T'\gamma$; putting $T = \dfrac{1}{\nu}$, $T' = \dfrac{1}{\nu'}$, we obtain the relation
$$\nu = \frac{\nu'}{\gamma}.$$

But from the expression
$$Z^2 = \frac{p^2 - p_0^2}{p^2 - \epsilon^2 p_0^2} = \frac{\gamma^2}{1 - \epsilon^2 (1 - \gamma^2)},$$

it follows
$$\gamma^2 = \frac{Z^2 (1 - \epsilon^2)}{1 - \epsilon^2 Z^2}, \quad 1 - \gamma^2 = \frac{1 - Z^2}{1 - \epsilon^2 Z^2}, \quad p^2 = p_0^2 \frac{1 - \epsilon^2 Z^2}{1 - Z^2};$$

hence
$$\nu = \frac{\nu'}{\gamma} = \frac{m_0 c^3}{2\pi e E} \frac{(1 - Z^2)^{\frac{3}{2}}}{Z \sqrt{1 - \epsilon^2}} \sqrt{1 - \epsilon^2 Z^2}.$$

We have thus expressed ν as a function of variables Z and ϵ (or of a and ϵ).

(3) Calculation of the kinetic potential. We have
$$F - m_0 c^2 = m_0 c^2 \sqrt{1 - \beta^2} = -\frac{m_0 c^2 (1 - \epsilon^2) Z}{(1 - Z^2 \epsilon^2) + (1 - Z^2) \epsilon \cos \psi},$$
$$E_{\text{pot}} = -\frac{eE}{r} = -\frac{m_0 c^2}{1 - \epsilon^2} \frac{1 - Z^2}{Z} (1 + \epsilon \cos \psi),$$
$$L = F - E_{\text{pot}}$$
$$= -\frac{m_0 c^2 (1 - \epsilon^2) Z}{(1 - Z^2 \epsilon^2) + (1 - Z^2) \epsilon \cos \psi} + \frac{m_0 c^2}{1 - \epsilon^2} \frac{1 - Z^2}{Z} (1 + \epsilon \cos \psi) + m_0 c^2.$$

Putting $\epsilon = 0$, we obtain the kinetic potential of a circle, which agrees with the result in (22).

We have found in (23), (25) the expression for dt. Substituting into (25), (24) the expressions
$$a = \frac{p_0}{m_0 c} \frac{Z}{1 - Z^2}, \quad \sqrt{p^2 - p_0^2} = p_0 \frac{Z \sqrt{1 - \epsilon^2}}{\sqrt{1 - Z^2}},$$

we obtain

$$dt = \frac{p_0}{m_0 c^2} \frac{\sqrt{1 - \epsilon^2}}{\sqrt{1 - Z^2}} \left\{ (1 - \epsilon^2) \frac{Z^2}{1 - Z^2} \frac{d\psi}{(1 + \epsilon \cos \psi)^2} + \frac{d\psi}{1 + \epsilon \cos \psi} \right\},$$

and

$$\int_0^{T'} L \, dt = - p_0 \frac{\sqrt{1 - \epsilon^2}}{\sqrt{1 - Z^2}} \left[\frac{(1 - \epsilon^2)^2 Z^3}{1 - Z^2} \right.$$

$$\times \int_0^{2\pi} \frac{d\psi}{(1 + \epsilon \cos \psi)^2 \{(1 - \epsilon^2 Z^2) + (1 - Z^2) \epsilon \cos \psi\}} - Z \int_0^{2\pi} \frac{d\psi}{1 + \epsilon \cos \psi}$$

$$+ (1 - \epsilon^2) Z \int_0^{2\pi} \frac{d\psi}{(1 + \epsilon \cos \psi) \{(1 - \epsilon^2 Z^2) + (1 - Z^2) \epsilon \cos \psi\}}$$

$$\left. - \frac{1}{1 - \epsilon^2} \int_0^{2\pi} \frac{1 - Z^2}{Z} d\psi \right].$$

Let us calculate some integrals.

$$(\mathrm{I}) \quad \int_0^{2\pi} \frac{d\psi}{1 + \epsilon \cos \psi} = \frac{2\pi}{\sqrt{1 - \epsilon^2}}, \ |\epsilon| < 1.$$

Putting $\epsilon = \beta : \alpha, |\alpha| > |\beta|$, we have

$$(\mathrm{II}) \quad \int_0^{2\pi} \frac{d\psi}{\alpha + \beta \cos \psi} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2}}.$$

Differentiating this integral in the parameter α, we obtain

$$(\mathrm{III}) \quad \int_0^{2\pi} \frac{d\psi}{(\alpha + \beta \cos \psi)^2} = 2\pi \alpha (\alpha^2 - \beta^2)^{-\frac{3}{2}}, \ |\alpha| > |\beta|.$$

Further

$$(\mathrm{IV}) \quad \int_0^{2\pi} \frac{d\psi}{(\alpha + \beta \cos \psi)(\gamma + \delta \cos \psi)}$$

$$= \frac{\beta}{\beta \gamma - \alpha \delta} \int_0^{2\pi} \frac{d\psi}{\alpha + \beta \cos \psi} - \frac{\delta}{\beta \gamma - \alpha \delta} \int_0^{2\pi} \frac{d\psi}{\gamma + \delta \cos \psi}$$

$$= \frac{2\pi}{\beta \gamma - \alpha \delta} \left[\frac{\beta}{\sqrt{\alpha^2 - \beta^2}} - \frac{\delta}{\sqrt{\gamma^2 - \delta^2}} \right]; \ |\alpha| > |\beta|, |\gamma| > |\delta|.$$

Differentiating this integral in the parameter α we obtain

$$(\mathrm{V}) \int_0^{2\pi} \frac{d\psi}{(\alpha + \beta \cos \psi)^2 (\gamma + \delta \cos \psi)} = \frac{2\pi \delta}{(\beta \gamma - \alpha \delta)^2} \left[\frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \right.$$

$$\left. - \frac{\delta}{\sqrt{\gamma^2 - \delta^2}} \right] + \frac{2\pi \alpha \beta}{(\beta \gamma - \alpha \delta)(\alpha^2 - \beta^2)^{\frac{3}{2}}}; \ |\alpha| > |\beta|, |\gamma| > |\delta|.$$

Hence $\quad L = \dfrac{1}{T'} \int_0^{T'} L \, dt = + m_0 c^2 \left[1 - Z^3 + \dfrac{1}{\sqrt{1 - \epsilon^2}} \dfrac{(1 - Z^2)^2}{Z} \right].$

Our general condition gives

$$\delta\left(nh\nu + n'h\nu' - \bar{L}\right) = 0.$$

Instead of varying in a and ϵ we may also vary in Z and ϵ. It will be more convenient to use new variables x and q and vary with respect to these. Putting

$$x = \frac{1-Z^2}{Z^2}, \; q^2 = \frac{1}{1-\epsilon^2}, \; \left(\frac{2\pi e^2}{hc} = \alpha, \; p_0 = \frac{eE}{c}\right),$$

we obtain

$$\delta\left\{\frac{m_0 c^2 e}{\alpha E}\left(\frac{x}{x+1}\right)^{\frac{3}{2}}\left[n\sqrt{1+q^2x} + n'\right] + \frac{m_0 c^2}{(x+1)^{\frac{3}{2}}}\left(1-q^2x\right) - m_0 c^2\right\} = 0.$$

The variation in x and in q gives these conditions

$$-\frac{3}{2}\frac{\alpha E}{e}\left(1-qx^2\right) + \frac{3}{2}x^{\frac{1}{2}}\left(n\sqrt{1+q^2x} + n'\right)$$
$$+ q\left(x+1\right)xn\left[\frac{x^{\frac{1}{2}}q}{2\sqrt{1+q^2x}} - 2\frac{\alpha E}{ne}\right] = 0 \quad \ldots(26),$$

$$\frac{q^2x}{1+q^2x} = \left(\frac{\alpha E}{ne}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots(27).$$

Substituting (27) into (26), we find

$$x = \frac{1}{Z^2} - 1 = \frac{\left(\frac{\alpha E}{e}\right)^2}{\left[n' + \sqrt{n^2 - \left(\frac{\alpha E}{e}\right)^2}\right]^2},$$

or
$$Z = 1 + \frac{W}{m_0 c^2} = \left\{1 + \frac{\left(\frac{\alpha E}{e}\right)^2}{\left[n' + \sqrt{n^2 - \left(\frac{\alpha E}{e}\right)^2}\right]^2}\right\}^{-\frac{1}{2}},$$

which also is identical with the Sommerfeld* equation.

In conclusion, I desire to express my thanks to Professors P. Ehrenfest, F. Záviška and J. Heyrovský for their kind interest and advice.

* A. Sommerfeld, *l.c.*, p. 330, form. (23); p. 521, form. (5).

The tide in the Bristol Channel. By Sir George Greenhill.

An interesting comparison with reality of the theory of the long flat tidal wave up an estuary is made by Mr G. I. Taylor in the *Proceedings of the Cambridge Philosophical Society*, Jan. 1921, p. 230, in the application to the tide in the Bristol Channel, taken as increasing slowly and uniformly in breadth and depth out to sea from an origin at Portishead.

Similar applications can be made to the remarkable tide in the Bay of Fundy, Nova Scotia; also to the Gulf of California, and to the Humber and others, as in the *Principia*, Book III, 24, 26, 27, p. 390.

Then there are the investigations by Chrystal of the Seiche in the Lake of Geneva, in the *Trans. R. S. Edinburgh*, 1905, leading to the same analytical treatment, to be rewritten in the Fourier notation.

Similar theoretical calculation was given in the *Phil. Mag.*, Nov. 1919, on the flat tidal wave in a channel, changing its section slowly according to some simple mathematical law, with the view of bringing forward the Clifford function (*Mathematical Papers*, p. 346), really the original Fourier function, and anterior to the Bessel function, to illustrate its advantage.

A memoir was read to the Mathematical Congress at Strasburg, Sept. 1920, in the Pedagogic Section, with the hope of attracting the attention of the mathematician to the Fourier function in preference to the usual Bessel function, and the application here to the tidal problem will serve as an illustration.

The conventional notation may be followed as adopted by Mr G. I. Taylor, except that the equations become more convenient and canonical, if we work to $\zeta = A\xi$, the flow through the cross section A of the horizontal displacement ξ, making the equation of continuity, for surface breadth B,

$$\frac{d\zeta}{dx} + B\eta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(I).$$

On the usual dynamical assumption that the pressure head in a long flat wave is the depth below the free surface, and so neglecting the effect of the vertical velocity as insensible, and supposing the liquid to oscillate in the swing over the ground of a pendulum of length l, the equation of motion is

$$\frac{d^2\xi}{g\,dt^2} = -\frac{\xi}{l} = -\frac{d\eta}{dx},$$

and the dynamical equation is written

$$\frac{d\eta}{dx} - \frac{\zeta}{Al} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(II),$$

and (I), (II) are the equations of the motion, in a canonical form.

By the alternate elimination of ζ and η

$$\frac{d}{dx}\left(A\frac{d\eta}{dx}\right) + \frac{B\eta}{l} = 0, \qquad \frac{d^2\eta}{dx^2} + \frac{d\log A}{dx}\frac{d\eta}{dx} + \frac{B\eta}{Al} = 0 \ldots(III),$$

$$\frac{d}{dx}\left(\frac{d\zeta}{Bdx}\right) + \frac{\zeta}{Al} = 0, \qquad \frac{d^2\zeta}{dx^2} - \frac{d\log B}{dx}\frac{d\zeta}{dx} + \frac{B\zeta}{Al} = 0 \ldots(IV),$$

two more canonical equations.

Assuming that A and B vary slowly as some power of x, $A = \alpha x^p$ and $B = \beta x^q$, and with $\frac{B}{Al} = hx^{q-p}$,

$$\frac{d^2\eta}{dx^2} + \frac{p}{x}\frac{d\eta}{dx} + hx^{q-p}\eta = 0, \qquad \frac{d^2\zeta}{dx^2} - \frac{q}{x}\frac{d\zeta}{dx} + hx^{q-p}\zeta = 0 \ldots(V);$$

and changing to the new variable

$$z = \frac{hx^{q-p+2}}{(q-p+2)^2},$$

making

$$x\frac{dy}{dx} = (q-p+2)z\frac{dy}{dz},$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = (q-p+2)^2\left(z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz}\right),$$

the two equations in (V) reduce to the Fourier form of his D.E. of order n,

$$z\frac{d^2y}{dz^2} + (1+n)\frac{dy}{dz} + y = 0 \ldots\ldots\ldots\ldots\ldots(VI),$$

with a solution $y = F_n(z)$, in which

$$F_0(z) = \Sigma \frac{(-z)^k}{(\Pi k)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$F_n(z) = \left(-\frac{d}{dz}\right)^n F_0(z) = \Sigma \frac{(-z)^k}{\Pi h \, \Pi(k+n)} \quad \ldots\ldots\ldots(2);$$

and F_{-n} denoting the nth integral of F_0,

$$F_{-n}(z) = z^n F_n(z), \quad \frac{dF_{-n}}{dz} = F_{-n+1} \quad \ldots\ldots\ldots\ldots(3).$$

Expressed by the Bessel function

$$F_0(z) = J_0(2\sqrt{z}), \quad F_n(z) = z^{-\frac{1}{2}n}J_n(2\sqrt{z})\ldots\ldots\ldots(4).$$

Then the solution of (V) by the Fourier function is

$$\eta = aF_n(z), \quad \zeta = bF_{-n-1}(z) \quad \dots\dots\dots\dots(5),$$

$$n = \frac{p-1}{q-p+2}, \quad n+1 = \frac{q+1}{q-p+2},$$

and the ratio of a to b would be assigned by (I) or (II).

Thus with uniform depth p, $p = q$, $n = \frac{1}{2}(p-1)$; and with similar cross section, $p = 2q$, $n = \dfrac{2q-1}{2-q}$.

In Mr G. I. Taylor's assumption of an estuary shallowing uniformly at one in m,

$$\frac{A}{B} = \frac{x}{m}, \quad p = q+1, \quad n = q; \quad \eta = aF_q\binom{mx}{l}, \quad \zeta = bF_{-q-1}\left(\frac{mx}{l}\right)\dots(6);$$

and further, with the breadth increasing uniformly,

$$q = 1, \ p = 2, \ n = 1, \ \eta = aF_1\left(\frac{mx}{l}\right), \ \zeta = bF_{-2}\left(\frac{mx}{l}\right) \ \dots(7),$$

agreeing with the Bessel function form given by Mr Taylor, with

$$\frac{J_1\,2\sqrt{(kx)}}{\sqrt{(kx)}} = F_1(kx), \ \ J_2\,2\sqrt{(kx)} = \frac{F_{-2}(kx)}{kx} = kxF_2(kx) \ \dots(8).$$

Then for the first three roots of

$$\left.\begin{array}{llll} F_0(z) = 0, & z = 1{\cdot}445, & 6{\cdot}945, & 18{\cdot}72 \\ F_1(z) = 0, & z = 3{\cdot}67, & 12{\cdot}30, & 25{\cdot}88 \\ F_2(z) = 0, & z = 6{\cdot}59, & 17{\cdot}71, & 33{\cdot}77 \end{array}\right\} \ \dots(9),$$

and so on. These give a place where the tide is a maximum, horizontal or vertical.

The tables calculated for the Bessel function $J(x)$ can then be utilised for the Fourier function $F(z)$ by entering a new column of the argument $z = \frac{1}{4}x^2$.

When n is half an integer, the Fourier and Bessel function is expressible by the circular function

$$\sqrt{\pi}F_{-\frac{1}{2}}(z) = \cos(2\sqrt{z}+\epsilon), \ \ \sqrt{\pi}F_{\frac{1}{2}}(z) = \frac{\sin(2\sqrt{z}+\epsilon)}{\sqrt{z}}\dots(10),$$

and so on, by differentiation and integration, including both solutions of the differential equation by the addition of the lead ϵ as an arbitrary constant.

The reciprocal of Mr Taylor's k is a length of $84{\cdot}74$ miles, the equivalent of our $\dfrac{l}{m}$, the fictitious depth of water at a distance l from the origin, and in his calculations the slope is about one in 3000, in round numbers.

In a tidal current the swing over the ground of the equivalent pendulum motion is on an arc of too great a radius to be expressed except in a unit such as R the radius of the Earth, 3438 geographical (G) miles, the circumference of the Earth being $360 \times 60 = 21600$ G miles.

The discount in gravity at the equator due to diurnal rotation μ is usually taken as one part in 289, in round numbers (*Principia*, lib. III, prop. xix); and then

$$g = 289R\mu^2 = l(N\mu)^2, \quad \frac{l}{R} = \left(\frac{17}{N}\right)^2 \ldots\ldots\ldots\ldots(11),$$

for an oscillation of l making N double beats a day.

Thus in a semi-diurnal tide, taking $N = 2$, it makes $l = 72R$; but with an average tide of 24 minutes over 12 hours, 12·4 hours, l is raised to over $77R$, making $m = kl = \dfrac{77R}{84\cdot74} = 3080$, agreeing fairly with the average of Mr Taylor's measurements.

With such a great value of l, it must not be taken as the radius of the arc of pendulum swing over the ground. The radiation of gravity must be taken into account from the centre of the Earth, and the swing of pendulum oscillation over the ground must be imitated by a truck over a line apparently straight and level, but bent downward slightly to a curvature of m' (minutes) per mile, a radius $a = R/m$ miles, with centre beyond the centre of the Earth, and then the truck will make $N = 17\sqrt{(1 - m)}$ oscillations, to and fro, in the day in the absence of resistance.

Because on a radius a, when the truck is displaced from equilibrium through a small angle θ, and gravity through an angle ϕ,

$$\frac{a d^2\theta}{g dt^2} = -\sin(\phi - \theta) = -\left(\frac{\sin\phi}{\sin\theta} - 1\right)\sin\theta = -\left(\frac{a}{R} - 1\right)\sin\theta \ldots(12),$$

$$\frac{d^2\theta}{g dt^2} = -\frac{1}{l}\sin\theta = -\left(\frac{1}{R} - \frac{1}{a}\right)\sin\theta, \quad \frac{1}{l} = \frac{1}{R} - \frac{1}{a} = \frac{1}{R}(1 - m) \ldots(13).$$

Thus if the line is level with the surface of the sea, $m = 1$, and the truck is everywhere in equilibrium.

But if straight like a Channel tunnel, $m = 0$, and the truck oscillates 17 times a day, in unison with the grazing satellite, or the free oscillation of water filling a diametral tunnel, as imagined by Socrates (*Phædo*).

For a semi-diurnal tide, with $N = 2$,

$$m = \frac{R}{a} = 1 - \frac{R}{l} = 1 - \frac{4}{289}, \quad a = \frac{289}{285}R \ldots\ldots\ldots(14).$$

If the line is curved upward, m is taken negative. Thus if curved upward to a radius R, $n = -1$, $l = \tfrac{1}{2}R$, $N = 17\sqrt{2}$, say 24, replacing 289 by 288, and the single pendulum beat is half an hour.

This radiation of gravity is ignored by Kelvin in his statement of the oscillation of a liquid globe, say of the radius of the Earth, if slightly disturbed in a spherical harmonic of order n; and then he finds

$$\frac{l}{R} = \frac{2n+1}{2n(n-1)},$$

and in the gravest mode when

$$n = 2, \quad l = \tfrac{5}{4}R.$$

But this l is not the radius of the arc of equivalent pendulum swing over the ground; the radius $a = 5R$ must be taken, or the arc must be curved downward at one minute in 5 miles.

A sailor calls the horizontal motion, ξ or ζ, the tide out at sea, estimated as a current, independent of the depth. But in soundings η the vertical motion is called the tide, to be considered in crossing shallows and entering a dock or port.

In a ground swell (*houle*) of long flat waves in shallow water, η will be small compared with ξ; so that although the pitching motion of a steamer may be insensible, the engines will race from the fluctuation of ξ and ζ, and the flow of water through the screw.

In nautical units of the G mile for length, and the hour for time-longitude, speed K is given in knots. Here 289 may be replaced for convenience by 288, allowing for discount in g due to diurnal rotation, the true number lying between these limits, thus making

$$g = 288R\mu^2 = 288R\left(\frac{2\pi}{24}\right)^2 = 2\pi^2 R = 21600\pi \text{ (knots an hour) } ...(15),$$

too large a number to be remembered easily, like the familiar $g = 32$ f/s². The grazing satellite velocity is then given by

$$G = \sqrt{(gR)} = \pi R\sqrt{2} = 10800\sqrt{2} \text{ knots.}$$

This value of g in cosmopolitan nautical units is suitable to employ in Newton's calculation applied to the Moon, when he tested his theory of gravitation.

With a parallax 57′·3, cosec. parallax 60, the range of the Moon $r = 60R$; and on the inverse square law, g must be reduced from 21600π on the Earth to 6π at the Moon; so that for a sidereal lunation of H hours,

$$\frac{4\pi^2 r}{H^2} = 6\pi, \quad H^2 = \frac{4\pi r}{6} = 40\pi R = 40 \times 10800, \quad H = 120\sqrt{30} \text{ hours,}$$

or $$5\sqrt{30} = 5 \times 5\cdot48 = 27\cdot4 \text{ days.}$$

This results at once from Kepler's Law III, with the grazing satellite taken as making $\sqrt{(288)} = 12\sqrt{2}$ revolutions a day, and 289 reduced to 288 when pure gravity is discounted for diurnal

rotation. Then for a satellite at the range of the Moon, $r = 60R$; and Law III makes the sidereal lunation

$$\frac{60\sqrt{60}}{12\sqrt{2}} = 5\sqrt{30} = 27 \cdot 4 \text{ days,}$$

as before.

Long free rollers of length λ out at the offing in the deep sea advance with velocity K, where

$$K^2 = \frac{g\lambda}{2\pi} = \pi R \lambda = 10800\lambda, \quad \left(\frac{K}{60}\right)^2 = 3\lambda \quad \text{.........(16)},$$

and then if M is the number of waves per minute,

$$M = \frac{K}{60\lambda}, \quad MK = \frac{K^2}{60\lambda} = 180 \quad \text{...............(17)}.$$

Thus, for example, if M Atlantic rollers were counted in the minute, breaking on the shore one day, and M' the day before, 24 hours earlier, the distance of the storm centre in the offing out at sea will be x miles, where

$$24 = \frac{x}{K} - \frac{x}{K'} = \frac{x}{180}(M - M'), \quad x = \frac{24 \times 180}{M - M'} \quad \text{......(18)}.$$

For instance, if $M = 6$, $K = 30$, $\lambda = \frac{1}{12}$; $M' = 2$, $K = 90$, $\lambda = \frac{3}{4}$; then $x = 1080$ miles, and the storm took place 36 hours earlier; a Smith's Prize question by Prof. Stokes.

In a further application of the Fourier function for Mr G. I. Taylor's consideration, take his *Report to the Aeronautical Committee on Dissipation of Eddies*, R. and M. 598, Dec. 1918.

Equation (1) there, on the assumption that ω the angular velocity of a ring of fluid of radius r dies out on the compound discount law, $\frac{1}{\omega}\frac{d\omega}{dt} = -k$, in consequence of viscosity μ,

$$\frac{d}{dr}\left(2\pi r^2 \mu r \frac{d\omega}{dr}\right) = 2\pi r^3 \rho \frac{d\omega}{dt} = -2\pi k r^3 \rho \omega \quad \text{......(19)},$$

$$r^2 \frac{d^2\omega}{dr^2} + 3r \frac{d\omega}{dr} + \frac{k\rho}{\mu} r^2 \omega = 0 \quad \text{...............(20)},$$

and this, with $\frac{k\rho r^2}{4\mu} = x$, becomes

$$x \frac{d^2\omega}{dx^2} + 2 \frac{d\omega}{dx} + \omega = 0 \quad \text{..................(21)}.$$

Fourier's D.E. for $n = 1$, having a solution

$$\omega = F_1(x) = F_1\left(\frac{k\rho r^2}{4\mu}\right) = \frac{J_1(\lambda r)}{\lambda r}, \quad \lambda^2 = \frac{k\rho}{\mu} \quad \text{......(22)}.$$

The rest of the investigation can easily be supplied, as M and θ there are then expressed by $F(x)$.

The Definition of an Envelope. By E. H. NEVILLE[*].

1. The remark that circles of curvature at neighbouring points of a plane curve do not as a rule intersect was a sentence of death on the classical definition of the envelope of a family of real plane curves; if a definition does not allow a curve to be the envelope of its own circles of curvature, so much the worse for the definition.

The gist of the definition by which de la Vallée Poussin replaced the classical definition is that the envelope is the locus of points where the members of the family are unnaturally close together. To render the definition precise, the members of the family are associated with the values of a real variable α. It is then shewn that for small values of β the shortest distance from a point Q on the curve α_0 to the curve $\alpha_0 + \beta$ is *in general* of the first order in β, but may be of a higher order. An ordinary point on α_0 at which the distance is of a higher order than the first is called a characteristic point of α_0, and the envelope is defined as the aggregate of the characteristic points of α for those values of α for which the curve α does not consist entirely of characteristic points.

This definition is unsatisfactory in certain respects. In the first place, the change from α to a new variable ξ defined by

$$\alpha - \alpha_0 = (\xi - \xi_0)^3$$

gives $\beta = \eta^3$ if $\alpha_0 + \beta = (\xi_0 + \eta)^3$. A distance which is of the first order in β is of order higher than the first in η, and therefore if the family is associated with the variable ξ instead of with the variable α, the curve ξ_0, which is an *arbitrary* member of the family, is composed entirely of characteristic points, and according to the definition no points of the envelope can be found on ξ_0. That is to say, the envelope depends on the parameter used, and any point of the envelope can be removed by an appropriate choice of parameter.

Again, if the members of the family are the tangents to a given curve, any tangent whose order of contact is higher than the first proves to be composed entirely of characteristic points: the envelope is not the curve from which the family is derived, but is this curve deprived of its points of inflexion as well as of its singular points.

One modification of de la Vallée Poussin's definition is to *complete* the envelope as he defined it by the addition of all its limiting points. This process is effective in simple cases, but it may admit singular points whose exclusion is actually desirable, and it does not discriminate between points at which there is some genuine geometrical peculiarity and points omitted accidentally, that is, owing to some peculiarity in the variable associated with the family. Moreover, the method is inherently troublesome to use.

An alternative is to define Q as a characteristic point on the curve α_0 if the concentration of the family near Q is unnaturally close not merely in comparison with points in general but *in comparison with most points on the curve* α_0. This is the view that is developed here.

2. We suppose the family of curves to be

$$f(x, y, \alpha) = 0, \qquad \ldots\ldots 2\cdot 1$$

referred to rectangular axes, and we consider a point Q on the curve α_0; for the present we assume Q to be an ordinary point of α_0. With this hypothesis, $f_x(x_Q, y_Q, \alpha_0)$ and $f_y(x_Q, y_Q, \alpha_0)$ are not both zero, and therefore the range of x, y, and α can be so restricted that the two square-roots of $f_x{}^2 + f_y{}^2$ are separate functions; we denote one of these functions by $g(x, y, \alpha)$, and we write

$$l(x, y, \alpha) = f_x(x, y, \alpha)/g(x, y, \alpha), \quad m(x, y, \alpha) = f_y(x, y, \alpha)/g(x, y, \alpha).$$

If the point P is on the curve α, then $l(x_P, y_P, \alpha)$, $m(x_P, y_P, \alpha)$ are the ratios of a direction normal to α at P. But since it is essential that $2\cdot 1$ should not be regarded as expressing α in terms of x and y, the functions l, m must be taken primarily as functions of three variables, not of two.

3. Instead of dealing with the shortest distance from Q to the curve $\alpha_0 + \beta$, we deal with the distance from Q to the nearest point in which this curve cuts the normal QN to α_0; the two distances are asymptotically equivalent on the assumption that l, m are continuous functions of x, y, and α. An arbitrary point P on QN may be represented as $(x_Q + rl_Q, y_Q + rm_Q)$, where l_Q, m_Q stand for $l(x_Q, y_Q, \alpha_0)$, $m(x_Q, y_Q, \alpha_0)$ and r is the distance QP, and if we write

$$F(r, \beta) \equiv f(x_Q + rl_Q, y_Q + rm_Q, \alpha_0 + \beta),$$

the condition for P to be on the curve $\alpha_0 + \beta$ is

$$F(r, \beta) = 0. \qquad \ldots\ldots 3\cdot 1$$

Since Q is on α_0, $\qquad F(0, 0) = 0. \qquad \ldots\ldots 3\cdot 2$

Also for all values of r and β

$$F_r(r, \beta) \equiv l_Q f_x(x_Q + rl_Q, y_Q + rm_Q, \alpha_0 + \beta) \\ + m_Q f_y(x_Q + rl_Q, y_Q + rm_Q, \alpha_0 + \beta);$$

hence $F_r(0, 0) = (l_Q{}^2 + m_Q{}^2) g(x_Q, y_Q, \alpha_0) = g(x_Q, y_Q, \alpha_0),$

and therefore $\qquad\qquad F_r(0, 0) \neq 0.$ $\qquad$ ......3·3

But 3·2 and 3·3 together imply that 3·1 is equivalent for small values of r and β to a relation

$$r = \Phi(\beta)$$

in which $\Phi(\beta)$ is a single-valued function expressible for any value of n as

$$\sum_{s=1}^{n} (\Phi_s/s!) \beta^s + O(\beta^{n+1});$$

Φ_s is independent of β but depends of course on the numbers x_Q, y_Q, α_0 which have been treated as constants. To calculate Φ_s, we have only to determine $d^s r/d\beta^s$ from the implicit relation

$$f(x_Q + rl_Q, y_Q + rm_Q, \alpha_0 + \beta) = 0,$$

and to give r and β zero values; 3·3 implies that each coefficient is determinate and finite.

The order of r as a function of β, which will be called the α-order of concentration of the family at Q, is the index of the first of the coefficients Φ_1, Φ_2, Φ_3, ... that is not zero, and because $F_r(0, 0)$ is not zero, this is merely the index of the first of the derivatives $F_\beta(0, 0)$, $F_{\beta\beta}(0, 0)$, $F_{\beta\beta\beta}(0, 0)$, ... that is not zero. In other words,

Th. I. *The α-order of concentration of the family of curves $f(x, y, \alpha) = 0$ at an ordinary point Q of the curve α_0 is the index of the first of the derivatives f_α, $f_{\alpha\alpha}$, $f_{\alpha\alpha\alpha}$, ... that does not vanish for the values x_Q, y_Q, α_0 of x, y, α.*

4. To compare orders of concentration at different points, we have only to regard the derivatives $f_\alpha, f_{\alpha\alpha}, \ldots$ as functions of the three variables x, y, α instead of fixing our attention on the values which these derivatives have for given arguments x_Q, y_Q, α_0. Keeping α_0 constant and allowing x, y to vary subject to the relation $f(x, y, \alpha_0) = 0$, we have one analytical form of the definition of characteristic points under consideration:

Th. II. *The characteristic ordinary points of the member α_0 of the family of curves $f(x, y, \alpha) = 0$ are those ordinary points of α_0 whose coordinates satisfy the first of the equations*

$$f_\alpha(x, y, \alpha_0) = 0, \quad f_{\alpha\alpha}(x, y, \alpha_0) = 0, \ldots$$

that is not satisfied at every point of α_0.

Let us write $f^{(n)}(x, y, \alpha)$ as an abbreviation for $\partial^n f(x, y, \alpha)\,\partial\alpha^n$. It is possible, though exceptional, for a function $f^{(n)}(x, y, \alpha)$ to have a factor independent of x and y; to meet this case let us suppose

$$f^{(n)}(x, y, \alpha) = h^{(n)}(\alpha) H^{(n)}(x, y, \alpha),$$

where $H^{(n)}(x, y, \alpha)$ can not vanish independently of x and y for any value of α.

The vanishing of $f^{(s)}(x, y, \alpha_0)$ along the curve α_0 may be due either to the vanishing of $H^{(s)}(x, y, \alpha_0)$ along the curve or to the vanishing of $h^{(s)}(\alpha_0)$, that is, to the presence of a factor $(\alpha - \alpha_0)^p$ in $h^{(s)}(\alpha)$. If $h^{(s)}(\alpha)$ is expressible as $(\alpha - \alpha_0)^p j(\alpha)$, then, for all values of r from 0 to $p - 1$, $f^{(s+r)}(x, y, \alpha)$ has a factor $(\alpha - \alpha_0)^{p-r}$, and $f^{(s+p)}(x, y, \alpha_0)$ reduces to a numerical multiple of $j(\alpha_0) H^{(s)}(x, y, \alpha_0)$. If $f^{(1)}, f^{(2)}, \ldots f^{(s-1)}$ are known to vanish along α_0, the α-order of concentration at a point Q of α_0 is $s + p + q$, where $p + q$ is the smallest value of u for which

$$\partial^u \{(\alpha - \alpha_0)^p j(\alpha) H^{(s)}(x, y, \alpha)\}/\partial \alpha^u$$

does not vanish along α_0, that is, where q is the smallest value of v for which $\partial^v \{j(\alpha) H^{(s)}(x, y, \alpha)\}/\partial \alpha^v$

does not vanish along α_0; if then $j(\alpha_0)$ is not zero, q is the smallest value of v for which $\partial^v H^{(s)}/\partial \alpha^v$ does not vanish at every point of α_0, and Q is or is not a characteristic point of α_0 according as $\partial^q H^{(s)}(x_Q, y_Q, \alpha_0)/\partial \alpha^q$ is or is not zero. That is to say, if we wish only to find the characteristic points and not to determine their α-orders, *we may simply ignore any factor $h^{(s)}(\alpha)$ whether or not we are dealing with a curve for which this factor is zero.*

The application to a change from the variable α to another variable ξ is obvious. If

$$f(x, y, \alpha) \equiv k(x, y, \xi),$$

then $f_\alpha = k_\xi (d\xi/d\alpha), \quad f_{\alpha\alpha} = k_{\xi\xi} (d\xi/d\alpha)^2 + k_\xi (d^2\xi/d\alpha^2),$

and so on. If k_ξ is not zero along a curve of the family, the characteristic points of this curve are determinable equally from $f_\alpha = 0$ and from $k_\xi = 0$, whether or not $d\xi/d\alpha$ is zero. If these equations are ineffective because k_ξ is zero along the curve, then *for this same reason* the equations $f_{\alpha\alpha} = 0$, $k_{\xi\xi} = 0$ are equivalent but for the factor $(d\xi/d\alpha)^2$, and this factor can be ignored; the argument can be prolonged to any necessary extent.

The difficulty with regard to points of inflexion also disappears. Taking the equation of the family of tangents* as

$$y - f(\alpha) - (x - \alpha)f'(\alpha) = 0,$$

where $f(\alpha)$ is a given function of α, we have the characteristic points given by

$$(x - \alpha)f''(\alpha) = 0,$$

and therefore on the line α_0 we must have $x = \alpha_0$ whether or not $f''(\alpha_0)$ is zero. Since on a line every point is ordinary, the envelope of the family of lines is

$$x = \alpha, \quad y = f(\alpha),$$

* Cf. Fowler, *op. cit.* p. 67.

that is, is $y = f(x)$, without subtraction of any point to which there is a value of α to correspond.

5. The definition of a characteristic point is applicable if the family is given as

$$x = \phi(t, \alpha), \quad y = \psi(t, \alpha), \qquad \ldots\ldots 5\cdot 1$$

where the functions are regular and uniform. We assume now that the derivatives $\phi_t(t, \alpha)$, $\psi_t(t, \alpha)$ are not both zero at a point Q of the curve α_0 and therefore are not both zero for any values of t, α near to t_Q, α_0. The ratios of the normal QN are definite numbers l_Q, m_Q, not both zero, such that

$$l_Q \phi_t(t_Q, \alpha_0) + m_Q \psi_t(t_Q, \alpha_0) = 0,$$

and therefore such that

$$m_Q \phi_t(t_Q, \alpha_0) - l_Q \psi_t(t_Q, \alpha_0) \neq 0. \qquad \ldots\ldots 5\cdot 2$$

In order that the point $(x_Q + rl_Q, y_Q + rm_Q)$ should be on the curve $\alpha_0 + \beta$, there must be a value of t such that simultaneously

$$x_Q + rl_Q = \phi(t, \alpha_0 + \beta), \quad y_Q + rm_Q = \psi(t, \alpha_0 + \beta). \quad \ldots 5\cdot 3$$

The condition for a pair of equations

$$\Phi(r, t, \beta) = 0, \quad \Psi(r, t, \beta) = 0$$

to define r and t as functions of β in the neighbourhood of a given set of values is

$$\partial(\Phi, \Psi)/\partial(r, t) \neq 0,$$

and if Φ, Ψ denote

$$x_Q + rl_Q - \phi(t, \alpha_0 + \beta), \quad y_Q + rm_Q - \psi(t, \alpha_0 + \beta)$$

this condition is simply $5\cdot 2$.

From $5\cdot 3$, treating r and t as functions of β, we have

$$l_Q(dr/d\beta) = \phi_t(t, \alpha_0 + \beta)(dt/d\beta) + \phi_\alpha(t, \alpha_0 + \beta),$$

$$m_Q(dr/d\beta) = \psi_t(t, \alpha_0 + \beta)(dt/d\beta) + \psi_\alpha(t, \alpha_0 + \beta),$$

and therefore
$$\frac{dr}{d\beta} = \left[\frac{\partial(\phi, \psi)/\partial(t, \alpha)}{m_Q \phi_t - l_Q \psi_t}\right]_{\alpha = \alpha_0 + \beta}. \qquad \ldots\ldots 5\cdot 4$$

Hence unless
$$\partial(\phi, \psi)/\partial(t, \alpha) = 0, \qquad \ldots\ldots 5\cdot 5$$

for $t = t_Q$, $\alpha = \alpha_0$, the α-order of concentration at Q is unity and Q is not a characteristic point on α_0.

Now if $5\cdot 5$ was satisfied *identically*, there would be a functional relation, independent of t and α, between $\phi(t, \alpha)$ and $\psi(t, \alpha)$, that is, between x and y, and the original pair of equations $5\cdot 1$ would represent not a family of curves but a single curve; this case therefore can not occur. Hence further, $5\cdot 5$ can not be satisfied at every

point of a curve α_0 except through the presence of a factor $(\alpha - \alpha_0)^p$ in $\partial (\phi, \psi)/\partial (t, \alpha)$. Writing

$$\partial (\phi, \psi)/\partial (t, \alpha) = (\alpha - \alpha_0)^p \{m_Q \phi_t (t, \alpha) - l_Q \psi_t (t, \alpha)\} F (t, \alpha)$$

we have $\qquad\qquad dr/d\beta = \beta^p F (t_Q, \alpha_0 + \beta),$

where $F (t, \alpha_0)$ is not zero independently of t. Since the successive derivatives of t with respect to β, as implied by 5·3, are all finite when β is zero, and since the successive derivatives of r with respect to β may be conceived as calculated from 5·4 by substitution for derivatives of t, a point Q on α_0 has α-order of concentration equal to p if $F (t_Q, \alpha_0)$ is not zero and greater than p if $F (t_Q, \alpha_0)$ is zero:

Th. III. *The characteristic ordinary points of the typical curve in the family*

$$x = \phi (t, \alpha), \quad y = \psi (t, \alpha)$$

are those for which t satisfies the condition

$$F (t, \alpha) = 0,$$

where $F (t, \alpha)$ is derived from $\partial (\phi, \psi)/\partial (t, \alpha)$ by the suppression of every factor that is a function of α alone.

For example, the envelope of the line

$$x = X (\alpha) + t X' (\alpha), \quad y = Y (\alpha) + t Y' (\alpha),$$

where $X (\alpha)$, $Y (\alpha)$ are given functions of α and $X' (\alpha)$, $Y' (\alpha)$ are their derivatives, is given by the *relevant* factor of

$$t \{X' (\alpha) Y'' (\alpha) - Y' (\alpha) X'' (\alpha)\} = 0,$$

that is, by $t = 0$, even for values of α for which $X'Y'' - Y'X''$ is zero, provided only that $\partial x/\partial t$ and $\partial y/\partial t$ are not both zero, that is, provided only that $X' (\alpha)$ and $Y' (\alpha)$ are not both zero.

6. The method of § 5 is more powerful than that of §§ 2–4, for some points that have to be treated as singular in a functional representation of a curve are ordinary in a parametric representation. For example, if the function $f (x, y, \alpha)$ happens to have the form $\{v (x, y, \alpha)\}^2$, every point is singular in the sense that f_x and f_y are zero, but the family $f = 0$ is geometrically indistinguishable from the family $v = 0$; a parametric substitution of the form 5·1 does not reduce f identically to zero without reducing v, and therefore preserves no record of the peculiar form of f. Again, for a multiple point through which all the branches are regular, the parametric representation separates the branches completely; the point may be characteristic on some branches and not on others, or may not be characteristic at all, but unless it is a cusp or a transcendental singularity on one of the branches the general analysis needs no modification.

To associate the two methods, we have to suppose that the functions $f(x, y, \alpha)$, $\phi(t, \alpha)$, $\psi(t, \alpha)$ are such that

$$f\{\phi(t, \alpha), \psi(t, \alpha), \alpha\} = 0 \qquad \ldots\ldots 6\cdot 1$$

for all values of t and α. Then identically

$$f_x \phi_t + f_y \psi_t = 0,$$
$$f_x \phi_\alpha + f_y \psi_\alpha + f_\alpha = 0. \qquad \ldots\ldots 6\cdot 2$$

From the first of these relations it follows that there is a function $G(t, \alpha)$ such that, for all values of t and α,

$$f_x = G\psi_t, \quad f_y = -G\phi_t; \qquad \ldots\ldots 6\cdot 3$$

since ϕ_t and ψ_t are not simultaneously zero, and f_x and f_y are finite, G is finite. Substituting in 6·2 from 6·3 we have

$$f_\alpha = G\, \partial(\phi, \psi)/\partial(t, \alpha).$$

All the characteristic points are to be found from factors of $\partial(\phi, \psi)/\partial(t, \alpha)$ that do not vanish independently of t; hence we can ignore factors of f_α that vanish in virtue of the equation $f = 0$, and factors that vanish independently of t for particular values of α, without considering whether these factors belong to the function G or to the Jacobian $\partial(\phi, \psi)/\partial(t, \alpha)$.

Thus we are justified, for the family

$$f(x, y, \alpha) \equiv \{v(x, y, \alpha)\}^2 = 0,$$

in neglecting the factor v in f_α and deriving the characteristic points from $v_\alpha = 0$. For a multiple point through which there are only regular branches, G is necessarily zero; hence f_α is zero there, but the vanishing of f_α does not indicate that the point is characteristic. If $G(t, \alpha_0)$ is not identically zero, the order of a value t_Q as a zero of $G(t, \alpha_0)$ is the smaller of the orders of t_Q as a zero of $f_x\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0\}$ and as a zero of $f_y\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0\}$, and therefore Q is a characteristic point on the branch of α_0 represented by values of t near t_Q if and only if the smaller of these orders is less than the order of t_Q as a zero of

$$f_\alpha\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0\},$$

supposing this last function not to be identically zero; this condition is of little use in general, for if the functions ϕ, ψ are known, reference to f is superfluous. But the case in which $G(t_Q, \alpha_0)$ is not zero, that is, in which $f_x(x_Q, y_Q, \alpha_0)$ and $f_y(x_Q, y_Q, \alpha_0)$ are not both zero, is that of §§ 2–4, which is in this way subordinated to that of § 5.

7. It is perhaps 6·1 that gives the simplest basis for examining the classical definition of an envelope. To suppose the curve α_0 to be intersected by its neighbours is to assume that for every sufficiently small value of β the equation

$$f\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0 + \beta\} = 0 \qquad \ldots\ldots 7\cdot 1$$

is satisfied by some real value of t. It is impossible for 7·1 to be an *identity* for all small values of β, and if n is the index of the first of the functions $f^{(1)}(x, y, \alpha), f^{(2)}(x, y, \alpha), \ldots$ that is not *identically* zero for the values $\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0$ of x, y, α, there is some number $\gamma(t, \alpha_0, \beta)$ between 0 and β such that

$$f\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0 + \beta\} - f\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0\}$$
$$= \beta^n f^{(n)}\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0 + \gamma(t, \alpha_0, \beta)\}/n!.$$

From 6·1, this last identity implies that 7·1 is equivalent to

$$f^{(n)}\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0 + \gamma(t, \alpha_0, \beta)\} = 0. \quad \ldots\ldots 7\cdot2$$

As β tends to zero, $\gamma(t, \alpha_0, \beta)$ tends to zero uniformly with respect to t, and therefore* any root of 7·2 necessarily tends to some root of

$$f^{(n)}\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0\} = 0. \quad \ldots\ldots 7\cdot3$$

Th. IV. *The ordinary points of α_0 that are limits of points in which α_0 is intersected by a variable neighbouring curve of the family are among the characteristic points of α_0.*

But there is no reason to suppose that every root of 7·3 is a limit of a root of 7·2, or that if $\delta(t)$ is a *given* function of t there is a value of β such that $\gamma(t, \alpha_0, \beta)$ is $\delta(t)$: on both accounts the argument is irreversible, and we must discuss otherwise whether in general characteristic points are limits of points of intersection.

If we write

$$\Psi(t, \beta) = f\{\phi(t, \alpha_0), \psi(t, \alpha_0), \alpha_0 + \beta\},$$

then for the curve α_0 to be intersected by neighbouring curves, the equation

$$\Psi(t, \beta) = 0, \quad \ldots\ldots 7\cdot4$$

which is 7·1, must represent an effective relation between t and β for sufficiently small values of β. In examining this equation, we have to remember that $\Psi(t, 0)$ is *identically* zero, and that therefore

$$\Psi_t(t, 0) = 0, \quad \Psi_{tt}(t, 0) = 0, \ldots.$$

Supposing that we are to expand t in powers of β, we must find the values of the derivatives $dt/d\beta, d^2t/d\beta^2, \ldots$ when β is zero. From 7·4,

$$0 = \Psi_t t' + \Psi_\beta,$$
$$0 = \Psi_t t'' + \Psi_{tt} t'^2 + 2\Psi_{t\beta} t' + \Psi_{\beta\beta},$$
$$0 = \Psi_t t''' + 3\Psi_{tt} t' t'' + 3\Psi_{t\beta} t''$$
$$+ \Psi_{ttt} t'^3 + 3\Psi_{tt\beta} t'^2 + 3\Psi_{t\beta\beta} t' + \Psi_{\beta\beta\beta},$$

and so on. Hence it is impossible for t' to be finite when $\beta = 0$ unless $\Psi_\beta = 0$, that is, unless $f_\alpha = 0$, but if this condition is satisfied by a value t_Q of t, and if

$$\Psi_{t\beta}(t_Q, 0) \neq 0, \quad \ldots\ldots 7\cdot5$$

* See § 9 below.

then the first equation is satisfied identically, the second determines a unique finite value for t', the third a unique finite value for t'', and so on. On the assumption that $f_x(x_Q, y_Q, \alpha_0)$ and $f_y(x_Q, y_Q, \alpha_0)$ are not both zero, it is easy to identify the condition 7·5 with

$$[\partial(f, f_a)/\partial(x, y)]_{x_Q, y_Q, \alpha_0} \neq 0;$$

hence

TH. V. *If the Jacobian* $\partial(f, f_a)/\partial(x, y)$ *does not vanish at a characteristic ordinary point Q of the curve* α_0, *then Q is a limit of intersections of* α_0 *by neighbouring curves of the family* $f(x, y, \alpha) = 0$.

8. Concentration and intersection are not more closely associated with the idea of an envelope than is contact. With the notation we are using, the envelope is given parametrically by

$$x = \phi(t, \alpha), \quad y = \psi(t, \alpha) \qquad \text{......8·1}$$

when t, α are connected by a relation which implies

$$f^{(n)}\{\phi(t, \alpha), \psi(t, \alpha), \alpha\} = 0, \qquad \text{......8·2}$$
$$\partial(\phi, \psi)/\partial(t, \alpha) = 0. \qquad \text{......8·3}$$

If r is any current parameter on the envelope, $d\alpha/dr$ is not zero everywhere, since no α-curve is formed wholly of characteristic points; hence to suppose $d\alpha/dr$ not zero is only to exclude isolated points of the envelope. That is, we may use α for the parameter of the envelope without any sensible loss of generality.

Let Q be a characteristic ordinary point on the curve α_0, and assume that Q is ordinary also on the envelope and that α is available as the parameter of the envelope near Q. Let x_s, y_s be the direction ratios of the tangent to α_0 at Q, and x_T, y_T those of the tangent to the envelope there. Then

$$x_s : y_s = \partial x/\partial t : \partial y/\partial t$$

where the derivatives come from 8·1 on the assumption that α is constant; since t is a regular variable on α_0 and Q is an ordinary point on α_0, these values of $\partial x/\partial t$ and $\partial y/\partial t$ are not both zero. Similarly

$$x_T : y_T = \delta x/\delta\alpha : \delta y/\delta\alpha$$

where the derivatives come from 8·1 on the assumption that t and α are connected by 8·2; since on the envelope α is a regular variable and Q is an ordinary point, the values of $\delta x/\delta\alpha$ and $\delta y/\delta\alpha$ are not both zero. Hence the condition

$$x_s y_T - y_s x_T = 0$$

is *equivalent* to

$$(\partial x/\partial t)(\delta y/\delta\alpha) - (\partial y/\partial t)(\delta x/\delta\alpha) = 0,$$

that is, to

$$\phi_t\{\psi_t(\delta t/\delta\alpha) + \psi_a\} - \psi_t\{\phi_t(\delta t/\delta\alpha) + \phi_a\} = 0, \quad \text{......8·4}$$

which is satisfied in virtue of 8·3.

Thus at any characteristic point Q that is ordinary both on the α-curve through it and on the envelope, these two curves touch provided only that α can be used as a parameter on the envelope near Q. The last restriction can be removed immediately; any point R at which the restriction operates must be a limit of points at which it does not operate. If R is ordinary on the α-curve through R, any point Q sufficiently near to R is ordinary *on its own α-curve*, and the limit of the tangent QS_Q as Q tends to R *in any way* (not necessarily along the α-curve through R) is the tangent RS_R. If Q tends to R along the envelope, RS_R appears as the limit of QT_Q, the tangent to the *envelope* at Q, and if R is an ordinary point *on the envelope*, this limit is the tangent to the envelope at $\dot{R}$.

TH. VI. *At any characteristic point that is ordinary both on the curve to which it belongs and on the envelope, the envelope touches the curve.*

The steps by which we have proved this theorem are valid to establish the converse. Suppose that a regular curve touches at every point a member of the family. Then the curve is determined by a functional relation

$$\varpi\,(t,\,\alpha) = 0, \qquad\qquad \dots\dots 8\cdot5$$

and the current point on the curve has its coordinates given by $8\cdot1$ with $t,\,\alpha$ subject to this relation. Arguing as before we can begin by considering points near which α can be used as the parameter, and the condition for contact at such points takes the form $8\cdot4$, where now $\delta t\,\delta\alpha$ is to be determined from $8\cdot5$. But whatever the form of $\varpi\,(t,\,\alpha)$, the condition $8\cdot4$ is equivalent to $8\cdot3$, and therefore the point of contact is either a characteristic point, a cusp, or a transcendental singularity on the α-curve to which it belongs. The restriction involving α may again be removed, and therefore

TH. VII. *If a curve touches at each of its points one member α_0 of a family of curves, then if the point of contact is an ordinary point on the branch of α_0 which is touched, the curve is part of the envelope of the family.*

9. In § 7 we have used a lemma of which proof is perhaps necessary:

TH. VIII. *If as u tends to zero the function $\phi\,(x,\,u)$ tends to zero uniformly with respect to x, and if for all sufficiently small values of u the equation*

$$f\,\{x,\,u,\,\phi\,(x,\,u)\} = 0$$

has roots, then any value of x which is a limit of roots of this equation as u tends to zero is a root of the equation

$$f\,(x,\,0,\,0) = 0,$$

provided that the function $f(x, u, v)$ is a continuous function of the three variables on which it depends.

To suppose that l is a limit of roots of

$$f\{x, u, \phi(x, u)\} = 0$$

is to suppose that given any positive numbers ξ, η there are numbers X, U such that

$$|X - l| < \xi, \quad 0 < U < \eta, \quad f\{X, U, \phi(X, U)\} = 0.$$

If $f(l, 0, 0) = m \neq 0$, then because $f(x, u, v)$ is continuous there is a number ζ_m such that if simultaneously

$$|x - l| < \zeta_m, \quad |u| < \zeta_m, \quad |v| < \zeta_m,$$

then

$$|f(x, u, v) - f(l, 0, 0)| < m$$

and therefore

$$f(x, u, v) \neq 0.$$

Because $\phi(x, u)$ tends to zero uniformly, it is possible to find a number δ such that

$$|\phi(x, u)| < \zeta_m$$

for all values of x and for all values of u subject to

$$|u| < \delta.$$

Now take ξ equal to ζ_m, and η equal to the smaller of ζ_m and δ; there are values X, U satisfying

$$|X - l| < \zeta_m, \quad 0 < |U| < \eta, \quad f\{X, U, \phi(X, U)\} = 0.$$

If V denotes $\phi(X, U)$, then $|V| < \zeta_m$ because $\eta \leqslant \delta$; also $|U| < \zeta_m$ because $\eta \leqslant \zeta_m$. Hence

$$|X - l| < \zeta_m, \quad |U| < \zeta_m, \quad |V| < \zeta_m, \quad f(X, U, V) = 0,$$

and this contradicts the definition of ζ_m. That is, the assumption that $f(l, 0, 0) \neq 0$ is untenable.

The important feature of this lemma is that the restriction on $\phi(x, u)$ is slight; this function is not supposed to be regular or continuous or even single-valued.

Note on the number of primes of the form $n^2 + 1$. By A. E. WESTERN, Sc.D.

Prof. G. H. Hardy and Mr J. E. Littlewood[*] have suggested, but have not been able to obtain a rigorous proof of, the following asymptotic formula for $P(x)$, the number of primes less than x of the form

$$an^2 + bn + c,$$

where a, b, and c are given integers, a is positive, $a + b$ and c are not both even, $\Delta = b^2 - 4ac$ is not a square, and n is any integer:

$$P(x) \sim \tfrac{1}{2}\epsilon C a^{-\frac{1}{2}} \prod \frac{\varpi'}{\varpi' - 1} \operatorname{Li}(x^{\frac{1}{2}}),$$

wherein $\epsilon = 1$ if $a + b$ is odd, $\epsilon = 2$ if $a + b$ is even,

$$C = \prod_{\varpi \geqslant 3}^{x} \left(1 - \left(\frac{\Delta}{\varpi}\right)\frac{1}{\varpi - 1}\right),$$

and ϖ represents all primes not divisors of a, and ϖ' all odd prime common factors of a and b.

For the particular case of the form $n^2 + 1$, this formula becomes

$$P(x) \sim \tfrac{1}{2} C \operatorname{Li}(x^{\frac{1}{2}}),$$

where

$$C = \prod_{\varpi \geqslant 3}^{\infty} \left(1 - \left(\frac{-1}{\varpi}\right)\frac{1}{\varpi - 1}\right) = \prod_{p} \left(1 - \frac{1}{p - 1}\right)\prod_{q}\left(1 + \frac{1}{q - 1}\right),$$

and p and q represent all primes of the forms $4n + 1$ and $4n - 1$ respectively.

In this case a more thorough verification is possible than in any other case, because Lt.-Col. A. Cunningham, R.E. has computed a table of all primes of the form $n^2 + 1$ up to $n = 15,000$[†], and has at my request kindly made the enumerations of such primes given in the table at the end of this note.

The infinite product C is convergent but, in the form given above, it converges too slowly to be convenient for calculation. I have calculated C by the use of a transformation suggested by Mr Littlewood, which reduces to

$$7C = 1024\pi^{-4}a_3 b_2 d_4^{-1} d_6^{-1} (\prod \phi(p))^{-1},$$

where

$$a_3^{-1} = \zeta(3), \quad b_2 = 1 - 3^{-2} + 5^{-2} - \ldots,$$
$$d_4 = \prod(1 - q^{-4}), \quad d_6 = \prod(1 - q^{-6}),$$

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[*] G. H. Hardy and J. E. Littlewood, "Some problems of 'partitio numerorum,' III: On the expression of a number as a sum of primes," *Acta Mathematica*, vol. XLIV. (to appear shortly).

[†] *Messenger of Mathematics*, vol. XXXVI. p. 152; and A. Cunningham, *Binomial Factorisations*, vol. I., in print, but not yet published.

and
$$\phi(p) = \frac{(p-1)^3(p+1)(p^3-1)^2}{p^9(p-2)}.$$

As a_3 and b_2 are known, the evaluation of C depends only upon d_4, d_6, and $\Pi\phi(p)$, all of which converge rapidly. In fact, it is easily proved that

$$1 + 3p^{-4} < \phi(p) < 1 + (3\cdot3)\,p^{-4},$$

when $p \geqslant 29$. I find that $\frac{1}{2}C = 0\cdot68641$.

The table given below shews the total number of primes of the form $n^2 + 1$ up to various limits, chosen because values of $\mathrm{Li}(x)$ were known for them, and finally up to $2\cdot25 \times 10^8$, which is the limit of Cunningham's table. It will be seen from the last column of the table that the ratio of the predicted to the actual number shews a satisfactory tendency to converge towards 1, that there are three minima within the range of this table, and that the magnitude of the oscillations of the ratio tends to diminish as the variable y increases.

If Cunningham's table had not existed, this table would have had to end at about $y = 8$, which corresponds to the end of the published factor-tables, and one would then have been tempted to draw the incorrect inference that the ratio tends steadily towards 1.

The number of primes of form $n^2 + 1$ less than e^{2y} and comparison with Hardy and Littlewood's conjectured formula.

y	e^y (integer next below)	$\mathrm{Li}(e^y)$	The formula $\frac{1}{2}C\,\mathrm{Li}(e^y)$	Actual number of such primes less than e^{2y} (counted by A. Cunningham)	Ratio of predicted to actual number
7·0	1096	191·50	131·45	121	1·086
7·2	1339	225·69	154·92	143	1·083
7·4	1635	266·30	182·79	173	1·051
7·6	1998	314·57	215·92	208	1·038
7·8	2440	372·01	255·35	248	1·030
8·0	2980	440·38	302·28	301	1·004
8·2	3640	521·83	358·19	351	1·020
8·4	4447	618·92	424·83	425	1·000
8·6	5431	734·71	504·31	508	0·993
8·8	6634	872·89	599·16	603	0·994
9·0	8103	1037·88	712·41	706	1·009
9·2	9897	1234·96	847·69	833	1·018
9·4	12088	1470·51	1009·37	993	1·016
9·6	14764	1752·14	1202·68	1186	1·014
9·6158	15000	1776·62	1219·49	1199	1·017

On some new and rare Jurassic Plants from Yorkshire. V : fertile specimens of Dictyophyllum rugosum *L. and H.* By H. HAMSHAW THOMAS.

(Plate I.)

[*Read* 6 February 1922.]

The genus *Dictyophyllum* was founded by Lindley and Hutton for certain pinnatifid leaves from the Yorkshire Oolite which had a somewhat characteristic nervation and were named *Dictyophyllum rugosum**. The original authors regarded it as a doubtful Dicotyledon, but it became clear by comparison with other forms subsequently discovered that it should be considered as a fern. Some years later attention was drawn by Prof. Seward† and others to the similarity which existed between these leaves and the fronds of the modern *Dipteris*, a view which has been widely discussed. In 1841 Göppert ‡ described the sporangia of a somewhat similar fern which he named *Thaumatopteris Munsteri*, but which was subsequently transferred to the genus *Dictyophyllum* by Nathorst. Later Schenk§ described fertile fronds of another species, *Dictyophyllum obtusilobum*, from Franconia, and Nathorst‖ made a careful study of the sporangia of *Dictyophyllum exile* (Brauns), discovered in the Rhaetic beds of Skania. Zeiller found two species bearing sporangia among the plants which he described from Tonkin in 1903¶. Recently Prof. Halle** of Stockholm has made a careful re-investigation of the sporangia of *D. exile* and of fertile specimens belonging to the allied genera *Hausmannia* and *Thaumatopteris*, using modern methods of study. No sporangia have, however, been yet described for the original English species.

While most recent authors have agreed that *Dictyophyllum* may be regarded as allied to the modern Dipterids, they have differed as to the closeness of this relationship, and meanwhile the study of the phylogeny of the ferns as indicated by the investigation of living forms has made great progress, mainly by the work of Prof. Bower. He has investigated many forms †† and finds that among

   * *Fossil Flora*, II. Pl. CIV, 1833.
   † *Jurassic Flora of Yorkshire*, p. 123, 1900.
   ‡ *Die Gattungen der fossilen Pflanzen*, Lief. 1 and 2, p. 1, Tab. I–III, 1841.
   § *Die foss. Flor. des Keupers und Lias Frankens*, p. 77, 1867.
   ‖ *Svenska Vet. Akad. Handl.* Bd. 41, No. 5, p. 12, 1906.
   ¶ *Flor. Foss. des Gîtes de Charbon du Tonkin*, pp. 102, 114, Pl. XXI, Fig. 1*b*, Pl. XXVI, Fig. 3, 1903.
   ** *The Sporangia of some Mesozoic Ferns.* Arkiv för Botanik, Bd. 17, No. 1, 1921.
   †† *Annals of Botany*, XXIX. p. 495, 1915 and XXXI. p. 1, 1917.

modern ferns other types, such as *Platycerium, Leptochilus tricuspis* and *Cheiropleuria,* must be regarded as derivatives of the Dipterid stock, though their sporangia are not arranged in distinct sori but distributed on the fertile fronds in the 'Acrostichoid' manner. A series of forms has been described which illustrate progressive advance from circumscribed sori to an Acrostichoid spread of the sporangia over the whole leaf-surface, which are of great phylogenetic interest. He has thus introduced the conception of a wider alliance, termed the Dipteroideae, and has brought into it some forms which have hitherto been included in that unwieldy and probably unnatural group, the Polypodiaceae.

It is consequently of some interest to consider the extent to which the Mesozoic Dipterids agree with this wider conception and to look for early indications of the spread of the soral areas to give the Acrostichoid type. The discovery of fertile specimens of *Dictyophyllum rugosum* throws some light on these questions.

Occurrence. While collecting from the plant-bearing beds of the Yorkshire coast in 1912 and 1913, I obtained a few fragments of pinnules with a characteristic nervation bearing fairly well-preserved sporangia. The best specimens (see Figs. 2 and 4) were obtained from the bed at Cloughton Wyke, which occurs on the shore between tide-marks and contains singularly well-preserved examples of Ptilophyllum fronds, strobili of Williamsonias, *Solenites* (*Czekanowskia*) and a few other forms. Other specimens were obtained in the Gristhorpe plant bed to the south of Scarborough* (Figs. 1 and 3). It is probable that in both cases the plants are of the same age, belonging to the Middle Estuarine Series of the Lower Oolite. They were preserved in fine-grained shale which has formed moulds of the fertile surface, showing clearly the positions of the sporangia after the carbonaceous remains of the actual plant tissue have been removed (see Figs. 1 and 3).

All the specimens are small, the largest being 3 cms. long and just over 1 cm. broad. They appear to have been parts of frond segments similar to that figured by Prof. Seward†. They can be readily recognised by their shape and by their characteristic venation. Fortunately, only one species with this type of venation is known to occur in these Yorkshire beds, so that the attribution of these examples to *D. rugosum* is fairly certain.

Description. In spite of the conversion of the plant tissue into black coaly material, the specimens retain clearly their chief external characters. As collected, the upper side of the frond is usually seen, the lower side with its sporangia adhering more firmly to the matrix (Fig. 2). In one case the fertile side was seen (Fig. 4)

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* While this paper was in the press I was fortunate in discovering about ten additional specimens in Gristhorpe Bay. Some of them are beautifully preserved and show the sporangia very clearly. They confirm the observations previously made.

† *Jurassic Flora of Yorkshire,* Pl. XIII, Fig. 3, 1900.

and it is also possible to detach from the rock (by means of plasticine) portions of the coaly leaf-material with the sporangia.

From such examples as well as from the moulds left in the matrix it is seen that the whole of the surface of the lamina was covered with sporangia, only the midrib and some of the larger veins being free. The sporangia do not appear to be grouped into sori but scattered without trace of regular arrangement. The margin of the lamina seems to have been slightly reflexed. As far as can be ascertained the sporangia were all of the same size with a fully developed annulus, and when some groups were removed from the rock and treated chemically the spores in each case appeared to be in the same stage of development. Thus it seems probable, though not absolutely certain, that *Dictyophyllum* should be classed with the Simplices. I have been able to examine a specimen of *D. exile* from Bjuf in Sweden, sent by Nathorst some time ago, and here also the sporangia, which are large and are clearly seen, appear to have arisen simultaneously in the sori (Fig. 5).

The individual sporangium was probably more or less lenticular in shape. On the frond they are seen in various positions, but are seldom observed lying flat with the annulus in the horizontal plane; this may be an indication that they originally possessed a short massive stalk. They were small, and though it is difficult to make accurate measurements, the diameter of the annulus was between ·35 mm. and ·5 mm. The spore masses can be measured more accurately and are ·25–·28 mm. Thus the sporangia are about half the size of those seen in the Cambridge specimen of *D. exile*. A part of the annulus is always clearly seen, but in no case can we determine absolutely whether it was complete or not, nor can we count the number of the cells composing it*. There can, however, be no doubt that it was oblique. We get indications of this by examination of the specimens with a binocular microscope and better by the treatment of sporangia with chromic acid, which gradually dissolves the black substance, leaving, however, parts of the indurated cells of the annulus; the cells of the annulus are then seen to lie slightly to one side of the lenticular spore mass. These preparations show the remains of 8 to 10 cells of the annulus extending over rather less than half the diameter of the spore mass. From the most complete examples, it would appear that near the stalk of the sporangium the cells of the annulus were either wanting or incompletely indurated.

It is somewhat difficult to remove single sporangia from the specimens, but if a small group is detached from the matrix and treated with a warm solution of chromic acid for some hours, the

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* A few sporangia in the newly discovered specimens are lying almost flat. The annulus then appears to be complete and oblique, its cells are not so distinct near the short thick stalk, but it is safe to say that their number was between 20 and 24.

individual sporangia become separated and their walls begin to disappear, the spores still remaining firmly united together. When this stage is reached the separated groups of spores may be treated for a short time with Schultz's macerating fluid followed by dilute ammonia or ammonium carbonate solution; the spores then separate from one another and can be counted. The counts gave the following as the numbers of spores per sporangium: 99, 121, 121, 92 (in the last count a few spores were known to have been lost in manipulation). From this it would appear that the original typical number was 128, the nearest power of 2.

The individual spores have a somewhat tetrahedral shape and show the usual triradiate marking, indicating that they originated in tetrads. They are about ·036 mm. in size.

I have been unable to detect any definite relation between the arrangement of the sporangia and the venation. I have, however, observed in one specimen that the sporangia do not entirely cover the surface but still show no tendency to be grouped in circular sori. In one place a line of sporangia appears to extend between two larger veins, possibly an elongated sorus. In another case there are indications of fine veins above the fertile areas, but I have not been able to confirm the presence of additional veins.

Comparison with other forms. When we compare the structures described above with the fertile specimens previously known as species of *Dictyophyllum*, we may at once notice several important points of difference. The form *D. exile* (Brauns) has been very carefully studied by Halle, and I have had a specimen before me for comparison (Fig. 5). Here the sori, though closely approximated, are quite distinct and contain 4–7 sporangia, whereas *D. rugosum* shows no distinct sori but an Acrostichoid spread of the sporangia over the whole surface from margin to margin. The individual sporangia in the Rhaetic species are double the size of the Yorkshire examples and they generally contained four times as many spores. Our knowledge of the frond form of *D. rugosum* is not yet complete, and, of course, we have no information as to the internal structure of either of these types, but the important differences above noted clearly call for the generic separation of the species *exile* and *rugosum*. The other species, such as *Munsteri* of Göppert and *Nathorsti* of Zeiller, which show general agreement with *exile*, must also be separated from the Yorkshire type.

Since the genus *Dictyophyllum* was instituted for the Yorkshire species, it must be retained for this type and another generic name must be found for the Continental species. It will probably be necessary to revive Göppert's name of *Thaumatopteris*, notwithstanding the fact that it has been used in a restricted sense by Nathorst. I fear that the changes in nomenclature involved will be somewhat numerous and it is impossible to consider them in detail here.

Nathorst used the name *Thaumatopteris* for a species from the
Rhaetic of Skania, *T. Schenki*. This has also been found in a fertile
condition*, possessing very distinct sori of 8–10 sporangia, each
containing typically 128 spores and having a diameter of ·2–·3 mm.
This form resembles *D. rugosum* in the size of its sporangia but
differs in their arrangement. If the differences in frond form and
in spore output are to be considered as sufficient to separate
T. Schenki and *D. exile* generically, then a new generic name will
have to be found for the former.

Another fossil Dipterid genus is *Hausmannia*. We are again
indebted to Prof. Halle† for our knowledge of its sporangia. He
has found that they cover the whole of the lower surface of the
fertile frond, as in the examples described above. They were,
however, small (·18–·24 mm. in diameter), producing typically 64
spores but sometimes 128. Half the annulus showed 10–12 cells.
This type is then closely allied to *D. rugosum*, though it is generically
distinct in the size, texture and venation of its fronds.

Fertile specimens of the allied genus *Clathropteris* were described
by Schenk‡. They seem to have had very distinct sori of 7–9
spherical sporangia.

The sporangia of the genus *Camptopteris* have been described
by Nathorst for *C. spiralis*§, and from his figures we may judge
that this form also was non-soral. The sporangia are stated to be
large, 0·44–0·5 mm. in diameter, but their spore content is not
given. Nathorst states that they cover the entire surface of the
frond as in *Dictyophyllum*, but this does not indicate whether they
were grouped in sori or not, for in his species of *Dictyophyllum* there
were distinct sori. Further information on this genus is desirable.

The relationship of Dictyophyllum to living ferns. In comparing
this group of Jurassic ferns with their possible modern relatives,
we must no longer limit our observations to *Dipteris* but must
notice also those newly recognised members of the Dipterid alliance
Cheiropleuria and *Platycerium*.

In the general features of frond construction and nervation, a
comparison is possible between the Jurassic forms and the three
genera just mentioned. The indications which we noted of an
occasional elongated sorus and the possible presence of fine veins
additional to the series normally seen in sterile fronds, are of interest
in connection with the fertile leaves of *Platycerium* described by
Prof. Bower.

In the spread of the sporangia over the surface of the frond,
D. rugosum resembles *Platycerium* and *Cheiropleuria* more closely
than it does *Dipteris*, but we cannot of course determine whether,

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* Halle, *The Sporangia of some Mesozoic Ferns, ibid.* p. 22.
† *Ibid.* p. 19.
‡ Schenk, *ibid.* p. 83, Taf. XVI, Fig. 9 *b*, XVII, Fig. 3.
§ *Svenska Vet. Akad. Handl.* Bd. 41, No. 5, p. 17, Taf. VII, Figs. 12–14.

as in *Platycerium*, the distribution of the sporangia is really due to the elongation and crowding of the sori, or to their becoming confluent; in both cases a similar outward appearance is produced.

Most, if not all, of the species referred to *Dictyophyllum* should be classed with the Simplices, while it has generally been thought that *Dipteris*, together with *Cheiropleuria* and *Platycerium*, showed typical mixed sori. However, Prof. Bower* reports that in *Dipteris Lobbiana* the sporangia arise simultaneously as in the Yorkshire form and thus we have a further point of resemblance rather than of difference.

In respect of the size of the sporangia and spore output *Dictyophyllum rugosum* comes nearer to *Cheiropleuria* than to *Dipteris* or *Platycerium*. Owing to the kindness of Prof. Bower, who has sent me material from his collection, I have been able to examine the sporangia of *Cheiropleuria bicuspis, Dipteris conjugata (Horsfieldii), D. Lobbiana* and also *Leptochilus tricuspis*, a form which has also been shown by Bower to be referable to the Dipterid alliance. In each case it has been possible to treat the more or less mature sporangia in the same way as the fossil sporangia and to count the number of spores contained in them for comparison with the spore output of the fossil forms. The actual number of spores seen may have been less than the theoretical and probable number owing to the failure of all the spores to reach maturity, to low visibility after treatment, or to mistakes in the counting, but the typical number is clear in each case.

The typical spore outputs are as follows, the actual numbers recorded being given in brackets:

Cheiropleuria bicuspis (Bl.)	128	(124, 123, 108)
Dipteris conjugata Reinw.	64	(62, 60, 61)
Dipteris Lobbiana (Hook.)	64	(52, 50, 47)†
Leptochilus tricuspis (Hook.)	64	(63, 61, 60)
Platycerium alcicorne (Sw.)	64	(62, 60)

As might be expected from the fact that *Cheiropleuria* is nearest to *Dictyophyllum rugosum* in spore output, it is also most similar in the size of the sporangia which are about 0·37 mm. in diameter, while those of *Dipteris conjugata*, as has been previously recorded, are about 0·2 mm. If we may judge from Prof. Bower's figures, the sporangia of *D. Lobbiana* may sometimes be rather larger than 0·2 mm. though the specimens which I examined, and which were somewhat immature, were smaller than those of *D. conjugata*.

It should be noticed that among the recent species examined only *Cheiropleuria* has spores which develop in a tetrahedral manner as in the fossil forms.

* *Annals of Botany*, xxix. p. 514, 1915.
† These spores were not quite mature and their visibility was low.

Prof. Halle's examination of *Thaumatopteris* (*Dictyophyllum*) *exile* led him to the conclusion that the size of its sporangia, the number of spores and the greater extension of the annulus precluded the actual classification of *Dictyophyllum* under either the Dipteridineae or the Matoniaceae, a view previously held by Nathorst on less complete evidence. The present work, however, seems to strengthen considerably the suggestion made by Prof. Seward and others, that not only *Dictyophyllum* but other related Mesozoic forms should be considered as closely allied to the modern Dipterid ferns. From the consideration of the fertile fronds of this wider alliance we obtain strong confirmation of Prof. Bower's phyletic conclusions. One of the oldest known forms, which we must now call *Thaumatopteris* (*Dictyophyllum*) *exile* (Brauns), had sori of large sporangia which may be compared to the Gleicheniaceous type. The Rhaetic *Thaumatopteris Schenki* Nath. had distinct sori of smaller and more numerous sporangia. The Jurassic species *Dictyophyllum rugosum* and probably the genus *Hausmannia* had small sporangia which had apparently spread over the whole surface of the frond and the soral arrangement seems to have been lost, though the sporangia still appear to have arisen simultaneously. Among the modern forms the mixed arrangement of sporangia has come in generally, and while *Dipteris* retains distinct sori, *Platycerium* and *Cheiropleuria* show the Acrostichoid condition.

In so far as we can judge from the sporangial characters and the nervation, it seems probable that the modern *Cheiropleuria bicuspis* is the nearest living relative to the Jurassic *Dictyophyllum rugosum* described in this paper.

DESCRIPTION OF PLATE I.

Photographs by the author.

Dictyophyllum rugosum L. and H.

Fig. 1. Impression of part of a fertile pinna from Gristhorpe Point, showing the non-soral spread of the sporangia over the whole surface. × 5.

Fig. 2. Part of a fertile pinna from Cloughton Wyke. Some of the original leaf substance remains with a well-preserved upper surface. Where this has been removed the moulds of the sporangia are visible. × 4.

Fig. 3. Moulds of sporangia from Gristhorpe Point. × 8·5.

Fig. 4. Photograph of a group of sporangia from Cloughton Wyke. Most of these sporangia show an annulus which is not clearly shown in the figure. There are no indications of a soral arrangement. × 12.

Thaumatopteris (*Dictyophyllum*) *exile* (Brauns).

Fig. 5. Part of a fertile pinna from the Rhaetic of Bjuf (Sweden) showing the distinct soral arrangement of the sporangia, which are much larger than in *Dictyophyllum rugosum*. × 4.

Fig. 1
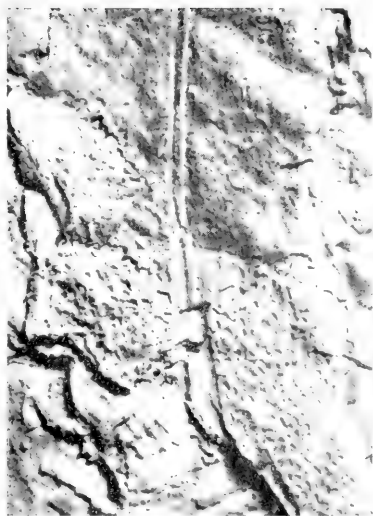
Fig. 2
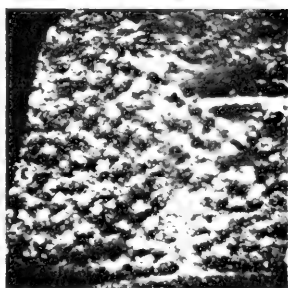
Fig. 3
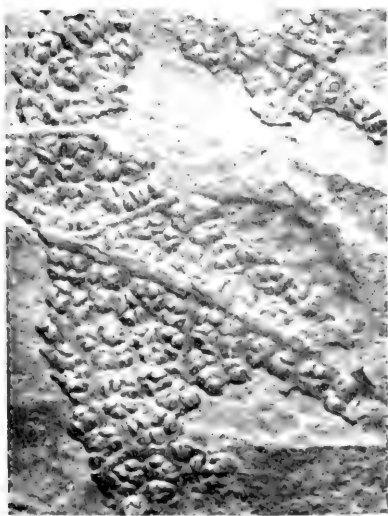
Fig. 4

Fig. 5
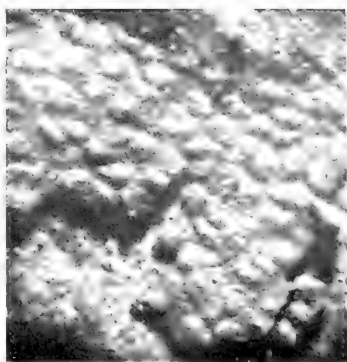
Dictyophyllum and Thaumatopteris.

A Proof of the Impossibility of the Coexistence of two Mathieu Functions. By E. L. INCE, M.A., Trinity College.

§ 1. *The four types of Mathieu functions.*

Mathieu functions* are defined as solutions of Mathieu's equation

$$\frac{d^2y}{dx^2} + (a + 16q \cos 2x)\, y = 0 \qquad \ldots\ldots(1\cdot1)$$

which admit the period 2π (or in certain cases, π). They are of four types:

$$C_c = \sum_{r=0}^{\infty} a_r \cos 2rx,$$

$$C_o = \sum_{r=0}^{\infty} a_r \cos (2r+1)\, x,$$

$$S_o = \sum_{r=0}^{\infty} a_r \sin (2r+1)\, x,$$

$$S_e = \sum_{r=0}^{\infty} a_r \sin (2r+2)\, x,$$

corresponding to Whittaker's notation $ce_{2n}(x)$, $ce_{2n+1}(x)$, $se_{2n+1}(x)$, $se_{2n}(x)$ respectively. The equation admits such solutions when, and only when, a and q are related in a definite manner indicated by the vanishing of an infinite determinant.

The second solutions which correspond to the Mathieu functions have been investigated; they are in general not periodic, but it has been suggested by several writers that for particular numerical values of q, the second solution might be periodic and therefore fall within the above classification. The supposed second periodic solution corresponding to C_c would be of type S_c and *vice versa*, and that corresponding to C_o would be of type S_o and *vice versa*.

Thus the second solution which corresponds to the Mathieu function

$$ce_0(x) = 1 + 4q \cos 2x + 2q^2 \cos 4x + q^3 \left(\tfrac{4}{9} \cos 6x - 28 \cos 2x\right) + \ldots$$

with $\qquad\qquad a = -32q^2 + 224q^4 - \ldots$

is

$$x\, ce_0(x) - 4q \sin 2x - 3q^2 \sin 4x + q^3 \left(-\tfrac{22}{27} \sin 6x + 54 \sin 2x\right) \quad \ldots$$

and cannot be periodic. Similarly, the second solution which corresponds to

$$ce_1(x) = \cos x + q \cos 3x + q^2 \left(\tfrac{1}{3} \cos 5x - \cos x\right) - \ldots$$

with $\qquad a = 1 - 8q - 8q^2 + 8q^3 - \tfrac{8}{3}q^4 - \ldots$

* Whittaker and Watson, *Modern Analysis*, chap. XIX.

is

$$- 8q \left(1 - 3q^2 + 6q^3 + \tfrac{31}{9}q^4 + \dots \right) x\, ce_1(x)$$
$$+ \sin x + q \sin 3x + q^2 \left(\tfrac{1}{3}\sin 5x + 5 \sin 3x\right) + \dots.$$

This would be periodic if q were a root of the transcendental equation

$$1 - 3q^2 + 6q^3 + \tfrac{31}{9}q^4 + \dots = 0,$$

assuming such a root to exist.

The purpose of this paper is to show that these second periodic solutions, expressible as convergent trigonometrical series, cannot exist apart from the trivial case $q = 0$, which will be excluded in all that follows.

§ 2. *The case of C_o and S_o.*

The two cases may be treated together, for if $C_o(x, q)$ be a Mathieu function $C_o(\tfrac{1}{2}\pi - x, -q)$ will be a Mathieu function of type $S_o(x, q)$.

By direct substitution of C_o in Mathieu's equation, the following relations between the coefficients are obtained

$$(a - 1 + 8q)\, a_0 + 8qa_1 = 0 \qquad \dots\dots(2{\cdot}1),$$
$$[(2n + 1)^2 - a]\, a_n = 8q\, (a_{n+1} + a_{n-1}) \qquad \dots\dots(2{\cdot}2).$$

These relations determine the coefficients, and show that all the coefficients remain finite except, possibly, when a is the square of an odd integer. The solution then reduces to the trivial case corresponding to $q = 0$, viz. $\cos (2r + 1) x$ where $a = (2r + 1)^2$.

Let a second periodic solution be supposed to exist for the same value of a, and let it be

$$S_o = \sum_{r=0}^{\infty} a_r{}' \sin (2r + 1) x.$$

The coefficients are found to be determined by the relations

$$(a - 1 - 8q)\, a_0{}' + 8qa_1{}' = 0 \qquad \dots\dots(2{\cdot}3),$$
$$[(2n + 1)^2 - a]\, a_n{}' = 8q\, (a'_{n+1} + a'_{n-1}) \qquad \dots\dots(2{\cdot}4).$$

From $(2{\cdot}1)$ and $(2{\cdot}3)$ is found

$$\begin{vmatrix} a_0 & a_1 \\ a_0{}' & a_1{}' \end{vmatrix} = 2a_0 a_0{}'$$

and from $(2{\cdot}2)$ and $(2{\cdot}4)$

$$a_n\, (a'_{n+1} + a'_{n-1}) = a_n{}'\, (a_{n+1} + a_{n-1})$$

or

$$\begin{vmatrix} a_n & a_{n+1} \\ a_n{}' & a'_{n+1} \end{vmatrix} = \begin{vmatrix} a_{n-1} & a_n \\ a'_{n-1} & a_n{}' \end{vmatrix}$$

whence, for all values of n,

$$\begin{vmatrix} a_n & a_{n+1} \\ a_n{}' & a'_{n+1} \end{vmatrix} = 2a_0 a_0{}' \qquad \dots\dots(2{\cdot}5).$$

The equations (2·1) to (2·4) show that a_0 and a_0' cannot reduce to zero except in the excluded case where $q = 0$ and a is the square of an odd integer. Hence the determinant in (2·5) must preserve a constant non-zero value for all values of n. But the convergence of the series in C_o renders it necessary that

$$a_n \to 0 \text{ as } n \to \infty,$$

leading to a contradiction. The non-existence of a second periodic solution is thus established.

§ 3. *The case of C_e and S_e.*

The presence of a constant term in the series for C_c renders it *a priori* almost evident that no periodic second solution can exist. Nevertheless the outlines of a proof similar to that of the previous section will be given.

Let
$$C_e = \sum_{r=0}^{\infty} a_r \cos 2rx$$

and
$$S_e = \sum_{r=1}^{\infty} a_r' \sin 2rx,$$

satisfy Mathieu's equation simultaneously, then

$$aa_0 + 8qa_1 = 0 \qquad \qquad ......(3·1),$$
$$(4n^2 - a)\, a_n = 8q\,(a_{n+1} + a_{n-1}) \qquad ......(3·2),$$
$$(a - 4)\, a_1' + 8qa_2' = 0 \qquad ......(3·3),$$
$$(4n^2 - a)\, a_n' = 8q\,(a'_{n+1} + a'_{n-1}) \qquad ......(3·4),$$

and the trivial case where a is the square of an even integer is excluded. From these relations it follows that

$$\begin{vmatrix} a_n & a_{n+1} \\ a_n' & a'_{n+1} \end{vmatrix} = \begin{vmatrix} a_{n-1} & a_n \\ a'_{n-1} & a_n' \end{vmatrix} = ... = \begin{vmatrix} a_1 & a_2 \\ a_1' & a_2' \end{vmatrix} = 2a_0a_1'.$$

Both a_0 and a_1' are non-zero, apart from the trivial case, which leads to the same contradiction as before.

§ 4. *Functions whose period is 4π.*

It was shown recently by Mr E. G. C. Poole[*] that Mathieu's equation may admit, for the same value of a, two solutions having the period 4π. They are not Mathieu functions in the accepted sense of the term since they appear as series of sines or cosines of odd multiples of the half argument. It is, however, interesting to investigate this case in the light of the preceding paragraphs.

Let
$$C = \sum_{n=0}^{\infty} a_n \cos(n + \tfrac{1}{2})\, x,$$

$$S = \sum_{n=0}^{\infty} a_n' \sin(n + \tfrac{1}{2})\, x$$

[*] *Proc. Lond. Math. Soc.* (2) 20 (1922), p. 378.

be supposed to satisfy Mathieu's equation for the same value of a, then

$$[(n + \tfrac{1}{2})^2 - a] \, a_n = 8q \, (a_{n+2} + a_{n-2}) \quad n = 0, 1, 2 \ldots \quad (4\cdot1)$$

with the proviso that

$$a_{-2} = a_1, \quad a_{-1} = a_0$$

and

$$[(n + \tfrac{1}{2})^2 - a] \, a_n' = 8q \, (a'_{n+2} + a'_{n-2}) \quad n = 0, 1, 2 \ldots \quad (4\cdot2)$$

with
$$a_{-2}' = -a_1', \quad a_{-1}' = -a_0'.$$

Hence

$$a_1 a_3' - a_1' a_3 = a_0 a_1' + a_0' a_1,$$
$$a_0 a_2' - a_0' a_2 = a_0 a_1' + a_0' a_1.$$

From $(4\cdot1)$ and $(4\cdot2)$

$$\begin{vmatrix} a_n & a_{n+2} \\ a_n' & a'_{n+2} \end{vmatrix} = \begin{vmatrix} a_{n-2} & a_n \\ a'_{n-2} & a_n' \end{vmatrix} = \ldots = a_0 a_1' + a_0' a_1 \quad (4\cdot3)$$

whether n be even or odd.

Now the first of the equations in $(4\cdot1)$, viz. that where $n = 0$, holds between the three undetermined constants a_0, a_1, a_2, and each subsequent equation introduces a further constant from the set a_n. Hence a_0 and a_1, and for a similar reason a_0' and a_1', may be regarded as arbitrary. Let them therefore be chosen in such manner that

$$a_0 a_1' + a_0' a_1 = 0.$$

It is the possibility of this choice that renders possible the simultaneous existence of two periodic solutions, for now the conditions

$$a_n \to 0, \ a_n' \to 0 \ \text{as} \ n \to \infty$$

are not violated.

The equations in $(4\cdot3)$ show that

$$\frac{a'_{n+2}}{a_{n+2}} = \frac{a_n'}{a_n}, \quad \frac{a_1'}{a_1} = -\frac{a_0'}{a_0}.$$

Let $a_0' = a_0$, then $a_n' = (-1)^n a_n$. Hence

$$C(\pi - x) = S(x).$$

The existence of periodic simultaneous solutions of period $2s\pi$ may be demonstrated in the same manner.

The interpretation of β-ray and γ-ray spectra. By C. D. Ellis.

[*Read* 1 May 1922.]

The typical β-ray disintegration involves the emission of both β-rays and γ-rays. The statistical result of the β-ray emission of a large number of disintegrating atoms is very complicated since the energies of the emitted β-particles vary within wide limits. The work of Rutherford* and Chadwick† showed that the emission should really be considered as consisting of two parts. The first is a continuous distribution in which the number of electrons ejected with any assigned energy varies continuously with this energy. This may be analysed by a magnetic field into a general spectrum and will be denoted by this name. The other part consists of a series of groups of electrons, the electrons in any one group having all the same energy. These groups are termed collectively the magnetic line spectrum from the type of photograph obtained when this part of the emission is analysed by a magnetic field.

Rutherford‡ pointed out that this line spectrum was almost certainly due to the ejection of electrons from the radioactive atoms by the γ-rays which are also emitted in the disintegration, the energy of the β-rays being connected with the frequency of the γ-rays by the quantum relation. Later experiments§ on the magnetic spectra of the electrons ejected from various metals by the γ-rays have shown this to be undoubtedly the correct explanation. In general, the radioactive element emits a number of monochromatic γ-rays which may be converted in any of the electronic levels of the radioactive atom itself. Each γ-ray will give rise to a series of groups of energy $h\nu - A$, where A is the absorption energy of the K, L, M, etc. levels. In this way it has been possible to account for the observed β-ray line spectra of several bodies and, what is most important, to deduce the frequencies of the γ-rays from these spectra.

It is clear then that the primary results of the β-ray disintegration are the emission of monochromatic γ-rays and the general spectrum of β-rays. It is quite certain that these last electrons, which may be called the disintegration electrons, come from the nucleus and there is very strong evidence that the γ-rays also have their origin there. What is not clear, however, is the connection

* *Phil. Mag.* 26, p. 717 (1913).
† *Verh. der Deut. Physik. Gesell.* xvi. No. 8, p. 383 (1914).
‡ *Phil. Mag.* 34, p. 153 (1917).
§ Ellis, *Proc. Roy. Soc.* A, 99, p. 261 (1921).

between the general β-ray spectrum and the γ-rays. For instance, there is not sufficient evidence to decide whether the disintegration electron is emitted first and stimulates the γ-rays or whether conversely the γ-ray comes first and subsequently the disintegration electron is emitted. This is a very difficult point and has been treated by the author from one point of view in another place*. It was there pointed out that the evidence from the γ-ray spectra showed that it was simpler to assume the γ-ray came first but did not prove it. The great difficulty in this view is in accounting for the wide range of energy of the disintegration electrons. I have shown that it is probable that the quantum dynamics applies to the nucleus and that the various constituent particles occupy some of a series of positions of definite energy. Any disintegration would therefore at first sight appear to give an electron with a definite characteristic energy. The alternative view of the γ-ray emission being after the disintegration would give a possibility of a varying loss of energy in stimulating the γ-ray.

It would seem reasonable, however, provisionally to assume that, if there is any γ-ray, it immediately precedes the disintegration. The actual emission of α- and β-particles would then be essentially the same process and the fact that α-particles appear with definite energies and β-particles with energies varying over a wide range would be referred to the difference in the structure of the α-particle and the electron.

In whichever way this particular point may be settled the general picture of a β-ray disintegration is clear. Supposing the γ-ray emission is the primary phenomenon the process would be as follows. One of the electrons in the nucleus must be considered as a result of the previous disintegration to occupy one of the higher energy stationary states out of the series of such states that it can occupy. After an interval depending only on the conditions inside the nucleus this electron will make transitions to lower energy states, emitting γ-rays in the process. Some of these γ-rays are absorbed in the electronic structure of the same atom and eject electrons with characteristic energies. These electrons form the β-ray line spectrum. Finally, the electron arrives in a stationary state in which it is not permanently stable and it flies out from the nucleus. The kinetic energy of this electron must be considered to depend on other factors besides those of the stationary state, and the variable kinetic energy is possibly connected with the two facts that the nuclear field must vary considerably in distances comparable with the diameter of the electron and that the electron cannot be considered as rigid under these conditions.

The β-ray spectra of radium B and C and thorium D have already been treated from this point of view. Some measurements

of the β-ray spectra of thorium B, radium D and the lower energy part of radium B will be given here because in these cases the results are slightly peculiar and have led Frl. Meitner to propose quite a different theory of β-ray disintegration. The peculiarity of these spectra lies wholly in the fact that they are mainly due to one intense γ-ray and not to several less intense ones. In fact, the γ-rays are sufficiently strong to enable the β-ray groups from the M and N levels to be measured.

The β-rays of all these bodies have been measured before, but the experiments have been repeated and some new lines have been found and some of the old values altered slightly. All the measurements refer to Rutherford and Robinson's determination of the β-rays of radium B as standard and were carried out in the same way.

The β-rays of ThB.

The β-ray groups of thorium B are given in Table I. They were measured from an active source prepared by electrolysis of a thorium X solution. These measurements differ considerably from those of Frl. Meitner*, her values for the energies of groups numbers 2 and 3 being greater. It is possible that in her experiments the faster rays penetrated into the weaker regions of the magnetic field and so travelled in a larger circle. In addition, three new lines (Nos. 1, 4, 6) are included. It is certain that these lines come from thorium B since, although during the experiment some thorium D was formed, only the strongest of the thorium D lines were visible and there are not even weak lines of these energies in its spectrum.

Column V refers to Table II and shows which γ-ray gives the β-ray lines. Column VI shows the origin of the line and column VII the energy calculated from the absorption voltages given in Table III.

TABLE I. *β-ray groups of ThB.*

I No.	II $H\rho$	III Intens.	IV Energy obs. volts	V γ-ray	VI Origin	VII Energy calc.
1	2015	m.f.	$2\cdot813 \times 10^5$	2	L_3	2·80
2	1798	m.s.	2·327	1	M	2·32
3	1738	s.	2·197	1	L_3	2·20
4	1677	m.	2·066	2	K	2·07
5	1382	v.s.	1·475	1	K	1·47
6	1110	m.	0·993	...	...	...

* Meitner, *Zeit. f. Physik*, Bd. 9, p. 131 (1922).

TABLE II.			TABLE III.	
γ-rays of ThB.			*Absorption Voltages at No. 82.*	

No.	Energy in volts	γ in A.U.
1	$2\cdot36 \times 10^5$	0·0523
2	2·96	0·0417

K	$0\cdot891 \times 10^5$
L_3	0·158
L_2	0·152
L_1	0·130
M	0·03 to 0·04
N	about 0·004

It can be seen that excellent agreement is obtained on the assumption that two γ-rays acting on the levels shown produce these lines. It is certain therefore that ThB emits at least these two γ-rays. Some evidence of two lines giving a third γ-ray of energy $5\cdot05 \times 10^5$ volts was obtained, but there is some doubt whether it should be assigned to thorium B and it is not included. There is also one other line $H\rho$ 1913 m.f. about which there is a similar doubt.

The γ-ray No. 1 is exceptionally strong and it is due to this fact that it is possible to observe a line from the M level. The relative intensities of lines 5, 3 and 2 is exactly what one would anticipate from the general laws of absorption and lends support to the analysis.

Frl. Meitner has already shown that thorium B emits a γ-ray of wave-length 0·052 A.U. and that the line 5 is due to the conversion of this γ-ray in the K level. In other respects her account of the thorium B, β's and γ's, is very different from that given here.

The β-rays of Radium D.

These β-rays have been measured by Danysz*. I have been able to confirm his measurements and find one new line No. 3. There do not appear to be any more strong lines up to $3\cdot0 \times 10^5$ volts. The values are given in Table IV and the origin of the lines shown. The absorption voltages are those for a body of atomic No. 82 and were given in Table III.

All these lines appear to be due to one γ-ray acting on the $L, M, N \ldots$ levels. The agreement that is obtained is convincing proof of the correctness of the analysis. The first line, however, does not agree so well.

* *Le Radium*, Jan. 1913, p. 1.

TABLE IV. *β-ray groups RaD due to one γ-ray of energy*
0.467×10^5 *volts* $= \lambda$ 0.264 A.U.

No.	$H\rho$	Intens.	Energy obs.	Origin	Energy calc.
1	742	m.f.	0.466×10^5	N	0.463 approx.
2	717	m.s.	0.436	M	0.432 approx.
3	628	m.f.	0.338	L_1	0.337
4	605	m.	0.314	L_2	0.315
5	600	v.s.	0.309	L_3	0.309

It is interesting to note that all the L levels are represented, and that the line from the L_3 level is the strongest. The relative intensity of the groups from the L, M and N levels is in agreement with the X-ray evidence.

Radium D appears to emit only this one γ-ray of wave-length 0.264 A.U. As Frl. Meitner has already pointed out, the absorption measurements of Rutherford and Richardson confirm this conclusion strongly. Frl. Meitner deduced a value 0.26 A.U. for the wave-length of this γ-ray from slightly different measurements of three of these lines (Nos. 1, 2, 5) by considering No. 5 to come from the L_1 level. In view of the undoubted existence of lines 3 and 4 and the agreement that is shown in the above table it seems very probable that the analysis given here is correct.

Low Energy β-rays of RaB.

In a note to a previous paper[*] I mentioned that these lines were probably due to one γ-ray of wave-length about 0.229 A.U. The analysis is shown in Table V. The atomic number of radium B is 82, so the absorption voltages given in Table III apply here. These lines have been measured by Rutherford and Robinson[†]. Line No. 4 was not given, but its existence is quite certain. Danysz[‡] measured it and I have found it also. I obtained some evidence of a line from the L_1 level, but accurate measurements could not be made. Line No. 1 was measured by Rutherford and Robinson as $H\rho$ 798, I found slightly lower values about 794 by comparison with lines 2 and 4. The value given is the mean.

[*] *Proc. Roy. Soc.* A, vol. 101, p. 1 (1922).
[†] *Phil. Mag.* 26, p. 717 (1913).
[‡] *Loc. cit.*

TABLE V. *β-rays of RaB due to one γ-ray of energy*
0.533×10^5 *volts* $= \lambda\ 0.231$ A.U.

No.	$H\rho$	Intens.	Energy obs.	Origin	Energy calc.
1	796	m.s.	0.533×10^5	N	0.529 approx.
2	770	s.	0.500	M	0.498 approx.
3	668	m.s.	0.380	L_2	0.381
4	663	v.s.	0.375	L_3	0.375

It can be seen that the agreement is good and that again the relative intensities support the analysis.

It is very interesting to note that this γ-ray was actually measured by the crystal method by Rutherford and Andrade*, their value being 0.229 A.U. This is very strong evidence in favour of this view of the β-ray lines and of the applicability of the quantum theory. Owing to the difficulties of the experiment Rutherford and Andrade were unable to determine whether this γ-ray came from radium B or radium C. A hypothesis that I have proposed† about the existence of certain stationary states in the radium B nucleus, however, suggested that this γ-ray came from radium B. This direct evidence from the β-ray lines therefore supports this hypothesis about the radium B nucleus.

In the case of both radium B and radium D the β-ray line from the L_3 level is stronger than those from the L_2 and L_1 levels. Similar results are found for thorium D†. For γ-rays of this and higher frequencies it would appear that the L_3 group has a greater absorption than the L_2 or L_1. This justifies the use of the L_3 level in the analysis of the thorium B and similar spectra. At first sight this might seem to be in contradiction with the X-ray evidence, but the relative absorbing powers of the L_3, L_2 and L_1 levels is only found by X-ray absorption measurements for frequencies immediately adjoining the critical absorption frequencies. There is no evidence at all about the relative absorbing powers at higher frequencies except this evidence from the β-ray lines and this admits only of the interpretation I have given.

Discussion.

At the commencement of this paper I gave an account of the general view of a β-ray disintegration that has evolved from the researches of Rutherford and others on this problem. It has been possible to account for the details of five different β-ray spectra

* *Phil. Mag.* 28, p. 263 (1914). † *Loc. cit.*

(RaB, RaC, RaD, ThB, ThD) and so it would appear that there is very strong evidence for the essential correctness of the view.

In a recent paper*, however, Frl. Meitner has proposed a very interesting and completely different theory of β-ray disintegration. She supposes that the initial phenomenon in the disintegration is the emission of a β-particle from the nucleus with a definite characteristic energy. Its initial energy 'inside' the nucleus may be denoted by E_1. This disintegration electron may escape with no other loss of energy except that, 'w,' of separating itself from the nucleus. In this way the disintegration electrons will give a β-ray line of energy $E_1 - w = E_2$. Another process may happen, however, and that is, that this electron may convert part of its energy into γ-ray energy. The amount it converts is determined by a special assumption and is $E_1 - w$ giving a γ-ray of frequency

$$\nu = (E_1 - w)/h = E_2/h.$$

This leaves the electron 'inside' the nucleus with energy

$$E_1 - (E_1 - w) = w,$$

which is just sufficient to take it to the surface of the atom. The γ-ray will give secondary β-ray lines in the usual way due to conversion in the K, L, M ... levels. I shall not here discuss the arbitrary nature of the assumption determining the frequency of the γ-ray.

This theory predicts that the total emission should consist of one series of definite groups and of one γ-ray. The energies of these groups should be

$$E_1, \qquad h\nu - K_{\text{abs}}, \qquad h\nu - L_{\text{abs}} \ldots, \text{ etc.}$$

which in virtue of the assumption $E_1 = h\nu$ become

$$h\nu, \qquad h\nu - K_{\text{abs}} \ldots, \text{ etc.}$$

Of these the first group comes from the nucleus, the remainder are secondary and due to the conversion of the γ-ray.

Frl. Meitner claims that groups having energies related in this way actually exist. I shall return to this point. The line she associates with the disintegration electrons (energy $h\nu$) is always the weakest. This would mean that the disintegration electron rarely escapes with any energy, but usually excites γ-rays. Even allowing for the secondary electrons ejected by the γ-rays it is clear that the total number of electrons emitted would be less than the number of atoms disintegrating. This seems to be directly contrary to experiment, since Danysz and Duane† found that radium B + C emitted about three electrons for every pair of electrons disintegrating.

Again a most serious objection is that this theory gives no possibility of explaining the general spectrum of β-rays, in fact it appears to deny its existence. This general spectrum certainly does exist* and no theory which disregards it can be correct.

The proof that Meitner gives of her theory is that groups having energies related in the manner predicted by the theory actually exist in the β-ray spectra of thorium B, radium D and radium B.

Consider the thorium B spectrum. Leaving aside the question of the extra lines Nos. 1, 4 and 6, her theory predicts that there should be a line of energy $2\cdot36 \times 10^5$ volts. The measurements I have given have a relative accuracy of 1 in 300 and it can be seen from the table that this line does not occur.

In the case of the radium D and the low energy part of the radium B spectra lines do occur having energies agreeing well with $h\nu$. Although the agreement is not very good, I have suggested that these lines were due to conversion of the γ-ray in levels near the surface of the atom like the N level. This last explanation is natural and one would anticipate that N level lines might be found in cases like this where the energy of the γ-ray is nearer the energy of the N ring. Also, as I pointed out, the relative intensity is in accordance with what one would expect.

The thorium D and radium C spectra provide no evidence at all for Meitner's theory and the case of the radium B spectrum is equally definite. Whatever else it may emit it is certain that radium B emits three prominent γ-rays, and according to this theory there should be found three β-ray groups of energies $h\nu_1$, $h\nu_2$, $h\nu_3$. The second group is quite definitely absent; there are, however, two groups agreeing with $h\nu_1$ and $h\nu_3$. These are the lines $H\rho$ 1815 and 2295. If Meitner's theory were correct these groups would be primary and would originate in the radium B nucleus. Not being due to γ-rays they could not be excited in other metals. Groups of precisely these energies, however, are excited in the isotropic element lead. Rutherford, Robinson and Rawlinson† observed the 1815 group from lead and I have repeated the observations and also found evidence of the group 2295.

There would appear therefore to be no evidence in favour of this theory, but on the contrary very direct evidence against it.

Note added. The N_{abs} energy may be greater than that given. It is possible that the first groups of Tables IV and V are composite and due to faint O-level lines superimposed on the stronger N-level lines.

* Chadwick, *loc. cit.*
† *Phil. Mag.* 38, p. 281 (1914).

PROCEEDINGS

Note on the Curved Tracks of β Particles. By P. L. KAPITZA,
Fellow of the Physics Technical Institute, Petrograd. (Communicated by Mr C. G. Darwin.)

In the *Phil. Mag.* of February, 1921, Prof. Compton gives an
explanation of the curvature in the paths of β particles in the
photographs taken by the Wilson expansion apparatus. The fact
that the particle does not move in a straight line is, indeed, of
great interest if this effect is not due to some stray cause; for
instance, it is not yet clear from the published experiments what
may be the influence of the motion of the air currents which probably exist in the expansion chamber. But the fact that the tracks
of the α particles obtained in the same way are nearly straight shows
that the curvature in the case of the β particle may very likely be
due to some other cause. The explanation of this curvature may be
given either by some anisotropy of the space in which the β particle
is moving, or by some new property of the β particle itself. Considering the modern view of the gaseous state it is very doubtful
whether we may expect in a gas any kind of anisotropy over such
distances as the path of a β particle. The second explanation which
considers an asymmetry in the β particle is much more probable.
Such an asymmetry can be imagined in two ways. Firstly, the
particle may not be spherical but possess an oblate form. In this
case it may be possible that if the β particle is not moving along
its longest axis of symmetry it would describe a curved path. But
this explanation has the objection that such an oblate β particle
would possess two electromagnetic masses, along the short and
long axes respectively. But no such difference in the masses of
β particles has been found although the determination of e/m has
been made with great accuracy.

Evidently the more probable explanation is that given by
Compton, who assumes that the β particle has a magnetic moment.
This new property of the β particle is rather to be expected, for if
the β particle is set in rotation about any axis it will possess a
magnetic moment along that axis. It is indeed difficult to imagine
how the electron can be set in rotation, but once it is rotating we
possess no means to stop its motion.

Such a magnetic β particle moving through matter will create
a magnetic polarisation in it. This means that there will be superimposed on the magnetic field due to the β particle itself an additional

field due to the surrounding matter. Owing to its electric charge the path of the β particle in this additional field will be a helix. This is the Compton explanation of the curved tracks. But in the approximate calculation (*loc. cit.*) a very doubtful assumption is made in determining the magnetisation of the surrounding matter by simply using the value of the susceptibility obtained by ordinary experiments in a static magnetic field. I do not consider this justifiable in this case. The magnetic field surrounding the β particle diminishes very rapidly, as the third power of the distance, so that only the molecules nearest to the β particle would be affected. As the β particle is moving with an extremely high velocity the time during which the molecules are under the action of the β particle is very short. According to the modern view of magnetism there are two ways in which magnetisation may be produced in a body. The first is a paramagnetic phenomenon which is due to the change in the orientation of the molecules which possess magnetic moments. The time required for such a change in the orientation of molecules is clearly much greater than the period during which the β particle affects them. So the paramagnetic effect will not occur. The second method is diamagnetic, and is due, on Langevin's theory, to the change in the velocity of rotation of the electrons in their orbits. This change is produced at the moment when the magnetic field is created, and constitutes the only type of magnetic polarisation which can occur in the space surrounding the β particle. But the diamagnetic susceptibility measured in ordinary experiments is a mean value taken over a time which includes a great number of revolutions of the electrons and is calculated on the assumption that the magnetisation is uniform in the body. It is obvious that neither of these assumptions can be made in the case of the β particle. For the β particle affects the atom for such a short interval that in it the electrons can only cover a very small part of their orbits. In addition the non-uniformity of the field must be taken into account.

The purpose of this short note is to make the calculation more accurately and to determine the magnetic moment which a particle must possess in order to produce an appreciable curvature in its path.

Suppose the β particle is at *A* (Fig. 1).

Let it possess charge *e*, velocity *V* and magnetic moment *M*.

Fig. 1.

Suppose that one of the electrons is at *B*, its position being determined by *R*, which is the distance *AB*, and by a unit vector *r*, giving the direction of *AB*.
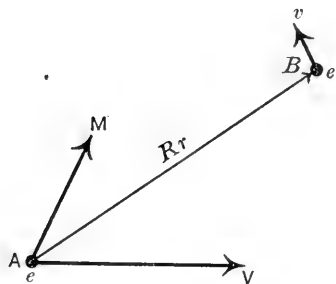
Now if the β particle is in motion there is an electric force at B, and the electron will be set in motion. If H is the magnetic field due to the magnetic moment of the β particle, the electric force E created by the moving H will be given by

$$\operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}.$$

Instead of H we shall use U, the vector potential, given by

$$\operatorname{curl} U = H.$$

Integrating we have

$$E = -\frac{1}{c}\frac{\partial U}{\partial t} + \operatorname{grad} \phi \qquad \ldots\ldots(1).$$

The part of the electric force $-\dfrac{1}{c}\dfrac{\partial U}{\partial t}$ is due to the magnetic moment of the β particle, the other part $\operatorname{grad} \phi$ to its electric charge.

We shall consider the electron at B to be free and at rest, as is usually assumed. These assumptions are quite legitimate, because the orbital motion is negligibly slow, and the force from the nucleus is small. Then the velocity of the electron at B will be

$$v = \frac{e}{m}\int_0^t E\,dt.$$

Using (1) we find

$$v = \frac{e}{m}\int_0^t \operatorname{grad} \phi\,dt - \frac{e}{mc}U.$$

The vector potential from a magnetic doublet is

$$U = \frac{1}{R^2}[M, r] \qquad \ldots\ldots(2).$$

Using this expression we get

$$v = \frac{e}{m}\int_0^t \operatorname{grad} \phi\,dt - \frac{e}{mcR^2}[M, r].$$

Divide v into two parts v_1, v_2 so that

$$v_1 = -\frac{e}{mcR^2}[M, r] \qquad \ldots\ldots(3),$$

and

$$v_2 = \frac{e}{m}\int_0^v \operatorname{grad} \phi\,dt \qquad \ldots\ldots(4).$$

The component v_1 is due to the magnetic part of the action, and v_2 is due to the ordinary electrostatic effect. Now we will assume that the displacement of the electron is small compared with R, and shall therefore neglect it. This assumption will permit the calculation of v_1 independently of v_2, but this is not always

quite legitimate, because if the distance R is of the order of 10^{-10} cm., the total displacement of the electron is of the same order of magnitude. But it is easy to see that the force grad ϕ acts in such a way as to increase R, and to bring the electron into a region where U is smaller. So the actual velocity v_1 will be less than that calculated. We shall see later that v_2 contributes nothing to the effect under consideration and that the curvature of the path will be proportional to v_1. Hence the magnetic moment so calculated will be less than the true moment.

Using first the velocity v_1, the electron at B moving with this velocity will create a magnetic field at A whose value is given by

$$H_A = -\frac{e}{c} \frac{[r, v_1]}{R^2} \qquad \text{......(5)}.$$

The β particle at A suffers in time dt a change of velocity given by the Lorentz formula

$$dW = \frac{e}{mc} [V, H_A] dt \qquad \text{......(6)}.$$

Now from (3), (5), (6), we have

$$dW = \frac{e^3}{c^3 m^2} \frac{dt}{R^4} \left[V \left[r [M, r] \right] \right].$$

Performing the multiplication we get

$$dW = \frac{e^3}{c^3 m^2} \frac{dt}{R^4} \{ [V, M] - (M, r)[V, r] \} \qquad \text{......(7)}.$$

Take rectangular coordinates with axes as shown in Fig. 2.
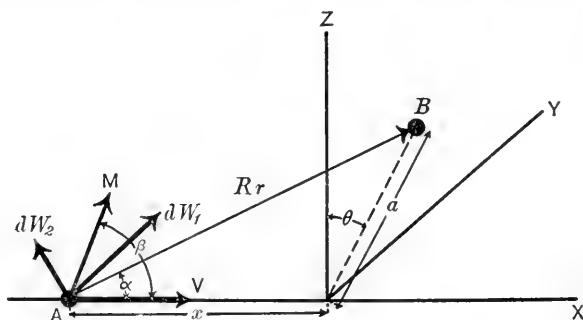
Fig. 2.

The axis of X coincides with V, M lies in the plane ZOX, and the electron in the plane ZOY at a distance a from O. The angle between M and V is β, between r and V, α, and the angle in the

plane ZOY between a and the axis OZ, θ. Now in (7) the first term which we shall denote by

$$dW_1 = \frac{e^3}{c^3 m^2} \frac{dt}{R^4} [V, M] \qquad \dots\dots(8),$$

does not depend on θ, because it is independent of r, and is parallel to OY. The second term which we shall call

$$dW_2 = -\frac{e^3}{c^3 m^2} \frac{dt}{R^4} (r, M) [V, M] \qquad \dots\dots(9),$$

has two components parallel to OZ and OY.

These are

$$dW_{2\,(Z)} \propto VM [\cos \beta \cos \alpha \sin \alpha \sin \theta$$
$$+ \sin \beta \cos \theta \sin \theta \sin^2 \alpha] \quad \dots(10),$$

and

$$dW_{2\,(Y)} \propto - VM [\cos \beta \cos \alpha \sin \alpha \cos \theta$$
$$+ \sin \beta \cos^2 \theta \sin^2 \alpha] \quad \dots(11).$$

In order to make the calculation simpler we shall assume that for a given value of a all values of θ are equally probable. Taking the mean value of dW_2 for all possible values of θ and calling this mean $\overline{dW_2}$, where

$$\overline{dW_2} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} dW_2 \, d\theta,$$

we get, by using (10) and (11)

$$\overline{dW_{2\,(Z)}} = 0,$$
$$\overline{dW_{2\,(Y)}} = -\frac{e^3}{c^3 m^2} VM \frac{dt}{2R^4} \sin^2 \alpha \sin \beta.$$

From (8) we find that the increase of velocity is parallel to OY and is equal to

$$\overline{dW_{(Y)}} = \frac{e^3}{c^3 m^2} VM \sin \beta \left(1 - \tfrac{1}{2} \sin^2 \alpha\right) \frac{dt}{R^4} \qquad \dots\dots(12).$$

Now the total increase of velocity during one "collision" is

$$W_{(Y)} = \int_{-\infty}^{+\infty} \overline{dW_{(Y)}}.$$

Using
$$dx = V dt$$
to change the variable of integration, we have

$$W_{(Y)} = \frac{e^3}{c^3 m^2} M \sin \beta \left\{ \int_{-\infty}^{+\infty} \frac{dx}{(a^2 + x^2)^2} - \tfrac{1}{2} \int_{-\infty}^{+\infty} \frac{a^2}{(a^2 + x^2)^3} \, dx \right\}.$$

Performing the integration we get

$$W_{(Y)} = \frac{5\pi}{16} \frac{e^3}{c^3 m^2} \frac{M \sin \beta}{a^3} \qquad \dots\dots(13).$$

Now we have to take into consideration not only one electron but all the electrons contained in the space surrounding the β particle.
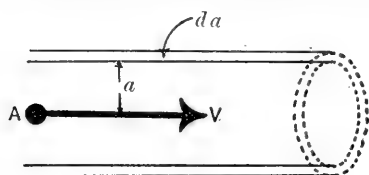
Fig. 3.

Let the number of electrons per unit volume be n. Instead of the change of velocity in a collision consider the change of velocity per unit time, i.e. the acceleration.

Then in a second the β particle will meet in a cylindrical shell of radius a and thickness da (Fig. 3), dn electrons where

$$dn = 2\pi V n a \, da.$$

The acceleration will be equal to dA where

$$dA = W dn = \frac{5\pi^2}{8} \frac{e^3}{c^3 m^2} \frac{M \sin \beta}{a^2} n \, da \qquad \ldots\ldots(14).$$

If the nearest distance between an electron and a β particle is a_0, we get by integration

$$A = \frac{5\pi^2}{8} \frac{e^3 n}{c^3 m^2} [MV] \int_{a_0}^{\infty} \frac{da}{a^2} = \frac{5\pi^2}{8} \frac{e^3}{c^3 m^2} \frac{n}{a_0} [MV] \ldots(15).$$

This is an equation of helical motion.

Before going further we shall prove that the motion v_2 of the electron (see (4)) will contribute nothing to the curvature of the path of the particle. This is the result of the symmetry of the force

$$\operatorname{grad} \phi = \frac{e}{R^2} r \qquad \ldots\ldots(16).$$

Consider in Fig. 2 an electron with coordinates z and y and velocities $v_{2(X)}, v_{2(Y)}, v_{2(Z)}$. Then from (16) another electron with coordinates $-z$, and $-y$, will have velocities $-v_{2(X)}, -v_{2(Y)}, -v_{2(Z)}$. The magnetic force at A from the two electrons is equal to zero, as is easily seen from (4).

Returning to (15) it follows that the β particle will describe a helix drawn on a cylinder of radius b given by

$$b = \frac{V^2 \sin^2 \beta}{A} \qquad \ldots\ldots(17).$$

The angle of inclination of the helix will be equal to β the angle between M and V (Fig. 4).

Fig. 4 shows how the β particle will move. M keeps its direction fixed in space.

From (17) and (15) we can find the magnetic moment of the β particle

$$M = \frac{V \sin \beta \, a_0}{b \, \dfrac{5\pi^2}{8} \, \dfrac{e^3}{c^3 m^2} \, n} \qquad \ldots\ldots(18).$$

Now if we try to introduce numerical values, we get a very large estimate for M. In (18) e, c, m, V, are known accurately.

The minimum distance between the β particle and an electron on the theory of collisions cannot be less than 10^{-10}, while n depends on the surrounding space and for oxygen molecules, which contain 16 electrons, is $16 \times 2\cdot7 \times 10^{19}$. Now b and $\sin \beta$ have to be taken from the Wilson photographs, but no definite numerical values have yet been given. But we may say that for appreciable curvature in the paths b must lie roughly between 1 and 10 cms. and $\sin \beta$ between $0\cdot1$ and 1. This gives for the two limits $1\cdot9 \times 10^{-18}$ and $1\cdot9 \times 10^{-16}$. This is a very large magnetic moment and we can scarcely believe that the β

Fig. 4.

particle possesses it. It is sufficient to notice that the magnetic field at a distance of 10^{-8} cms., i.e. at intermolecular distances will be millions of Gauss and would affect the structure of the atoms. If we calculate the energy of the electric field due to the motion of the magnetic field of the β particle, and determine as is done for the electromagnetic mass the new "magneto-electric" mass, taking M of the order found above, we find that the latter is more than 1000 times the ordinary mass. Hence we see that it is extremely difficult to explain quantitatively the curvature in the paths of β particles by means of magnetic moments in the particles.
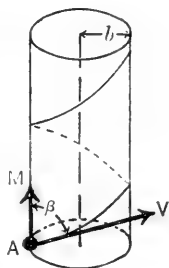
The author has also made similar calculations along other lines. In these it was supposed that the magnetic field of the β particle sets the surrounding electrons in rotation. Each electron will then possess a magnetic moment. The magnetic field created by the electrons will produce the same effect as that described above but the magnetic moment of the β particle comes out a million times greater than that calculated from (18). So this assumption is equally of no avail.

My best thanks are due to Mr C. G. Darwin for reading this paper in manuscript.

Waves of Permanent Type on the Interface of two Liquids.
By Dr H. LAMB.

The literature of waves of permanent type is already extensive, but the investigations have related mostly to the case of a single liquid with a free surface*. In the present communication I consider the case of waves on the horizontal interface between two fluids, both of great depth. This is of some interest in its application to atmospheric waves on the common boundary of two strata of slightly different densities.

Taking the axis of y vertical, with the positive direction upwards, and reducing the problem to one of steady motion in the usual way, the stream-function for the lower fluid may be written

$$\frac{\psi}{c} = -y + A_1 e^{ky} \cos kx + A_2 e^{2ky} \cos 2kx + A_3 e^{3ky} \cos 3kx + \dots$$
$$\dots\dots(1),$$

where $2\pi/k$ is the wave-length, and c the corresponding wave velocity, to be found. For the upper fluid

$$\frac{\psi'}{c} = -y + A_1' e^{-ky} \cos kx + A_2' e^{-2ky} \cos 2kx + A_3' e^{-3ky} \cos 3kx + \dots$$
$$\dots\dots(2).$$

In the very simple treatment by the late Lord Rayleigh† of the case of a single fluid it is shewn that if we neglect terms of higher order in the amplitude than the third, the conditions of the problem can all be satisfied on the supposition that $A_2 = 0$, $A_3 = 0$. In the present question this is no longer possible, and a more elaborate scheme of approximation is necessary.

Taking the origin in the mean level of the common surface, the equation of the profile may be written

$$y = C_1 \cos kx + C_2 \cos 2kx + C_3 \cos 3kx + \dots \quad \dots(3).$$

We have to express that when this value of y is substituted in (1), the resulting value of ψ is constant. We find with sufficient approximation in each case‡

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* See, however, the note at the end of this paper.

† *Phil. Mag.* (5) vol. 1, p. 251 (1876); *Papers*, vol. 1, p. 251.

‡ A provisional assumption, to be verified in the sequel, is here made. If C_1 be regarded as of the first order it is assumed that the remaining coefficients in (1), (2) and (3) are at least of the orders indicated by the suffixes. Terms of higher order than the third are neglected throughout this paper.

$$e^{ky} \cos kx = \tfrac{1}{2}kC_1 + (1 + \tfrac{1}{2}kC_2 + \tfrac{3}{8}k^2C_1{}^2) \cos kx + \tfrac{1}{2}kC_1 \cos 2kx$$
$$+ (\tfrac{1}{2}kC_2 + \tfrac{1}{8}k^2C_1{}^2) \cos 3kx \qquad \ldots\ldots(4),$$
$$e^{2ky} \cos 2kx = kC_1 \cos kx + \cos 2kx + kC_1 \cos 3kx \ldots(5),$$
$$e^{3ky} \cos 3kx = \cos 3kx \qquad \ldots\ldots(6).$$

Hence

$$\frac{\psi}{c} = -y + \tfrac{1}{2}kA_1C_1 + (A_1 + \tfrac{1}{2}kA_1C_2 + \tfrac{3}{8}k^2A_1C_1{}^2 + kA_2C_1) \cos kx$$
$$+ (A_2 + \tfrac{1}{2}kA_1C_1) \cos 2kx$$
$$+ (A_3 + \tfrac{1}{2}kA_1C_2 + \tfrac{1}{8}k^2A_1C_1{}^2 + kA_2C_1) \cos 3kx \quad \ldots(7).$$

Hence

$$\left.\begin{array}{l}
A_1 + \tfrac{1}{2}kA_1C_2 + \tfrac{3}{8}k^2A_1C_1{}^2 + kA_2C_1 = C_1 \\
A_2 + \tfrac{1}{2}kA_1C_1 = C_2 \\
A_3 + \tfrac{1}{2}kA_1C_2 + \tfrac{1}{8}k^2A_1C_1{}^2 + kA_2C_1 = C_3
\end{array}\right\} \quad \ldots\ldots(8).$$

From these we find

$$\left.\begin{array}{l}
A_1 = C_1 - \tfrac{3}{2}kC_1C_2 + \tfrac{1}{8}k^2C_1{}^3 \\
A_2 = C_2 - \tfrac{1}{2}kC_1{}^2 \\
A_3 = C_3 - \tfrac{3}{2}kC_1C_2 + \tfrac{3}{8}k^2C_1{}^3
\end{array}\right\} \qquad \ldots\ldots(9).$$

The corresponding quantities for the upper fluid are obtained by merely reversing the sign of k. Thus

$$\left.\begin{array}{l}
A_1' = C_1 + \tfrac{3}{2}kC_1C_2 + \tfrac{1}{8}k^2C_1{}^3 \\
A_2' = C_2 + \tfrac{1}{2}kC_1{}^2 \\
A_3' = C_3 + \tfrac{3}{2}kC_1C_2 + \tfrac{3}{8}k^2C_1{}^3
\end{array}\right\} \qquad \ldots\ldots(10).$$

We have still to secure equality of pressure on the two sides of the common boundary. If q, q' denote the velocities in the two fluids, and ρ, ρ' the densities, the requisite condition is

$$\rho (q^2 + 2gy) - \rho' (q'^2 + 2gy) = \text{const.} \qquad \ldots\ldots(11)$$

over the interface. Now

$$\frac{q^2}{c^2} = \frac{1}{c^2} \left\{ \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 \right\}$$
$$= 1 - 2kA_1 e^{ky} \cos kx - 4kA_2 e^{2ky} \cos 2kx - 6kA_3 e^{3ky} \cos 3kx$$
$$+ k^2A_1{}^2 e^{2ky} + 4k^2A_1A_2 e^{3ky} \cos kx \qquad \ldots\ldots(12).$$

Substituting from (4), (5) and (6), we find after some reduction

$$\frac{q^2}{c^2} = 1 - k^2A_1C_1 + k^2A_1{}^2$$
$$+ (- 2kA_1 - k^2A_1C_2 - \tfrac{3}{4}k^3A_1C_1{}^2 - 4k^2A_2C_1 + 2k^3A_1{}^2C_1$$
$$+ 4k^2A_1A_2) \cos kx$$
$$- (k^2A_1C_1 + 4kA_2) \cos 2kx$$
$$- (k^2A_1C_2 + \tfrac{1}{4}k^3A_1C_1{}^2 + 4k^2A_2C_1 + 6kA_3) \cos 3kx \quad \ldots(13).$$

Introducing the values of A_1, A_2, A_3, from (9), we have finally

$$\frac{q^2}{c^2} = 1 + (-2kC_1 + 2k^2C_1C_2 + k^3C_1{}^3) \cos kx$$
$$+ (k^2C_1{}^2 - 4kC_2) \cos 2kx$$
$$+ (-6kC_3 + 4k^2C_1C_2 - \tfrac{1}{2}k^3C_1{}^3) \cos 3kx \quad (14).$$

The value of q'^2 follows by reversing the sign of k. Hence, substituting in (11), and equating separately to zero the coefficients of $\cos kx$, $\cos 2kx$, $\cos 3kx$, we find

$$2g(\rho - \rho') - 2kc^2(\rho + \rho') + 2k^2c^2(\rho - \rho')C_2 + k^3c^2(\rho + \rho')C_1{}^2 = 0$$
$$\qquad\qquad\qquad \ldots\ldots(15),$$

$$\{2g(\rho - \rho') - 4kc^2(\rho + \rho')\}C_2 + k^2c^2(\rho - \rho')C_1{}^2 = 0 \quad \ldots\ldots(16),$$

$$\{2g(\rho - \rho') - 6kc^2(\rho + \rho')\}C_3 + 4k^2c^2(\rho - \rho')C_1C_2$$
$$- \tfrac{1}{2}k^3c^2(\rho + \rho')C_1{}^3 = 0 \ \ \ldots\ldots(17).$$

From (15) we have as a first approximation

$$c^2 = \frac{\rho - \rho'}{\rho + \rho'}\frac{g}{k} \qquad\qquad \ldots\ldots(18),$$

the well-known formula of Stokes. Using this, we have from (16)

$$C_2 = \frac{1}{2}\frac{\rho - \rho'}{\rho + \rho'}\,kC_1{}^2 \qquad\qquad \ldots\ldots(19),$$

and from (17) $\qquad C_3 = \frac{(3\rho - \rho')(\rho - 3\rho')}{8(\rho + \rho')^2}k^2C_1{}^3 \qquad\qquad \ldots\ldots(20).$

The next approximation to c^2 is obtained from (15); we find

$$c^2 = \frac{\rho - \rho'}{\rho + \rho'}\frac{g}{k}\left\{1 + \frac{\rho^2 + \rho'^2}{(\rho + \rho')^2}k^2C_1{}^2\right\} \qquad\qquad \ldots\ldots(21).$$

The values of A_1, A_2, A_3, etc. now follow from (9) and (10). Thus

$$A_1 = C_1 - \frac{5\rho - 7\rho'}{8(\rho + \rho')}k^2C_1{}^3, \ \ A_2 = -\frac{\rho'}{\rho + \rho'}\,kC_1{}^2,$$
$$A_3 = -\frac{\rho'(\rho - 3\rho')}{2(\rho + \rho')}k^2C_1{}^3 \ \ \ldots\ldots(22),$$

$$A_1' = C_1 - \frac{5\rho' - 7\rho}{8(\rho + \rho')}k^2C_1{}^3, \ \ A_2' = \frac{\rho}{\rho + \rho'}\,kC_1{}^2,$$
$$A_3' = -\frac{\rho(\rho' - 3\rho)}{2(\rho + \rho')}k^2C_1{}^3 \ \ \ldots\ldots(23).$$

Known results are reproduced by putting $\rho' = 0$. Thus

$$c^2 = \frac{g}{k}(1 + k^2C_1{}^2) \qquad\qquad \ldots\ldots(24),$$

and the equation of the wave-profile becomes

$$y = C_1 \cos kx + \tfrac{1}{2}kC_1^2 \cos 2kx + \tfrac{3}{8}k^2C_1^3 \cos 3kx \quad (25).$$

Also

$$A_1 = C_1(1 - \tfrac{5}{8}k^2C_1^2), \;\; A_2 = 0, \;\; A_3 = 0 \quad \ldots\ldots(26),$$

to our order of approximation, in agreement with Stokes and Rayleigh.

When, on the other hand, ρ and ρ' are nearly equal (as in the atmospheric application), we have

$$c^2 = \frac{\rho - \rho'}{\rho + \rho'}\frac{g}{k}(1 + \tfrac{1}{2}k^2C_1^2) \quad \ldots\ldots(27),$$

nearly. The limiting form of the profile is now

$$y = C_1 \cos kx - \tfrac{1}{8}k^2C_1^3 \cos 3kx \quad \ldots\ldots(28),$$

the coefficient of $\cos 2kx$ being negligible. The condition for permanent type is therefore much more nearly fulfilled by waves of simple harmonic profile than in the former case, to which (25) relates. Moreover, the formula (28) makes the elevations and depressions relatively to the mean level anti-symmetrical, whereas (25) gives higher and narrower crests and broader and shallower hollows*. We may also notice that

$$A_1 = C_1 + \tfrac{1}{8}k^2C_1^3, \;\; A_2 = -\tfrac{1}{2}kC_1^2, \;\; A_3 = \tfrac{1}{2}k^2C_1^3 \quad \ldots(29),$$
$$A_1' = C_1 + \tfrac{1}{8}k^2C_1^3, \;\; A_2' = \tfrac{1}{2}kC_1^2, \;\;\;\; A_3' = \tfrac{1}{2}k^2C_1^3 \quad \ldots(30),$$

confirming the statement previously made that A_2 and A_3 are no longer negligible†.

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* Cf. the diagram in *Hydrodynamics* (1916), p. 410. A similar representation of the curve (28), with $kC_1 = \tfrac{1}{2}$ would hardly shew the deviation from the simple harmonic form.

† The theory of waves of permanent type on the horizontal interface of two *currents* was attacked by Helmholtz by a difficult analysis in *Berl. Sitzb.* 1889 (*Wiss. Abh.*, Bd. 3, p. 309). The investigation was continued by Wien, see his *Lehrbuch d. Hydrodynamik* (1900), p 170. The theory includes the problems of this paper as a particular case, but I have not found it possible to compare results.

An asymptotic relation between the arithmetic sums $\sum\limits_{n \leq x} \sigma_r(n)$ *and* $x^r \sum\limits_{n \leq x} \sigma_{-r}(n)$. By B. M. WILSON. (Communicated by Prof. G. H. HARDY.)

§ 1. Ramanujan has proved that, if $r > 0$ and if $\sigma_r(n)$ denote, as usual, the sum of the rth powers of the divisors of n,

$$(1 \cdot 1) \quad \sum_{n \leq x} \sigma_r(n) = x^r \sum_{n \leq x} \sigma_{-r}(n) - \frac{r x^{1+r}}{1 + r} \zeta(1 + r)$$
$$+ \tfrac{1}{2} x^r \zeta(r) - \frac{r x}{1 - r} \zeta(1 - r) + O(m)^*,$$

where

$$(1 \cdot 11) \quad m = x^{\frac{1}{2}r} \ (r < 2), \quad m = x \log x \ (r = 2), \quad m = x^{r-1} \ (r > 2).$$

Since $\qquad \sigma_r(n) = \sum\limits_{d/n} d^r = \sum\limits_{d/n} \left(\frac{n}{d}\right)^r = n^r \sigma_{-r}(n),$

equation $(1 \cdot 1)$ clearly gives an asymptotic formula for the sum

$$(1 \cdot 2) \qquad \sum_{n \leq x} (x^r - n^r) \sigma_{-r}(n) \qquad (r > 0).$$

In the particular case $r = 1$ the asymptotic representation of this sum had already been discussed by Wigert† and Landau‡; Landau, reporting on Wigert's memoir showed, by transcendental methods, that the error-term in $(1 \cdot 1)$ is then of the form $O(x^{\frac{2}{5} + \epsilon})$, and, by using a special transformation into a series of Bessel Functions obtained by Wigert§, that this may be replaced by $O(x^{\frac{2}{5}})\|$.

§ 2. We write
$$(2 \cdot 1) \qquad \sum_{n \leq x} (x^r - n^r) \sigma_{-r}(n) = P(x, r) + R(x, r) \qquad (r > 0),$$

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* S. Ramanujan, "On certain trigonometrical sums and their applications in the Theory of Numbers," *Trans. Cambridge Phil. Soc.*, vol. xxii, pp. 259–276 (1918), page 276, equation (17·5).

† Wigert, "Sur quelques fonctions arithmétiques," *Acta Math.*, vol. xxxvii, pp. 113–140 (1914).

‡ E. Landau, *Göttingische Gelehrte Anzeigen*, 1915, pp. 377–414.

§ *Loc. cit.*, p. 140, equation (27).

‖ *Loc. cit.*, p. 414.

where

$$(2\cdot15) \quad P(x,r) = \frac{rx^{1+r}}{1+r}\,\zeta(1+r) - \tfrac{1}{2}x^r\,\zeta(r)$$
$$+ \frac{rx}{1-r}\,\zeta(1-r) + \tfrac{1}{2}\zeta(-r),$$

and obtain, in the present note, estimates of the order of the function $R(x,r)$ which are, for values of r not exceeding $\tfrac{3}{2}$, better than those given by Ramanujan. The methods used are, throughout, transcendental, and depend upon the representation of the left-hand member of (2·1) which is afforded by the following lemma:

Lemma 1. *If the Dirichlet's series*

$$(2\cdot2) \qquad\qquad Z(s) = \sum_{n=1}^{\infty} a_n\, n^{-s}$$

is absolutely convergent on the line $\sigma = \alpha > 0$, *and* r *is any positive number, then*

$$(2\cdot22) \quad \sum_{n \leqslant x} a_n (x^r - n^r) = \frac{r}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{x^{s+r}}{s(s+r)}\, Z(s)\, ds.$$

To prove this formula* one has only to express the integrand on the right as the sum of two partial fractions, and apply to each of the integrals so arising the well-known result that, if $\beta > 0$

$$\frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} \frac{y^s}{s}\, ds = 1\ (y > 1),\ = \tfrac{1}{2}\,(y = 1),\ = 0\,(0 < y < 1).$$

From equation (2·22) it follows, by an argument familiar in the establishment of asymptotic formulae†, that, under the conditions of Lemma 1, if ω denotes a real function of x, which tends to infinity with x, then

$$(2\cdot25) \sum_{n \leqslant x} a_n (x^r - n^r) = \frac{r}{2\pi i}\int_{\alpha - i\omega}^{\alpha + i\omega} \frac{x^{s+r}}{s(s+r)}\, Z(s)\, ds + O\left(\frac{x^{r+\alpha}}{\omega^2}\right).$$

It is in this form that Lemma 1 will be used in what follows.

We may place here also

Lemma 2. *If* γ *denotes a constant not less than* $-\tfrac{1}{2}$, *and* w *is independent of* u, *then*

$$(2\cdot3) \quad \left| \int_1^u u^\gamma\, e^{\pm iu(\log u - w)}\, du \right| < K U^{\frac{1}{2}+\gamma} \qquad (\gamma > -\tfrac{1}{2}),$$
$$< K \log U \qquad (\gamma = -\tfrac{1}{2}),$$

where K *is independent both of* U *and of* w.

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* The proof of a similar formula is given in full in Landau's report already quoted. We have not, of course, attempted to state the lemma in its most general form.

† Compare *e.g.* Landau, *Handbuch der Primzahlen*, page 184 and *passim*.

Landau has already shown the first inequality to hold if $\gamma \geqslant 0$*; so that, if we write

$$F(U) = \int_1^U e^{\pm iu(\log u - w)}\, du,$$

$$F(U) = O(U^{\frac{1}{2}}).$$

If then $0 > \gamma \geqslant -\frac{1}{2}$, an integration by parts gives

$$\int_1^U u^\gamma e^{\pm iu(\log u - w)}\, du = U^\gamma F(U) - \gamma \int_1^U u^{\gamma-1} F(u)\, du$$

$$= O(U^{\gamma+\frac{1}{2}}) - \gamma \int_1^U O(u^{\gamma-\frac{1}{2}})\, du,$$

whence, since $\gamma - \frac{1}{2} \geqslant -1$, the complete result stated in (2·3) follows at once.

§ 3. Since

$$(3\text{·}1) \qquad \sum_{n=1}^{\infty} \sigma_{-r}(n)\, n^{-s} = \zeta(s)\, \zeta(s+r), \quad (r > 0\; ;\; \sigma > 1);$$

it follows from (2·25) that, if α is greater than unity and ω is a real positive function of x which tends to infinity with x, then

$$(3\text{·}2) \quad \sum_{n \leqslant x} (x^r - n^r)\, \sigma_{-r}(n) = \frac{r}{2\pi i} \int_{\alpha-i\omega}^{\alpha+i\omega} \frac{x^{s+r}}{s(s+r)}\, \zeta(s)\, \zeta(s+r)\, ds$$

$$+ O\left(\frac{x^{r+a}}{\omega^2}\right),\; (r > 0\; ;\; \alpha > 1);$$

hence if $\beta < \alpha$, we obtain, by an application of Cauchy's Theorem,

$$(3\text{·}3) \quad \sum_{n \leqslant x} (x^r - n^r)\, \sigma_{-r}(n) = Q(x,\, r) + \frac{r}{2\pi i}\left\{ \int_{\alpha-i\omega}^{\beta-i\omega} + \int_{\beta-i\omega}^{\beta+i\omega} \right.$$

$$+ \left. \int_{\beta+i\omega}^{\alpha+i\omega} \frac{x^{s+r}}{s(s+r)}\, \zeta(s)\, \zeta(s+r)\, ds \right\} + O\left(\frac{x^{r+a}}{\omega^2}\right),\; (r > 0,\, \alpha > 1),$$

where $Q(x,\, r)$ denotes the sum of the residues of the function

$$\frac{r x^{s+r}}{s(s+r)}\, \zeta(s)\, \zeta(s+r)$$

at its poles inside the rectangle of vertices $\alpha \pm i\omega$, $\beta \pm i\omega$; so that, if

$$(3\text{·}4) \qquad\qquad\qquad \beta \leqslant -r,$$

$$(3\text{·}5) \qquad\qquad\qquad Q(x,\, r) = P(x,\, r),$$

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* E. Landau, "Über die Anzahl der Gitterpunkte in gewissen Bereichen," *Göttinger Nachrichten*, 1912, pp. 687–770, Hilfssatz 10, p. 707 *et seq*. Landau shows, in fact, that, for $\gamma = 0$, we may take $K = 23$. The inequality was applied by Landau to a discussion of Dirichlet's Divisor Problem, and his discussion served as a model for much of the present note. Full details of the method, not given here, will be found in Professor Landau's memoir.

where $P(x, r)$ is given by (2·15). The integrals along the lines $t = \pm \omega$ are most easily discussed; for, in virtue of (3·4), on the line $\sigma = \beta$

$$f(s) = \zeta(s)\,\zeta(s+r) = O\left(|t|^{1-2\beta-r}\right);$$

also on $\sigma = \alpha$ $f(s) = \zeta(s)\,\zeta(s+r) = O(1)$,

and $f(s)$ is regular for $\beta \leqslant \sigma \leqslant \alpha$, $|t| \geqslant t_0$.

Hence by Lindelöf's Theorem *

$$f(s) = O\left(|t|^{(1-2\beta-r)\frac{\alpha-\sigma}{\alpha-\beta}}\right)$$

uniformly for $\alpha \leqslant \sigma \leqslant \beta$, and, hence we find

$$(3\cdot6) \int_{\beta \pm i\omega}^{\alpha \pm i\omega} \frac{x^{s+r}}{s(s+r)}\,\zeta(s)\,\zeta(s+r)\,ds = O\left(\frac{x^{r+\alpha}}{\omega^2}\right) + O\left(\frac{x^{r+\beta}}{\omega^{1+2\beta+r}}\right).$$

We now have, therefore,

$$(3\cdot7) \sum_{n \leqslant x}(x^r - n^r)\,\sigma_{-r}(n) = P(x, r) + O\left(\frac{x^{r+\alpha}}{\omega^2}\right) + O\left(\frac{x^{r+\beta}}{\omega^{1+2\beta+r}}\right)$$

$$+ O\left(\int_{\beta-i\omega}^{\beta+i\omega} \frac{x^{s+r}}{s(s+r)}\,\zeta(s)\,\zeta(s+r)\,ds\right), \quad (\alpha > 1\,;\ \beta < -r < 0).$$

The constant β and the function $\omega(x)$ are still in large measure arbitrary; they are now to be chosen so as to make the aggregate order of the last three terms on the right of (3·7) as low as possible.

Inequality (3·4) ensures that the line $\sigma = \beta$ shall lie to the left of the critical strips of both the functions $\zeta(s)$, $\zeta(s+r)$. The integral along the line $\sigma = \beta$, can therefore, by transforming both ζ-functions by means of the functional equation, be written as the integral of a product of elementary functions by two absolutely and uniformly convergent series. On changing the order of integration and summation, we find, in fact

$$(3\cdot8) \qquad \int_{\beta-i\omega}^{\beta+i\omega} \frac{x^{s+r}}{s(s+r)}\,\zeta(s)\,\zeta(s+r)\,ds$$

$$= x^{\beta+r}\,\pi^{2\beta+r-1} \sum_{m,\,n=1}^{\infty} m^{\beta-1}\,n^{\beta+r-1} \int_{-\omega}^{\omega} \frac{(mn)^{it}}{(\beta+it)(\beta+r+it)}$$

$$\frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}\beta-\frac{1}{2}it\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\beta-\frac{1}{2}r-\frac{1}{2}it\right)}{\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}it\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}r+\frac{1}{2}it\right)}\,dt, \quad (\beta+r<0);$$

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* E. Lindelöf, "Quelques remarques sur la croissance de la fonction $\zeta(s)$," *Bull. de Soc. Math.* Ser. 2, vol. xxxii (1908, pp. 341–356 (346–348)). See also Landau, *loc. cit.*, p. 704, Hilfsatz 6.

and hence, approximating to the Γ functions by means of a general form of Stirling's formula*, we have

$$(3\cdot85) \qquad \int_{\beta-i\omega}^{\beta+i\omega} \frac{x^{s+r}}{s\,(s+r)}\,\zeta(s)\,\zeta(s+r)\,ds$$

$$= x^{3+r}\,\pi^{23+r-1}\,\Sigma\,m^{\beta-1}\,n^{3+r-1}\left\{ O\left(\int_1^\omega t^{-1-23-r}\,e^{-2ti\,(\log t - w)}\,dt\right)\right.$$

$$\left. + O\left(\int_1^\omega t^{-2-23-r}\,dt\right)\right\}.$$

In order that Lemma 2 may be applied effectively to the first integral on the right of (3·85) it is necessary that

$$-1-2\beta-r \geqslant -\tfrac{1}{2},\ \textit{i.e. that}$$

$$(3\cdot9) \qquad\qquad \beta \leqslant -\tfrac{1}{2}r - \tfrac{1}{4}.$$

A comparison of the two inequalities (3·4) and (3·9), which β is supposed to satisfy, shows that we have now to distinguish two cases, according as r is or is not less than $\tfrac{1}{2}$†.

§ 4. If $r < \tfrac{1}{2}$, (3·9) is the more stringent inequality, and we satisfy both conditions by taking

$$(4\cdot1) \qquad\qquad \beta = -\tfrac{1}{2}r - \tfrac{1}{4}\,;$$

(3·85) then gives, on applying Lemma 2 (with $\gamma = -\tfrac{1}{2}$),

$$\int_{\beta-i\omega}^{\beta+i\omega} \frac{x^{s+r}}{s\,(s+r)}\,\zeta(s)\,\zeta(s+r)\,ds = O\left(x^{\frac{1}{2}r-\frac{1}{4}}\log\omega\right)$$

so that (3·7) becomes

$$(4\cdot2)\quad R(x,\,r) = \sum_{n\leqslant x}\,(x^r - n^r)\,\sigma_{-r}(n) - P(x,\,r)$$

$$= O\left(\frac{x^{r+a}}{\omega^2}\right) + O\left(\frac{x^{\frac{1}{2}r-\frac{1}{4}}}{\omega^{\frac{1}{2}}}\right) + O\left(x^{\frac{1}{2}r-\frac{1}{4}}\log\omega\right),\quad (\alpha > 1).$$

The resultant order of the right-hand member of (4·2) is made as low as possible by now taking

$$\omega = x^p\,;\ \ p \geqslant \tfrac{1}{4}r + \tfrac{1}{2}\alpha + \tfrac{1}{8},$$

and in this way we arrive at the asymptotic equation

$$(4\cdot5) \qquad\qquad R(x,\,r) = O\left(x^{\frac{1}{2}r-\frac{1}{4}}\log x\right), \qquad (0 < r < \tfrac{1}{2}).$$

§ 5. If, on the other hand, $r \geqslant \tfrac{1}{2}$ the two conditions (3·4) and (3·9) are to be satisfied by taking

$$(5\cdot1) \qquad\qquad \beta = -r - \epsilon < -r\,;$$

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* See *e.g.* Landau, *loc. cit.*, pp. 701–702.
† I wish here to express my thanks to Professor Hardy for calling my attention to the problem presented by the order of the function $R(x,\,r)$, and for his criticisms and encouragement.

we now obtain from (3·85) and Lemma 2 (with $\gamma = r + 2\epsilon - 1 > -\frac{1}{2}$)

$$\int_{\beta - i\omega}^{\beta + i\omega} \frac{x^{s+r}}{s(s+r)} \zeta(s) \zeta(s+r)\, ds = O\left(x^{-\epsilon}\, \omega^{r+2\epsilon - \frac{1}{2}}\right)$$

so that now, from (3·7),

(5·2) $R(x, r) = O\left(\dfrac{x^{r+a}}{\omega^2}\right) + O\left(x^{-\epsilon}\, \omega^{r+2\epsilon - \frac{1}{2}}\right),$ $(\alpha > 1);$

and hence, if we write $\alpha = 1 + \epsilon > 1$, and take

$$\omega = x^{\frac{2(r+1)}{2r+3}},$$

we have

(5·5) $R(x, r) = O\left(x^{\frac{(r+1)(2r-1)}{2r+3} + \epsilon}\right),$ $(\tfrac{1}{2} \leqslant r).$

§ 6. The method used in § 5 is the natural generalization of that by which Landau showed that

$$R(x, 1) = O(x^{\frac{2}{3} + \epsilon})*.$$

Equation (5·5) gives an upper bound to the order of $R(x, r)$ which remains better than Ramanujan's (1·11) so long as

$$r < \frac{1 + \sqrt{17}}{4} = 1·28\ldots$$

It is, however, possible so to modify the method as to obtain an asymptotic equation for $R(x, r)$ which is better than (5·5) if $r > 1$, and better than (1·11) if $r < \dfrac{\sqrt{89} - 3}{4} = 1·61\ldots$

We return therefore to equation (3·3), which was, of course, proved independently of any particular hypothesis concerning r; but in this equation we take the line $\sigma = \beta$ to be situated no longer to the left of the critical strip for the function $\zeta(s + r)$ but actually *inside this strip*; *i.e.* we now suppose that

(6·1) $-r < \beta < 1 - r < 0.$

The integrals in (3·3) which are taken along the lines $t = \pm\, \omega$ are again readily estimated by means of Lindelöf's Theorem and well-known expressions for the order of the ζ-function inside and outside the critical strip; we thus find

(6·2) $\displaystyle\int_{\beta \pm i\omega}^{a \pm i\omega} \frac{x^{s+r}}{s(s+r)} \zeta(s) \zeta(s+r)\, ds = O\left(\frac{x^{r+a}}{\omega^2}\right)$
$$+ O\left(\frac{x^{r+\beta}}{\omega^{1 + \frac{1}{2}r + \frac{3}{2}\beta - \epsilon}}\right) \quad (\epsilon > 0).$$

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* See footnote, p. 140. For what concerns the natural analogue of Landau's "deeper" result that $R(x, 1) = O(x^{\frac{2}{6}})$, see below, § 9.

In the integral along $\sigma = \beta$ the factor $\zeta(s)$ is transformed by means of the functional equation, but the factor $\zeta(s+r)$ by means of an important equation recently discovered by Hardy and Littlewood and named by them the Approximate Functional Equation for the ζ-function. The Hardy-Littlewood Theorem gives the following representation for the ζ-function in the critical strip*: *if σ is fixed* and

(6·3) $0 < \sigma < 1, \ \xi > A, \ \eta > A, \ 2\pi\xi\eta = |t|,$

then

(6·35) $\zeta(s) = \underset{n<\xi}{\Sigma} n^{-s} + \chi \underset{n<\eta}{\Sigma} n^{s-1} + O(\xi^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma}\eta^{\sigma-1}),$

where A is a certain constant and

(6·37) $\chi = \left(\dfrac{|t|}{2\pi e}\right)^{-it} \left(\dfrac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} e^{\frac{1}{4}\pi i \operatorname{sgn} t}.$

Taking

(6·4) $\xi = \eta = \sqrt{(|t|/2\pi)}$

in this formula, carrying out the steps indicated, and approximating to the Γ functions as in §3, we find that, as x increases,

(6·5) $\displaystyle\int_{\beta-iB}^{\beta+i\omega} \frac{x^{s+r}}{s(s+r)} \zeta(s)\,\zeta(s+r)\,ds$

$= x^{9+r}\left[K_1 \overset{\infty}{\underset{m=1}{\Sigma}} \int_B^\omega \underset{n<\xi}{\Sigma} \dfrac{t^{-\frac{3}{2}-\beta}\,e^{-ti(\log t - w)}}{m^{1-\beta}\,n^{r+\beta}}\,dt \right.$

$\left. + K_2 \overset{\infty}{\underset{m=1}{\Sigma}} \int_B^\omega \underset{n<\xi}{\Sigma} \dfrac{t^{-1-r-2\beta}\,e^{-2ti(\log t - w)}}{m^{1-\beta}\,n^{1-r-\beta}}\,dt + O\left(\int^\omega t^{-\frac{3}{2}-\frac{3}{2}\beta-\frac{1}{2}r}\,dt\right)\right]$

$= x^{9+r}\{K_1 S_1 + K_2 S_2 + O(\omega^{-\frac{1}{2}-\frac{3}{2}\beta-\frac{1}{2}r})\},$

say. Here K_1, K_2 and B are constants, and w is independent of t.

In the integrals occurring in S_1 and S_2, ξ is a function of t, and we find, on inverting the order of integration and summation with regard to n, that, in virtue of (6·4)

(6·6) $S_1 = \overset{\infty}{\underset{m=1}{\Sigma}} \underset{n<\sqrt{\left(\frac{\omega}{2\pi}\right)}}{\Sigma} \displaystyle\int_B^{2\pi n^2} \dfrac{t^{-\frac{3}{2}-\beta}\,e^{-ti(\log t - w)}}{m^{1-\beta}\,n^{r+\beta}}\,dt,$

(6·65) $S_2 = \overset{\infty}{\underset{m=1}{\Sigma}} \underset{n<\sqrt{\left(\frac{\omega}{2\pi}\right)}}{\Sigma} \displaystyle\int_B^{2\pi n^2} \dfrac{t^{-1-r-2\beta}\,e^{-2ti(\log t - w)}}{m^{1-\beta}\,n^{1-r-\beta}}\,dt.$

For effective application of Lemma 2 to the integrals on the right of (6·6) and (6·65) it is necessary that both

(6·7) $-\frac{3}{2}-\beta \geqslant -\frac{1}{2}, \ i.e. \ \beta \leqslant -1$ and

(6·75) $-1-r-2\beta \geqslant -\frac{1}{2}, \ i.e. \ \beta \leqslant -\frac{1}{2}r-\frac{1}{4}.$

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* Hardy and Littlewood, *Proc. London Math. Soc.*, vol. xxi (1922), pp. 39–74; see also *Math. Zeitschrift*, vol. x (1921), page 301.

It is inequalities (6·1), (6·7), (6·75) which take the place of (3·4), (3·9) of our earlier method. On comparison of (6·7) and (6·75) it is seen that two cases have again to be distinguished, namely according as r is or is not less than $\frac{3}{2}$.

§ 7. If $1 < r \leqslant \frac{3}{2}$, all three inequalities are satisfied by taking $\beta = -1$. An application of Lemma 2 (with $\gamma = -\frac{1}{2}$) to (6·6) then gives

$$(7\cdot1) \quad S_1 = \sum_{m=1}^{\infty} \sum_{n < \sqrt{\left(\frac{\omega}{2\pi}\right)}} \frac{O(\log n)}{m^2\, n^{r-1}} = O(\omega^{1-\frac{1}{2}r} \log \omega).$$

Similarly Lemma 2 (with $\gamma = 1 - r \geqslant -\frac{1}{2}$) and (6·65) give

$$(7\cdot2) \quad S_2 = \sum_{m=1}^{\infty} \sum_{n < \sqrt{\left(\frac{\omega}{2\pi}\right)}} \frac{O(n^{3-2r})}{m^2\, n^{2-r}} = O(\omega^{1-\frac{1}{2}r}).$$

Thus, in virtue of (3·3), (6·2), (6·5) we now have

$$(7\cdot3) \quad \sum_{n \leqslant x} (x^r - n^r)\, \sigma_{-r}(n) = Q(x, r) + O\left(\frac{x^{r+\alpha}}{\omega^2}\right)$$
$$+ O(x^{r-1}\, \omega^{1-\frac{1}{2}r} \log \omega), \quad (\alpha > 1)$$

where, by the definition of $P(x, r)$ and $Q(x, r)$,

$$Q(x, r) = P(x, r) + O(1).$$

Taking now $\omega = x^{\frac{4}{6-r}}$ we obtain the equation

$$(7\cdot5) \quad R(x, r) = \sum_{n \leqslant x} (x^r - n^r)\, \sigma_{-r}(n) - P(x, r)$$
$$= O\left(x^{\frac{5r - r^2 - 2}{6-r} + \epsilon}\right) \quad (1 < r \leqslant \tfrac{3}{2}).$$

Since
$$\tfrac{1}{2}r - \frac{5r - r^2 - 2}{6 - r} = \frac{(r-2)^2}{2(6-r)}$$

it is seen by how much (7·5) improves upon Ramanujan's estimate (1·11) for $R(x, r)$.

§ 8. Finally, if $2 > r > \frac{3}{2}$, we satisfy all conditions by taking $\beta = -\frac{1}{2}r - \frac{1}{4}$, and so obtain, by exactly the same method as in § 7

$$S_1 = O(\omega^{\frac{1}{4}r - \frac{1}{8}}),$$
$$S_2 = O(\omega^{\frac{1}{4}r - \frac{1}{8}} \log \omega);$$

we are then led to take $\omega = x^{\frac{2(2r+5)}{2r+15}}$, and so arrive at the equation

$$(8\cdot5) \qquad R(x, r) = O\left(x^{\frac{(2r-1)(r+5)}{2r+15} + \epsilon}\right), \qquad (r > \tfrac{3}{2}).$$

It is noteworthy that the right-hand member of (8·5) is of lower order than (1·11) only if

$$r < \frac{\sqrt{89} - 3}{4} = 1 \cdot 61 \ldots$$

There would in fact appear to be an interval, $\dfrac{\sqrt{89} - 3}{4} < r < 2$, in which transcendental methods of the type here employed are incapable of giving an estimate as good as that obtained by Ramanujan by elementary methods.

§ 9. It is natural, both on general grounds and, in particular, on the analogy of the Landau "Abschätzung"

$$(9 \cdot 1) \qquad\qquad R(x, 1) = O(x^{\frac{2}{5}}),$$

to enquire whether the factors x^{ϵ} which appear in our asymptotic formula (5·5), (7·5), (8·5) for $R(x, r)$ when $r \geqslant \frac{1}{2}$ are not either entirely superfluous or at least capable of being replaced, by suitable refinements of the analysis, by logarithmic factors. On the other hand reference to Landau's report will show that (9·1) is an equation of considerable "depth"; and, in any case, the proof of it there given depends essentially on the particular value of r. And, in fact, the improvement of the general formulae in the direction indicated is found to be by no means easy : there is one case, however, in which it is almost immediate, that, namely of $r = \frac{1}{2}$. In this case we take, in (3·3) $\beta = -\frac{1}{2}$, and, in the integral along $\sigma = \beta$, transform the integrand by applying to the factor $\zeta(s)$ the Functional Equation, and to $\zeta(s + r)$ a formula of the type

$$(9 \cdot 2) \quad \zeta(it) = K \left| t \right|^{\frac{1}{2}} e^{-it (\log \ t \ - w)} \left\{ 1 + O\left(\frac{1}{|t|}\right) \right\} \sum_{n \leqslant A|t|} n^{-1+it},$$
$$(|t| \geqslant \delta > 0),$$

wherein A, K, w are independent of t. We thus obtain integrals of the type already discussed, leading to the equation

$$(9 \cdot 3) \qquad\qquad R(x, \tfrac{1}{2}) = O((\log x)^2).$$

In a similar manner taking $\beta = -1$ and using the equation

$$(9 \cdot 4) \qquad \zeta(1 + it) = \sum_{n \leqslant A|t|} n^{-1-it} + O\left(\frac{1}{|t|}\right), \ (|t| \geqslant \delta > 0)*,$$

we show that

$$(9 \cdot 5) \qquad\qquad R(x, 2) = O(x (\log x)^2).$$

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* For (9·4), see, for example, Hardy and Littlewood, "The zeros of Riemann's Zeta-Functions on the critical line," *Math. Zeitschrift*, vol. x (1921), pp. 283–317 (p. 285, lemma 2). Equation (9·2), is, of course, a consequence of (9·4) and the functional equation.

Finally, when $r > 2$, we take $\beta = -1$, and no transformation is then needed for the factor $\zeta(s+r)$ which is already expressible as an absolutely and uniformly convergent series; this leads to

$$(9{\cdot}6) \qquad R(x, r) = O(x^{r-1} \log x) \qquad\qquad (r > 2).$$

It is again noteworthy that each of the equations $(9{\cdot}5)$, $(9{\cdot}6)$ is, on comparison with $(1{\cdot}11)$ seen to contain a superfluous factor $\log x$.

§10. Thus, to sum up, the present note shows that transcendental methods enable us to replace Ramanujan's equation

$$(1{\cdot}11) \qquad R(x, r) = O(x^{\frac{1}{2}r}) \qquad\qquad (0 < r < 2)$$

by the set of equations

$$(4{\cdot}5) \qquad R(x, r) = O(x^{-\frac{1}{2}(\frac{1}{2}-r)} \log x) \qquad (0 < r < \tfrac{1}{2}).$$

$$(9{\cdot}3) \qquad R(x, \tfrac{1}{2}) = O((\log x)^2).$$

$$(5{\cdot}5) \qquad R(x, r) = O\left(x^{\frac{(2r-1)(r+1)}{2r+3}+\epsilon}\right) \qquad (\tfrac{1}{2} < r \leqslant 1).$$

$$(7{\cdot}5) \qquad R(x, r) = O\left(x^{\frac{5r-r^2-2}{6-r}+\epsilon}\right) \qquad (1 \leqslant r \leqslant \tfrac{3}{2}).$$

$$(8{\cdot}5) \qquad R(x, r) = O\left(x^{\frac{(2r-1)(r+5)}{2r+15}+\epsilon}\right)$$

$$\left(\tfrac{3}{2} \leqslant r < \frac{\sqrt{89}-3}{4}\right).$$

It would appear that for tolerably small values of r formulae $(2{\cdot}1)$, $(2{\cdot}15)$ give an unexpectedly close approximation to the value of the arithmetic function discussed, even for a quite small value of x. Thus, taking $r = \tfrac{1}{2}$, $x = 25$, I find, by computation, that

$$\sum_{n \leq x}(x^r - n^r)\,\sigma_{-r}(n) = 75{\cdot}7\ldots,$$
$$P(x, r) \qquad = 75{\cdot}5\ldots,$$
$$Q(x, r) \qquad = 76{\cdot}0\ldots.$$

On the Analytical Representation of Congruences of Conics. By
C. G. F. JAMES, Trinity College, Cambridge. (Communicated by
Professor H. F. BAKER.)

[*Read* 15 May, 1922.]

§ 1. A congruence of space curves is a system of curves de-
pending on two parameters; or, for brevity, a system ∞^2. A general
curve of the system will be subjected to conditions, two less in
number than those required to determine the curve as one of a
finite set of solutions. These will be in general conditions of contact
with a fixed surface, the *Focal Surface*; including, in special cases,
conditions of incidence with fixed points or curves. The *order* of the
system is defined as the number of curves which pass through a
given arbitrary point; the *class* as the number which have a given
line as chord. A *Singular Point* of such a system is a point through
which ∞^1 curves pass; a *Fundamental Point* is one through which
all the curves pass. The singular points, if any exist, will in general
lie on a curve, or curves; except in the special case when the order
is zero. In this case we have a doubly infinite system of curves on
a surface, every point of which is singular. The fundamental points
can be at most finite in number, provided all the curves do not
degenerate, and contain a fixed part.

In the same way there will be, in general, a congruence of
Singular Chords, or lines which are met twice by ∞^1 curves of the
system, and at most a finite number of *Fundamental Chords*, which
meet twice all the curves. Neither fundamental points nor chords
will exist in general.

§ 2. A congruence of conics will be formed by curves satisfying
six independent conditions. If the system is linear these conditions
are usually conditions of incidence. Such congruences may be ob-
tained in at least three ways:

(*a*) As defined directly by suitable conditions of incidence;

(*b*) As defined by the intersection, residual to a suitable base
group of

(1) Any pair of surfaces of two distinct pencils,

or　(2) A surface S of a pencil with any surface of a pencil ϖ,
belonging to a system ∞^2 of surfaces, not necessarily
linear, the pencil (S) and the system (ϖ) being in pro-
jective correspondence,

or   (3) Corresponding surfaces of two collinear nets*;

(*c*) As the residual base curve of pencils in a net of surfaces with a suitable base group.

Similar methods, of course, are applicable to systems of any order.

The discussion of congruences of conics appears to have been opened by Montesano†, who discusses synthetically the unique type of congruence of the first order and class. Pieri ‡ has considered congruences of conics with a singular conic. Some of these systems appear as special cases of congruences obtained in this paper. The bilinear congruence is also obtained, and will be quickly passed over. Congruences of conics of order 2 have also been discussed by various authors§.

§ 3. The object of this paper is to discuss a general class of congruence, to which we are led by the consideration of the matrix equation of a conic. Stuyvaert‖ has shewn that the matrix equation ¶

$$\left\| \begin{array}{ccc} a_x & b_x & c_x \\ a_x' & b_x' & c_x' \end{array} \right\| = 0 \quad \dots\dots\dots\dots\dots\dots(1),$$

where the a_x etc. are linear forms in $(x_1 \dots x_4)$, represents a congruence of space *cubics*, when these forms are also homogeneous forms in parameters $\alpha_1, \alpha_2, \alpha_3$. In particular he considers the case of forms linear in the α's, giving rise in general to a congruence of order 3, and finds that in six special cases this reduces to unity. Now this classification can plainly be used for congruences of curves of any order, and, in particular, for conics; for the order in x has no effect on the order of the system, which is determined solely by

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* A net of surfaces is a system of surfaces dependent linearly on three homogeneous parameters. The typical surface has an equation $a_1 u + a_2 v + a_3 w = 0$; where u, v, w, equated to zero represent fixed surfaces. Similar definitions hold for nets of plane curves, etc.

† "Un sistema lineare di coniche nello spazio," *Atti Accad. Torino*, t. xxvii, p. 660, 1892. A paper entitled "Sur un complexe remarquable etc.," Humbert, *Jour. Ecole. Poly.* Cr. 64, p. 123, 1894, has an important bearing on a special case.

‡ "Sopra alcune congruenze di coniche," *Atti Accad. Torino*, t. xxviii, p. 289, 1893. One of his congruences has been considered in greater detail, and apparently independently, by Stuyvaert "On Certain Twisted Sextics." *Royal Acad. of Sciences of Amsterdam* (English ed.), vol. xi, i, p. 346, 1908.

§ See "Sulle Congruenze [2, 1] di coniche nello spazio," Veneroni, *Rend. Ist. Lomb.*, Ser. 11, t. liv, p. 166, 1921. "Tipi particolari di sistemi [2, 1] di coniche nello spazio," *Id.*, *Id.*, p. 383.

‖ "Congruences de triangles, cubiques, etc.," *Crelle's Journal*, Bd. 132, p. 216, 1907. See also for later developments, and complete references a recent memoir, *Congruences de Cubiques Gauches*, Ghent, 1920.

¶ Throughout this paper a_{x^n} will represent an homogenous n-tic in four variables $x_1 \dots x_4$; for $n = 2$ ($a_x a_y$) will represent its first polar form with respect to y etc. Also a_x^1 will be replaced by a_x.

the number of variable points of intersections of curves of the net

$$| \; a_x \;\; a_x' \;\; A \; | = 0,$$

regarded as an equation in α, x being fixed*.

In the first part of this paper the matrix equation of a typical conic is taken as

$$\left\| \begin{array}{ccc} a_x^2 & a_x & a \\ a_x'^2 & a_x' & a' \end{array} \right\| = 0,$$

so that each of our six linear types will be sub-divided, on account of the lack of symmetry in the arrangement.

§ 4. The first type is

(I) $\left\| \begin{array}{ccc} \alpha_1 a_x^2 + \alpha_2 b_x^2 + \alpha_3 c_x^2 & \alpha_1 a_x + \alpha_2 b_x + \alpha_3 c_x & \alpha_1 a + \alpha_2 b + \alpha_3 c \\ d_x^2 & d_x & d \end{array} \right\| = 0$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots(2).$$

Being symmetrical from column to column this type is not sub-divided.

The second type gives rise to three congruences:

(II) $\left\| \begin{array}{cccc} \alpha_1 a_x^2 + \alpha_2 b_x^2 + \alpha_3 c_x^2 & \alpha_1 a_x + \alpha_2 b_x + \alpha_3 c_x & a \\ \alpha_1 a_x'^2 + \alpha_2 b_x'^2 + \alpha_3 c_x'^2 & \alpha_1 a_x' + \alpha_2 b_x' + \alpha_3 c_x' & a' \end{array} \right\| = 0 \ldots (3):$

(III) $\left\| \begin{array}{cccc} \alpha_1 a_x^2 + \alpha_2 b_x^2 + \alpha_3 c_x^2 & a_x & \alpha_1 a + \alpha_2 b + \alpha_3 c \\ \alpha_1 a_x'^2 + \alpha_2 b_x'^2 + \alpha_3 c_x'^2 & a_x' & \alpha_1 a' + \alpha_2 b' + \alpha_3 c' \end{array} \right\| = 0 \ldots (4);$

(IV) $\left\| \begin{array}{cccc} a_x^2 & \alpha_1 a_x + \alpha_2 b_x + \alpha_3 c_x & \alpha_1 a + \alpha_2 b + \alpha_3 c \\ a_x'^2 & \alpha_1 a_x' + \alpha_2 b_x' + \alpha_3 c_x' & \alpha_1 a' + \alpha_2 b' + \alpha_3 c' \end{array} \right\| = 0 \ldots (5);$

and so on. The remaining types lead only to special cases of the congruences given by types I and II of Stuyvaert classification. The results are stated briefly in an appendix, § 45. These expressions can be reduced in various ways when necessary. Thus the last column in III or IV can be replaced by α_1, α_2. As a rule, however, little is gained by this; and we shall generally use the forms as written on account of the symmetry of the results.

It is not possible to take over the class of the systems from the corresponding theory for cubics. However, they are almost immediately found from the geometry of the systems. They will, for brevity of reference, be denoted by $\Gamma_1 \ldots \Gamma_4$. Γ_1 and Γ_2 are both bilinear congruences of Montesano's Type, in each case of general type. The representation of both is very largely redundant, for they can plainly both be reduced to

$$\left. \begin{array}{l} \alpha_1 a_x^2 + \alpha_2 b_x^2 + \alpha_3 c_x^2 = 0 \\ \alpha_1 a_x + \alpha_2 b_x + \alpha_3 c_x = 0 \end{array} \right\} \quad \ldots\ldots\ldots\ldots(3\,\mathrm{a}).$$

* When only one row or column of a determinant or matrix is written, the usual convention adopted in this paper is that in the unwritten rows or columns, a is to be replaced by b and c respectively, in the variable letters.

However the matrices, as written, bring out new methods of generating the congruence; while the discussion is almost as simple. Hence we shall run through the principal properties of Γ_1, using the form as written. We can immediately express our results, to correspond to the simple form above, by writing

$$d_x{}^2 \equiv 0, \ d_x \equiv 0.$$

There is an almost perfect parallel between the theory of Γ_3 and that of Γ_4. It must be admitted that apart from this Γ_3 is not of much interest. A special case of Γ_4 occurs in the paper by Pieri (§ 7), but is not discussed in detail.

A second set of congruences is associated in the same way with a second matrix equation for a conic (§ 38 of this paper, *et seq.*). No new congruences, however, arise from this representation for the first two Stuyvaert types.

It might be remarked that the analytical results will hold equally for $(n-2)$-dimensional quadric varieties· in n-dimensional linear space. The interpretation, of course, will often be considerably different.

§ 5. *The Congruence* Γ_1.

$$\left\| \begin{array}{ccc} \alpha_1 a_x{}^2 + \alpha_2 b_x{}^2 + \alpha_3 c_x{}^2 & \alpha_1 a_x + \alpha_2 b_x + \alpha_3 c_x & \alpha_1 a + \alpha_2 b + \alpha_3 c \\ d_x{}^2 & d_x & d \end{array} \right\| = 0 \qquad \ldots\ldots(2).$$

The elements of the first row will occasionally be denoted, for brevity, by $g_x{}^2, g_x, g$, respectively. The conics of Γ_1 are given by the intersection of the planes of a star[*]

$$\left| \begin{array}{cc} g_x & g \\ d_x & d \end{array} \right| = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

the centre F of this star being

$$da_x - d_x a = db_x - d_2 b = dc_x - d_x c = 0,$$

with the corresponding quadric of a collinear net,

$$\left| \begin{array}{cc} g_x{}^2 & g_x \\ d_x{}^2 & d_x \end{array} \right| = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

whose base points are

$$\left\| \begin{array}{cccc} a_x{}^2 & b_x{}^2 & c_x{}^2 & d_x{}^2 \\ a'' & b'' & c'' & d'' \end{array} \right\| = 0.$$

Alternatively they lie on a net of cubic surfaces,

$$\left| \begin{array}{cc} g_x{}^2 & g_x \\ d_x{}^2 & d_x \end{array} \right| = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

residual to the lines of a plane pencil,

$$g_x = d_x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

[*] The figure formed by all the planes and lines through a fixed point, its centre.

§ 6. The parameters giving the conic through an arbitrary point (y) are obtained by solving

$$g_y{}^2 + \omega\, d_y{}^2 = g_y + \omega\, d_y = g + \omega d = 0.$$

Hence they are

$$\alpha_1 : \alpha_2 : \alpha_3 : \omega = \mid b_y{}^2 c_y d \mid : \mid c_y{}^2 a_y d \mid : \mid a_y{}^2 b_y d \mid : \mid a_y{}^2 b_y c \mid,$$

where, for instance, the first determinant is in full

$$\begin{vmatrix} b_y{}^2 & c_y{}^2 & d_y{}^2 \\ b_y & c_y & d_y \\ b & c & d \end{vmatrix}.$$

The plane of the conic, and the quadric on which it lies can be written down at once.

The solution becomes indeterminate for points on the curve

$$\begin{Vmatrix} a_x{}^2 & b_x{}^2 & c_x{}^2 & d_x{}^2 \\ a_x & b_x & c_x & d_x \\ a & b & c & d \end{Vmatrix} = 0 \quad \dots\dots\dots\dots\dots(10),$$

which, as the intersection of two cubic surfaces residual to a conic, is a curve of order 7 and genus 5. Such a curve will be denoted by $c_7{}^5$.

It contains the centre of the star of planes, and the base points of the nets of quadrics and cubics. Further we can plainly add to the first column of our matrix in (2), the second column multiplied by an arbitrary linear form ϕ_x. Thus the net of quadrics is to a large extent arbitrary, its base points describing on $c_7{}^5$ a linear series* $g_8{}^4$ of dimension 8 and freedom 4. For the further theory of this series we must refer to the paper by Montesano previously cited. In the same way we can vary the net of cubic surfaces.

§ 7. There is a second system of cubic surfaces of more importance in the theory of Γ_1, namely the system circumscribed to $c_7{}^5$

$$\mid a_x{}^2\ a_x\ a\ \lambda_a \mid = 0 \quad \dots\dots\dots\dots\dots(11).$$

This† is apparently a linear system ∞^3, but is essentially a net. Two surfaces of the system intersect again in a conic. Thus of the second surface being given by $\lambda_a{}'$ they intersect in

$$\parallel \lambda_a\ \lambda_a{}'\ a\ a_x\ a_x{}^2 \parallel = 0.$$

By combination of rows the first row can be replaced by

$$\{\bar\lambda\ \bar\lambda{}'\ g\ g_x\ g_x{}^2\},$$

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* Namely the intersection with the septimic of a linear family of surfaces ∞^4, such that, any fixed points being discarded, there are eight points regarded as variable.

† The convention in §§ 7–11 in these determinants is that the remaining rows have the same form in b, c, d respectively.

and putting $\bar{\lambda} = \bar{\lambda}' = \lambda_d = \lambda_d' = 0$, this matrix becomes

$$\left\| \begin{array}{ccc} g & g_x & g_x^2 \\ d & d_x & d_x^2 \end{array} \right\| = 0.$$

Thus this system of conics is, in fact, Γ_1. We deduce at once from the theory of curves on surfaces that the conics of Γ_1 are six-secant to c_7^5.

§ 8. Assigning a fixed point P, given by (y), the following surfaces are plainly the same:

(1) The locus of conics whose planes pass through P,

(2) The locus of points of intersection of those chords of conics of Γ_1, which pass through P, with these conics,

and (3) The locus of conics in the planes of the pencil, whose axis is PF.

This pencil is given by

$$A_y \alpha_1 + B_y \alpha_2 + C_y \alpha_3 = 0 \quad \dots\dots\dots\dots\dots(12),$$

where $A_y \equiv a_y d - a d_y$, etc.

Also we can find ω so that

$$d + \omega g = d_x + \omega g_x = d_x^2 + \omega g_x^2 = 0 \quad \dots\dots\dots(13).$$

Eliminating α and ω from (12) and (13) we obtain a cubic surface, passing through c_7^5 and PF. Its equation is

$$T^3 \equiv | a_x^2 \ a_x \ a \ A_y | = 0, \text{ where } D_y \equiv 0 \quad \dots\dots(14).$$

This can be easily modified into

$$T^3 \equiv | a_x^2 \ a_x \ a_y \ a | = 0 \quad \dots\dots\dots\dots\dots(15)$$

As an equation in y it represents the plane of the conic through x.

§ 9. If y lies on c_7^5 the same equation gives the surface filled by conics passing through P. In this case it acquires a node at P. Two such surfaces related to points P, P' on c_7^5 intersect again in the conic in the plane FPP'.

§ 10. The conic through y is given by equations

$$\left. \begin{array}{c} | a_x^2 \ a_y^2 \ a_y \ a | = 0 \\ | a_x \ a_y^2 \ a_y \ a | = 0 \end{array} \right\} \quad \dots\dots\dots\dots(16).$$

The tangent to this conic at y itself is

$$\left. \begin{array}{c} | (a_x a_y) \ a_y^2 \ a_y \ a | = 0 \\ | a_x \ a_y^2 \ a_y \ a | = 0 \end{array} \right\}.$$

In variables y this represents the intersection of a cubic and quartic residual to c_7^5.

It is therefore a quintic curve of genus two, locus of points of contact of tangents from x to conics of Γ_1. It passes through P, and has PF as trisecant, since PF is touched by two conics of Γ_1. It is also the curve of intersection of the surface T^3 for P and its first polar with respect to P.

The complex of tangents to Γ is thus of degree 4, containing doubly the star (F) of singular chords.

§ 11. If in (16) we write $x = \alpha + k\beta$ and eliminate k we obtain the surface generated by conics meeting the line (α, β), namely

$$T^9 \equiv |\, a_\alpha{}^2 \ a_y{}^2 \ a_y \ a \ |^2$$
$$- 2 \,|\, (a_\alpha a_\beta) \ a_y{}^2 \ a_y \ a \,|\,|\, a_\alpha \ a_y{}^2 \ a_y \ a \,|\,|\, a_\beta \ a_y{}^2 \ a_y \ a \,|$$
$$+ |\, a_\beta{}^2 \ a_y{}^2 \ a_y \ a \,|^2 \,|\, a_\alpha \ a_y{}^2 \ a_y \ a \,|^2 \ \ldots\ldots(17).$$

It has $c_7{}^5$ as triple curve; and as nodal conic the conic having (α, β) as chord.

Thus 9 conics of Γ_1 meet two arbitrary lines, and the conics which meet an arbitrary n-tic in space fill a surface of order $9n$.

From this result we can deduce that 10 conics of Γ_1 meet an arbitrary conic twice. In particular Γ_1 contains 10 circles.

§ 12. The system of degenerate conics is easily discussed from the geometrical configuration*.

The conics of Γ_1 cut an arbitrary plane in points generating a Geiser involution†. If, then, we regard our forms $a_x{}^2$, etc. as functions of three variables only we have an analytical representation of this involution.

It is interesting to note that if we take the system of base curves of a net of quadrics, we can relate them projectively to the lines of a star. The locus of intersection of such curves as meet is a $c_7{}^5$. This corresponds to the well-known generation of the space cubic by means of collinear stars‡.

§ 13. *The congruence Γ_2.*

$$\left|\begin{array}{cccc} \alpha_1 a_x{}^2 + \alpha_2 b_x{}^2 + \alpha_3 c_x{}^2 & \alpha_1 a_x + \alpha_2 b_x + \alpha_3 c_x & a \\ \alpha_1 a_x{}'^2 + \alpha_2 b_x{}'^2 + \alpha_3 c_x{}'^2 & \alpha_1 a_x' + \alpha_2 b_x' + \alpha_3 c_x' & a' \end{array}\right| = 0 \ldots(3).$$

This form of the matrix equation of Montesano's Congruence expresses that, in addition to the usual generation, the congruence can also be generated by the intersection of a star of planes with corresponding members of a projective quadratic system ∞^2 of

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* Montesano § 2, *loc. cit.* in note (†) p. 151.

† This is immediately obvious. For further details see the same paper.

‡ Reye, *Géométrie de Position* (French ed.), t. II, p. 99.

cubic surfaces. The residual intersections form a congruence [13] of chords of the cubic

$$\left\| \begin{array}{ccc} a_x & b_x & c_x \\ a_x{}' & b_x{}' & c_x{}' \end{array} \right\| = 0.$$

This cubic passes through the centre of the star of planes. The singular septimic is

$$\left\| \begin{array}{c} a_x{}^2 a' - a_x{}'^2 a \\ a_x a' - a_x{}' a \end{array} \right\| = 0 \quad \dots\dots\dots\dots(18).$$

Conversely, for a given congruence, we easily see that there is one such mode of generation for every cubic through F.

In the same way it can be described by the intersection of a net of quadrics with the system of cubics; the residual curve being in this case a member of the congruence of quartics

$$\Sigma \alpha_1 a_1{}^2 = \Sigma \alpha_1 a_x{}'^2 = 0.$$

These quartics are 16-secant to the curve

$$\left\| \begin{array}{ccc} a_x{}^2 & b_x{}^2 & c_x{}^2 \\ a_x{}'^2 & b_x{}'^2 & c_x{}'^2 \end{array} \right\| = 0,$$

which passes through the base group of the net of quadrics, and is also extremely arbitrary in a manner analogous to the cubic.

§ 14. *The congruence* Γ_3.

$$\left\| \begin{array}{cccc} \alpha_1 a_x{}^2 + \alpha_2 b_x{}^2 + \alpha_3 c_x{}^2 & a_x & \alpha_1 a + \alpha_2 b + \alpha_3 c \\ \alpha_1 x_x{}'^2 + \alpha_2 b_x{}'^2 + \alpha_3 c_x{}'^2 & a_x{}' & \alpha_1 a' + \alpha_2 b' + \alpha_3 c' \end{array} \right\| = 0 \dots(4).$$

The values $\qquad \alpha_1, \ \alpha_2, \ \alpha_3 = \left\| \begin{array}{ccc} a & b & c \\ a' & b' & c' \end{array} \right\|,$

are to be excluded, for, with these values the locus ceases to be a conic, and becomes the cubic surface

$$| (a_x{}^2 a_x{}' - a_x{}'^2 a_x) \ a \ a' | = 0 \dots\dots\dots\dots(19).$$

This surface passes through the singular curve (§ 15), and will be referred to as the special cubic surface.

The conics of Γ_3 lie in the planes of a pencil

$$\Sigma \alpha_1 (a_x a' - a_x{}' a) = 0,$$

and on the cubic surfaces

$$\Sigma \alpha_1 (a_x{}^2 a_x{}' - a_x{}'^2 a_x) = 0.$$

These have as axis and base line respectively the line l given by

$$a_x = a_x{}' = 0.$$

Alternatively they lie on quadrics of a quadratic system

$$\Sigma \alpha_1 (a_x{}^2 a' - a_x{}'^2 a) + \alpha \beta \, (a_x{}^2 b' + b_x{}^2 a' - a_x{}'^2 b - b_x{}'^2 a) = 0$$

which will be seen to reduce to a rational sequence of pencils, namely a sequence of pencils in which the base curves can be put into (1, 1) correspondence with the points of a line.

In each plane of the pencil (*l*) lie ∞^1 conics of Γ_3 forming a pencil. Hence the congruence is of class zero. The special linear complex of lines (*l*), counted twice, is the complex of tangents to the conics. It is also the system of singular chords, an illustration of the fact that these can form a complex only if the class is zero.

§ 15. *The Singular Curve* of Γ_3 is of order 10, and is given by

$$c_{10} \equiv \left\| \begin{array}{l} a_x^2 a_x' - a_x'^2 a_x \\ a a_x' \quad - a' a_x \end{array} \right\| = 0 \quad \ldots\ldots\ldots\ldots(20).$$

It must be the locus of base points of the pencil of conics in the planes of (*l*), and hence must have *l* as six-secant. It contains the 20 base points of the net of cubic surfaces. This net is to a large extent arbitrary, these base points varying in a g_{20}^8.

c_{10} lies on a net of quartic surfaces with, in addition, a double base line *l*,

$$| \, \lambda_1 \ a_x^2 a_x' - a_x'^2 a_x \ a a_x' - a' a_x \, | = 0 \quad \ldots\ldots\ldots(21).$$

Two such surfaces intersect in c_{10} and a conic of Γ_3,

$$\| \, \lambda_1 \ \lambda_1' \ a_x^2 a_x' - a_x'^2 a_x \ a a_x' - a' a_x \, \| = 0.$$

If (λ) is fixed, while (λ') varies, we get all the conics on the quartic λ. Conversely, if a definite conic is given it lies on all quartics for which

$$\alpha_1 \lambda_1 + \alpha_2 \lambda_2 + \alpha_3 \lambda_3 = 0,$$

namely, a pencil. Thus Γ_3 is generated by the residual base curves of pencils of a net of quartic surfaces $R^4 \equiv c_{10} l^2$.

We can write c_{10} in the form

$$\left\| \begin{array}{ccc} A_x^2 a_x' - A_x'^2 a_x & b_x^2 a_x' - b_n'^2 a_x & c_x^2 a_x' - c_x'^2 a_x \\ 0 & b a_x' - b' a_x & c a_x' - c' a_x \end{array} \right\| = 0,$$

where $\quad A_x^2 = \Sigma \, (bc' - b'c) \, a_x^2 \, ; \ A_x'^2 = \Sigma \, (bc' - b'c) \, a_x'^2.$

It is therefore also the intersection of

$$A_x^2 a_x' - A_x'^2 a_x = 0,$$

$$(b_x^2 a_x' - b_x'^2 a_x)(c A_x'^2 - c' A_x^2) - (c_x^2 a_x' - c_x'^2 a_x)(b A_x'^2 - b' A_x^2) = 0.$$

Hence c_{10} is given by the intersection of a cubic and a quintic residual to a line and a quartic of the first species, which do not meet. It is therefore of genus* 10.

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* Or from " The Curves which lie on a cubic surface," Baker, *Proc. Lond. Math. Soc.*, Ser. 2, XI, 1912, p. 290; since it is also the intersection of a cubic and a quartic surface, having a line of the cubic as double line.

§ 17. *The configuration of the degenerate conics.* In the first place the base points of the pencils of conics describe on C_{10} a linear series g_4^1. There will be[*] $N' = 2(p+3)$ double points of this series, arising from

(*a*) Proper nodes on c_{10}.

(*b*) From points whose tangent lines meet l. The latter number is equal to $R - 12$ where R is the rank of c_{10}. It is reasonable to suppose c_{10} has no proper cusps[†], and hence from the Salmon-Cayley Formulae, $R = 18 + 2p$.

Hence we can conclude that there are no proper nodes.

Each plane of the pencil (l) contains three degenerate conics, namely, the joins of the points c_{10} outside l; say, $P_1 \dots P_4$. They describe a scroll on which c_{10} is triple. It has in addition a nodal curve generated by the diagonal points R_1, R_2, R_3; and has l as a m-fold line, where m is the number of degenerate conics which pass through a point of l.

§ 18. *To find the equation of the nodal curve.* It will be shewn immediately (§ 22) that the quadrics associated with the plane $a_x + \lambda a_{x'} = 0$ form a pencil whose base is

$$\left\| \begin{array}{c} a_x^2 + \lambda a_{x'}^{'2} \\ a + \lambda a' \end{array} \right\| = 0 \quad \dots\dots\dots\dots\dots\dots(22).$$

By expressing that this plane touches the quadric, given by adding a row of constants $(A, B, 0)$ to (22), at a point ξ we obtain the equation of this curve. We have given this in full for the more interesting Γ_4 in § 31. It is a curve of order 14,

$$\| (a_\iota a_\xi' - a_\iota' a_\xi) \quad \{(b_\iota b_\xi) a_\xi' - (b_\iota' b_\xi') a_\xi\} \, \{(a_\iota a_\xi) a_\xi' - (a_\iota' a_\xi') a_\xi\} (c a_\xi' - c' a_\xi)$$
$$- \{(c_\iota c_\xi) a_\xi' - (c_\iota' c_\xi') a_\xi\} (a a_\xi' - a' a_\xi) \|$$
$$- \| (a_\iota a_\xi' - a_\iota' a_\xi) \quad \{(c_\iota c_\xi) a_\xi' - (c_\iota' c_\xi') a_\xi\} (b a_\xi' - b' a_\xi)$$
$$\times \{(a_\iota a_\xi) a_\xi' - (a_\iota' a_\xi') a_\xi\} \| = 0 \quad \dots\dots(24),$$

with the obvious conventions that corresponding determinants are to be taken together, and that ι takes the values $1 \dots 4$. This is symmetrical in the variable letters a, b, c; though written in an assymmetric form.

It is of genus 11, and must plainly meet l eleven times, and c_{10} 26 times.

§ 19. We can find *the degenerate conics passing through a point η on l.* Let us express that. $\eta + \mu x$ lies on the surfaces defining a conic (α) for all values of μ; η being a point of l.

$$\left| \begin{array}{cc} \Sigma \alpha_1 a_x^2 & \Sigma \alpha a \\ \Sigma \alpha_1 a_x'^2 & \Sigma \alpha a' \end{array} \right| = \left| \begin{array}{cc} a_x & \Sigma \alpha a \\ a_{x'} & \Sigma \alpha a' \end{array} \right| = 0.$$

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[*] Severi, *Lezioni di Geometria Algebrica*, § 68, p. 234.
[†] This is of course proved by the relations obtained.

We find on eliminating the α

$$\| \{(a_\eta a_x)\, a_x{}' - (a_\eta{}' a_x{}')\, a_x\}\ (a_x{}^2 a_x{}' - a_x{}'^2 a_x)\ (a_\eta{}^2 a_x{}' - a_\eta{}'^2 a_x)$$
$$\times (a\, a_x{}' - a'\, a_x)\,\| = 0\ \ldots\ldots(25).$$

This represents 11 lines, besides l.

Hence $m = 11$. This number is of course simply the number of apparent nodes for a point on l.

Hence *the scroll formed by the degenerate conics* is of order* 17. It has a triple $c_{10}{}^{10}$, a nodal $c_{14}{}^{11}$, and an 11-fold line, which meets c_{10} six times and c_{14} eleven times. The first six points are points at which three tangent planes coincide, while the latter are simple pinch points. We can find its equation by taking α and β as two fixed points on l, and eliminating k between

$$| A_x{}^3\ A_x\, (a_a a_x)\, a_x{}' - (a_a{}'\, a_x{}')\, a_x | + k | A_x{}^3\ A_x\, a_\beta a_x a_x{}' - (a_\beta{}' a_x{}')\, a_x | = 0,$$
$$| A_x{}^3\ A_x\, a_a{}^2 a_x{}' - a_a{}'^2 a_x | + 2k | A_x{}^3\ A_x\, (a_a a_\beta)\, a_x{}' - (a_a{}' a_\beta{}')\, a_x |$$
$$+ k^2 | A_x{}^3\ A_x\, a_\beta{}^2 a_x{}' - a_\beta{}'^2 a_x | = 0.$$

Here we have written

$$A_x{}^3 \equiv a_x{}^2 a_x{}' - a_x{}'^2 a_x,\ \text{etc.},$$
$$A_x \equiv a\, a_x{}' - a'\, a_x,\ \text{etc.}$$

This is plainly a surface of order 17, with a triple c_{10}.

§ 20. The conic of Γ_3 which passes through a point y is given by

$$\alpha_1,\ \alpha_2,\ \alpha_3 = \left\|\ \begin{array}{c} a_y{}^2 a_y{}' - a_y{}'^2 a_y \\ a\, a_y{}' - a'\, a_y \end{array}\ \right\|,$$

and hence is

$$| a\, a_x{}' - a_x{}'\, a\quad a_x{}^2 a_y{}' - a_y{}'^2 a_y\quad a\, a_y{}' - a'\, a_y | = 0\ \ldots(26),$$
$$| a_x{}^2 a_x{}' - a_x{}'^2 a_x\quad a_y{}^2 a_y{}' - a_y{}'^2 a_y\quad a\, a_y{}' - a'\, a_y | = 0\ \ldots(27).$$

The first equation factorizes into the special cubic surface, and

$$a_x a_y{}' - a_x{}' a_y = 0\ \ldots\ldots\ldots\ldots\ldots\ldots(26\text{a}).$$

The tangent at y to this conic is

$$a_x a_y{}' - a_y{}' a_x = 0,$$

$$| 2\, \{(a_x a_y)\, a_y{}' - (a_x{}' a_y{}')\, a_y\}\, .\, (a_y{}^2 a_y{}' - a_y{}'^2 a_y)\, .\, (a a_y{}' - a' a_y) |$$
$$+ | (a_y{}^2 a_x{}' - a_y{}'^2 a_x)\, .\, (a_y{}^2 a_y{}' - a_y{}'^2 a_y)\, .\, (a a_y{}' - a' a_y) | = 0.$$

Invariably these represent the locus of points of contact of tangents from x to conics of Γ_3, and is expressed as the intersection of a plane through l, and a surface having l as nodal line, of which, as can be easily verified, one tangent plane at every

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* Agreeing with a well known formula for the order of a scroll of chords of a curve meeting a line. Salmon, *Analytic Geometry of Three Dimensions.* Fifth ed., vol. II, p, 92.

point is fixed, and coincides with the plane. Thus the curve is a plane cubic, as it should be.

§ 21. When x lies on the line l, the equation (26) is evanescent, and (27) can be previously modified to give

$$\mid (a_x^2 a_y' - a_x'^2 a_y) \cdot (a_y^2 a_y' - a_y'^2 a_y) \cdot (aa_y' - a'a_y) \mid = 0 \ldots(28),$$

and hence represents in y *the surface filled by the conics through a point x on l.* It is a quintic surface $R^5 \equiv c_{10} l^3$. Thus three conics pass through two prescribed points of l. Taking $a_y \equiv y_1$, $a_y' \equiv y_2$, and $x_1 = x_2 = x_3 = 0$ we verify that it has a four-fold point at x.

§ 22. *The system of base curves of the pencils of quadrics associated with the given planes of (l).*

If the plane is $a_x + \lambda a_x' = 0$, then these quadrics are

$$\mid a_x^2 + \lambda a_x'^2, \quad a_y^2 + \lambda a_y'^2, \quad a + \lambda a' \mid = 0 \quad \ldots\ldots(29),$$

where, also, $\qquad a_y + \lambda a_y' = 0.$

They form a pencil having in common a quartic of the first species,

$$\left|\begin{array}{c} a_x^2 + \lambda a_x'^2 \\ a + \lambda a' \end{array}\right| = 0 \ldots\ldots\ldots\ldots\ldots\ldots(30).$$

We can verify that c_{10} meets the plane λ where it meets this curve by writing $\lambda = -a_x/a_x'$. This is certainly a family of quadrics cutting the pencil of conics on λ. We have to verify that it is actually the sub-system of our original quadratic system, which corresponds to this plane. This is immediate, writing,

$$\alpha_1 : \alpha_2 : \alpha_3 = \left\|\begin{array}{c} a_y^2 + \lambda a_y'^2 \\ \alpha + \lambda a' \end{array}\right\| .$$

Then the sub-system is

$$\left|\begin{array}{c} \mid a_x^2 \ \ a_y^2 + \lambda a_y'^2 \ \ a + \lambda a' \mid \mid a \ \ a_y^2 + \lambda a_y'^2 \ \ a + \lambda a' \mid \\ \mid a_x'^2 \ \ a_y^2 + \lambda a_y'^2 \ \ a + \lambda a' \mid \mid a' \ \ a_y^2 + \lambda a_y'^2 \ \ a + \lambda a' \mid \end{array}\right| = 0.$$

This contains the special surface as a superfluous factor, together with the surface in (29).

These quartics generate a surface, namely, using an obvious abbreviation,

$$R^{16} \equiv \mid A_x^2 \ H_x^2 \ G_x^2 \mid = 0 \ \ldots\ldots\ldots\ldots(31),$$

where $\quad A_x^2 \equiv \left|\begin{array}{cc} a_x^2 & a_x'^2 \\ a & a' \end{array}\right|$; $\ 2H_x^2 \equiv - \left|\begin{array}{cc} a_x^2 & b_x'^2 \\ a & b' \end{array}\right| - \left|\begin{array}{cc} a_x'^2 & b_x^2 \\ a' & b \end{array}\right|$ etc.

This is obtained by writing down the envelope of the quadratic system,

$$(A_x^2 \ldots H_x^2 \, \rangle\!\langle \, \alpha_1 \alpha_2 \alpha_3)^2 = 0.$$

The quartics are all 8-secant to a fixed quartic, which is 16-secant to c_{10}.

$$\mid a_x^2 \ a \ a' \mid \ = \ \mid a_x'^2 \ a \ a' \mid \ = 0 \ \ldots\ldots\ldots(32).$$

This is a nodal curve on R^{16}. The equation of R^{16} must in fact be expressible as a quadratic form in the surfaces defining the nodal quartic. This appears most readily by putting all the constants $(a \ldots c')$ equal to zero, except a and b', which are taken as unity.

§ 23. We can write instead of (27)

$$\mid A_x^3 \ (a_y^2 a_x' - a_y'^2 a_x) \ A_x \mid \ \equiv 0 \ \ldots\ldots\ldots(33).$$

In (33) and (26 a) write $y = \alpha + k\beta$. Eliminating k we obtain the surface filled by the conics through the points of a line g joining the points α and β,

$$\mid A_x^3 \ (a_\alpha^2 a_x' - a_\alpha'^2 a_x) \ A_x \mid \ (a_\beta a_x' - a_\beta' a_x)^2$$
$$-2 \mid A_x^3 \ \{(a_\alpha a_\beta) a_x' - (a_\alpha' a_\beta') a_x\} \ A_x \mid \ (a_\beta a_x' - a_\beta' a_x)(a_\alpha a_x' - a_\alpha' a_x)$$
$$- \mid A_x^3 \ (a_\beta^2 a_x' - a_\beta'^2 a_x) \ A_x \mid \ (a_\alpha a_x' - a_\alpha' a_x)^2 = 0.$$

This is a surface $R^7 \equiv c_{10} l^5 g$. Hence seven conics of Γ_3 meet two skew lines, while six conics meet an arbitrary conic twice. *In particular Γ_3 includes six circles.*

§ 24. *The conics of Γ_3 which touch a plane $l_x = 0$* in general position.

The tangents in the plane form a pencil whose centre is the point where l meets the plane. Each line of this pencil is touched by two conics. To find the locus of the point of contact we have plainly to express that the following planes are coaxial :

$$l_x = 0,$$
$$a_x a_\xi' - a_x' a_\xi = 0,$$

and the tangent plane at ξ to

$$\mid (a_x^2 a_\xi' - a_x'^2 a_\xi) \ (a a_\xi' - a' a_\xi) \ A \mid \ = 0.$$

The last column may be taken as $A, B, 0$. Hence

$$A \mid \{(b_x b_\xi) a_\xi' - (b_x' b_\xi') a_\xi\} \ \{(c_x c_\xi) a_\xi' - (c_x' c_\xi') a_\xi\} \mid$$
$$- B \mid \{(a_x a_\xi) a_\xi' - (a_x' a_\xi') a_\xi\} \ \{(c_x c_\xi) a_\xi' - c_x' c_\xi' a_\xi\} \mid$$
$$\equiv l_x + \omega \ (a_x a_\xi' - a_x' a_\xi).$$

Eliminating A, B, ω we obtain the equation of a surface ; which, in fact, is given by bordering the matrix of c_{14} with the coefficients of l_x. [§ 18, Eqn. 24.]

This is a sextic curve having l as a four-fold line, and passing once through c_{14}. The section by $l_\xi = 0$ gives the locus of points

of contact of conics of Γ_3 which touch the plane, a sextic curve with a four-fold point, and passing through the points of c_{14} on $l_\xi = 0$. It also passes through the points of intersection of c_{10} and $l_\xi = 0$; but the above surface does not contain this curve integrally. Thus among the conics through a point of l, four touch an arbitrary plane through it. So also of those, that pass through a point of c_{10}, one touches an arbitrary plane through it, as is obvious à priori. The points on c_{14} arise from improper contact by a degenerate conic.

Considered in its plane this sextic has 26 disposable constants, and has to satisfy 34 linear conditions. Thus every plane cuts the complex $l^4 c_{10} c_{14}$ in points on a sextic, while in the most arbitrary configuration, the curve of lowest order through them is a septimic.

The conics which touch the plane lie on a surface of order 12, on which l is 8-fold and c_{10} double.

§ 25. The congruence Γ_3 is included in a more general type of congruence of class zero. The planes of a congruence of class zero must touch a developable surface, in particular they must form a pencil if the system is linear.

Consider the plane $a_x + \lambda a_x' = 0$, and the system of pencils,

$$(\Phi + \lambda \Psi) + \mu (\Phi' + \lambda \Psi') = 0,$$

where Φ etc. represent surfaces of order $(n+2)$ with an n-fold line $a_x = a_x' = 0$.

The conic through y is,

$$\left| \begin{array}{cc} a_x & a_x' \\ a_y & a_y' \end{array} \right| = \left| \begin{array}{cc} h_x & h_x' \\ h_y & h_y' \end{array} \right| = 0.$$

Here $\qquad\qquad h_x \equiv \Phi(x)\, a_x' - \Psi(x)\, a_x.$

The Singular Curve is therefore $h_x = h_x' = 0$; or

$$\left\| \begin{array}{ccc} \Phi & \Psi & a_x \\ \Phi' & \Psi' & a_x' \end{array} \right\| = 0.$$

It is of order $4(n+2)$ and meets l $4(n+1)$ times.

Γ_3 can be reduced to this form; for, writing

$$\left. \begin{array}{l} \alpha a + \beta b + \gamma c = 1 \\ \alpha a' + \beta b' + \gamma c' = 1/\lambda \\ \alpha \qquad\qquad\quad = 1/\mu \end{array} \right\};$$

which is equivalent to a quadratic transformation in the α plane, the net of cubic surfaces reduces to

$$S_1 \lambda + S_2 \mu + S_3 \lambda \mu = 0,$$

where the S_i are cubic surfaces passing through l. Thus for Γ_3 we have $\Phi = 0$; $n = 1$. The singular curve is

$$a_y \Psi_y = a_y \Psi_y' - a_y' \Psi_y = 0.$$

This is, in fact, a c_{10} with a redundant conic*.

A further generalization is to the case when the system of $(n + 2)$-tics is arbitrary. The system is linear, only if μ enters in the first power. Otherwise in any plane of the pencil we have a non-linear system of conics, enveloping a curve. The locus of this curve is the focal surface, touched four times by each conic of the system; this reduction being caused by the presence of the fundamental chord l.

Of course, the most general congruence of class zero, is given by taking a curve, or system of curves, of order n, with a $(n - 4)$-secant. The conics in a plane through this secant form a pencil, whose base points are the remaining four intersections with the curve. The representation of such a system analytically is not, however, a simple matter. We may, of course, have special cases with variable, or fixed, base points on l itself.

§ 26. *The congruence* Γ_4.

$$\begin{Vmatrix} a_x^2 & a_1 a_x + a_2 b_x + a_3 c_x & a_1 a + a_2 b + a_3 c \\ a_x'^2 & a_1 a_x' + a_2 b_x' + a_3 c_x' & a_1 a' + a_2 b' + a_3 c' \end{Vmatrix} = 0 \ \ldots(5).$$

It is described as the intersection of quadrics of a pencil, whose base is the quartic,

$$a_x^2 = a_x'^2 = 0 \quad (c_4)$$

with a net of cubic surfaces, whose base is

$$\begin{Vmatrix} a_x^2 & a_x & b_x & c_x \\ a_x'^2 & a_x' & b_x' & c_x' \end{Vmatrix} = 0,$$

namely, c_4 and seven isolated points. By adding $u_x g$ to g_x the net of cubics can be varied, the base points describing a g_7^4 on the singular curve,

$$\begin{Vmatrix} a_x a_x'^2 - a_x' a_x^2 \\ a a_x'^2 - a' a_x^2 \end{Vmatrix} = 0 \quad (c_7) \ \ldots\ldots\ldots\ldots\ldots(34),$$

where a_x^2 and $a_x'^2$ are the same from column to column. It is of order 7. c_4 is of course also singular. The conics through a point y of c_7 fill the surface

$$\mid (a_x a_x'^2 - a_x' a_x^2) \ (a a_x'^2 - a' a_x^2) \ (a a_y'^2 - a' a_y^2) \mid = 0,$$

which is simply the quadric of the pencil through y together with a surface, passing through c_7 and c_4, namely,

$$\mid (a_x a_x'^2 - a_x' a_x^2) \ a \ a' \mid = 0 \ldots\ldots\ldots\ldots\ldots(35).$$

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* This is, in fact, a conic which, being a conic of the congruence, in no way special, is represented by an infinite set of values of these parameters, namely for $\lambda = 0$, and all values of μ. This is of frequent occurrence.

This corresponds to the values

$$\alpha_1, \ \alpha_2, \ \alpha_3 = \left\| \begin{array}{ccc} a & b & c \\ a' & b' & c' \end{array} \right\|,$$

which values are to be rejected as before.

§ 27. If we consider the planes associated with a definite quadric, $a_x{}^2 + \lambda a_{x'}{}^2 = 0$, we find as for Γ_3 that they are

$$| \ a_x + \lambda a_{x'} \ a_y + \lambda a_{y'} \ a + \lambda a' \ | = 0,$$

and form a pencil, whose axis is

$$\left\| \begin{array}{c} a_x + \lambda a_{x'} \\ a + \lambda a' \end{array} \right\| = 0.$$

Thus c_7 must meet c_4 in 12 points. This can in fact be easily verified. The *locus of these lines* is a ruled cubic R^3 obtained by eliminating λ between

$$| \ a_x + \lambda a_{x'} \ b_x + \lambda b_{x'} \ | = 0,$$
$$| \ a_x + \lambda a_{x'} \ c_x + \lambda c_{x'} \ | = 0,$$

the plane $a a_{x'} - a' a_x = 0$ being rejected.

The equation is most simply obtained by writing down the envelope of

$$(A_x \ldots H_x \chi \alpha_1 \alpha_2 \alpha_3)^2 = 0,$$

the complete system of planes where

$$A_x = \left| \begin{array}{cc} a_x & a_{x'} \\ a & a' \end{array} \right|; \ 2H_x = + \left| \begin{array}{cc} a_{x'} & b_x \\ a' & b \end{array} \right| + \left| \begin{array}{cc} b_{x'} & a_x \\ b' & a \end{array} \right|, \text{ etc.}$$

It is therefore

$$| \ A_x \ H_x \ G_x \ | = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots(36).$$

Hence *the system of conics is obtained as the intersection of corresponding elements in a pencil of quadrics, and a projective rational series of axial pencils, whose bases describe a ruled cubic R^3.*

c_7 lies integrally on R^3, and meets every generator in two points. It is therefore hyperelliptic. Its genus is 3. This will be found after we have considered the degenerate conics or we can verify it as for Γ_3.

The class of the system is plainly three. Thus the conics having a given line as chord are given by the three generators of R^3 which meet the line. The singular chords must meet an infinity of generators, or coincide with one of them. Hence the only other singular chords besides the generators of R^3 are its directrices d_1, d_2. Any plane ϖ through the simple directrix d_1 contains two generators, and hence a conic associated with each intersecting in the four points $c_4 \varpi$. The equations of d_2 are

$$| \ a_x \ a \ a' \ | = | \ a_{x'} \ a \ a' \ | = 0$$

for the axis λ lies on

$$| \; a_x + \lambda a_x' \quad a + \lambda a' \quad a \; | = 0,$$

or $\qquad | \; a_x + \lambda a_x' \quad a \quad a' \; | = 0.$

This plane contains one generator, and hence the line in question is the double directrix, provided the surface in question is general; and this is so, for the axis λ is the intersection of

$$K_x + \lambda K_x' = 0,$$

and $\qquad \left| \begin{array}{cc} b_x + \lambda b_x' & c_x + \lambda c_x' \\ b + \lambda b' & c + \lambda c' \end{array} \right| = 0,$

where K_x, K_x' are the planes defining d_2. This second plane envelops a quadric cone. This is equivalent to the correlative of the well known construction* by means of a conic and a line in birational correspondence. Clearly this cone will not, in general, be touched by d_2, and hence R^3 is a cubic scroll of general type.

The directrix d_2 is not strictly a singular chord, for the inter-section of the plane

$$| \; a_x + \lambda a_x' \quad a \quad a' \; | = 0,$$

with $\qquad a_x^2 + \lambda a_x'^2 = 0,$

plainly lies on the special surface (35).

It is clear that d_2 must be a 5-secant of c_7. Similarly d_1 must be a trisecant of the same curve. Again R^3 has two singular generators s_1, s_2. The conics of Γ_4 associated with these lines may in a certain sense be considered as double conics of the congruence.

§ 28. The finding of d_1 presents no theoretical difficulty, but the algebra is involved as long as symmetry is retained. The method most in keeping with the rest of this discussion is perhaps as follows. The generator λ lies on an infinite number of plane systems

$$| \; a_x + \lambda a_x' \quad a + \lambda a' \quad A \; | = 0,$$

where A, B, C are assumed constants. Under what conditions does this reduce to a pencil ? In the first place if $A = a + \mu a'$, etc., we get the pencil (d_2). Putting this aside, we have to express that the following three planes have a line in common

$$| \; a_x \quad a \quad A \; | = 0 \dots\dots\dots\dots\dots\dots(37),$$

$$| \; a_x' \quad a \quad A \; | = 0 \dots\dots\dots\dots\dots\dots(38),$$

$$\Sigma A \left\{ \left| \begin{array}{cc} b_x' & c_x \\ b' & c \end{array} \right| + \left| \begin{array}{cc} b_x & c_x' \\ b & c' \end{array} \right| \right\} = 0 \dots\dots\dots\dots(39).$$

This gives a matrix of four rows and three columns, whose elements

* Jessop, *Treatise on the Line Complex*, chap. v, § 67, or Reye, *op. cit.* p. 265.

are linear in A, B, C. Regarding these as coordinates in a plane, the cubics represented by the first minors have a line in common. Hence the matrix vanishes for a single point only. Substituting in (37) and (38), we get two planes meeting in d_1.

As this result will not be used we do not consider it worth while to record any details.

§ 29. The pairs of conics in the planes of the pencil (d_1) fill a surface of order $4 + x$, where x is the multiplicity of d_1 on it. It has in addition two conic tropes*, c_4 as nodal curve, and passes simply through c_7^3. To find x we may suppose the planes of (d_1) given by a parameter μ, and that ξ denotes the position of a point on d_1. Then we shall have relations of the form $(\lambda^2\mu) = 0$, $(\xi^2\lambda) = 0$; and hence $(\xi^1\mu) = 0$. Thus $x = 1$. Hence the surface is of order 5. It cuts R^3 in a 15-tic, of which we know components, c_7 and d_1, of total order 8. There remains a 7-tic which is plainly the locus of intersection of the conics in the planes of the pencil (d_1) with the generator in that plane, other than the one with which the conics were primarily associated. It meets d_2 in the same five points as c_7^3, and is trisecant to d_1.

Thus we can associate with the original congruence a second congruence of the same type given by associating with the quadric $\lambda = a$ generator $\lambda' = \alpha\lambda + \beta/(\gamma\lambda + \delta)$, where α, β, γ, δ are certain definite constants. The congruences have in common ∞^1 conics filling a 5-tic surface. The two sets of double conics are also common to each.

The special cubic surface of each congruence is of the form

$$a_x^2 K_x + a_x'^2 K_x' = 0,$$

where K_x and K_x' pass through d_2. Each of these cuts the cubic scroll in d_2, counted twice, and in the corresponding c_7. There are therefore ∞^3 congruences of this type, associated with a given scroll and pencil of quadrics, each being determined uniquely by its singular curve, or by its special surface. The system of special surfaces is linear, and therefore so also is the system of septimics. All these loci are associated in pairs, as explained above. All these 7-tics have in common the 12 points where c_4 meets R^3.

§ 30. Let us turn to the study of *the degenerate conics of the congruence*. Each quadric of the pencil contains two pairs of lines forming a degenerate conic of Γ_4. They intersect in the points where the polar line l' of the homologous ray l of R^3 meets the quadric.

They will form a scroll with the following singularities:

(a) The curve c_7^3, which is a doubly directrix curve.

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* Namely the planes through the singular lines, they cannot give nodal conics, for then the quintic surface would break up.

(b) The curve c_{11}^5, generated by the double points Q_1, Q_2 of the conics, of which the same is true. Its order and genus will be verified immediately. It is hyperelliptic and in $(2, 2)$ correspondence with c_7. There are 8 branch places of the involution on c_7, namely
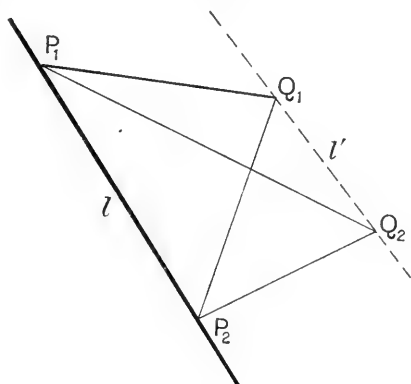
Fig. 1

the points H, where $P_1 P_2$ touches the correspondence quadrics. There are 12 on c_{11}, namely these same points, and the vertices of the 4 cones of the pencil of quadrics.

(c) The degenerate conics on these four cones, each line being a double generator.

The lines l' describe a rational quintic scroll, R^5, whose generators join the points of the involution on c_{11}. All these facts are obtained immediately from the parametric representation of the generators of R^3, namely,

$$p_y = (1, \lambda)^3.$$

The genera follow from the general formula used before (*loc. cit.* §17, p. 10). To find the order of c_{11} we consider any plane section. On it we have a rational 5-tic projective to a pencil of conics. Thus on the quintic we have a correspondence $(1, 10)$. The 11 united points are the points of c_{11} in the plane. The same method gives the order of c_7 directly, without appeal to analysis. We proceed to find the equations of c_{11}^5 and R^5; but it is useful to have these orders thus directly derived in order to have a check on redundant elements.

§ 31. Let us express that
$$| \ a_x + \lambda a_x' \quad a + \lambda a' \quad A \ | = 0,$$
touches the quadric defined by λ at a point (ξ). We have to identify it with
$$(a_x a_\xi) + \lambda (a_x' a_\xi') = 0.$$

Hence

$$(a_\iota a_\xi) + \lambda \, (a_\iota' a_\xi') + |\ a_\iota + \lambda a_\iota' \quad a + \lambda a'\quad A\ | = 0, \ \iota = 1 \ldots 4 \ldots (40),$$

or

$$(a_\iota a_\xi)\, a_\xi'^2 - (a_\iota' a_\xi')\, a_\xi^2 + |\ (a_\iota a_\xi'^2 - a_\iota' a_\xi^2)\ (a a_\xi'^2 - a' a_\xi^2)\ A\ | = 0,$$
$$\iota = 1 \ldots 4.$$

Taking $C = 0$, and eliminating A and B, we obtain, in ξ, the equation of c_{11}^5,

$$\left\|\ \{(a_\iota a_\xi)\, a_\xi'^2 - (a_\iota' a_\xi')\, a_\xi^2\}\quad \begin{vmatrix} B_\iota \xi^2 & C_\iota \xi^2 \\ B_\xi^2 & C_\xi^2 \end{vmatrix}\quad \begin{vmatrix} A_\iota \xi^2 & C_\iota \xi^2 \\ A_\xi^2 & C_\xi^2 \end{vmatrix}\quad \right\| = 0,$$

where $\quad A_\iota \xi^2 = a_\iota a_\xi'^2 - a_\iota' a_\xi^2; \ A_\xi^2 = a a_\xi'^2 - a' a_\xi^2$; etc., etc.

This is apparently a 13-tic. However, it vanishes redundantly for the quadric $C_\xi^2 = 0$, since for a point on this quadric the last two columns are the same, to a factor. We can, however, by a simple modification reject this as a factor, and we obtain finally an 11-tic*.

$$\|\ \{(a_\iota a_\xi)\, a_\xi'^2 - (a_\iota' a_\xi')\, a_\xi^2\}\ \ B_\iota \xi^2\ \ A_\iota \xi^2 C_\xi^2 - A_\xi^2 C_\iota \xi^2\ \|$$
$$- \|\ \{(a_\iota a_\xi)\, a_\xi'^2 - (a_\iota' a_\xi')\, a_\xi^2\}\ \ B_\xi^2 C_\iota \xi^2\ \ A_\iota \xi^2\ \| = 0 \ldots (41).$$

The equation of R^5 is given by eliminating A, B, and λ from (40). It contains a factor $(cc_x' - c'c_x)$ to be rejected, but the result does not appear worth recording. R^5 contains a rational quartic, and ∞' rational quintics, loci respectively of poles of planes of the pencil (d_1), and the general generating cone of R^5, namely

$$\Phi_x{}^{A\lambda^2} = |\ a_x + \lambda a_x' \quad a + \lambda a \quad A\ | = 0,$$

so that these curves are given parametrically by

$$\xi_1 = |\ k_1 + \lambda k_1' \quad a_{12} + \lambda a_{12}' \quad a_{13} + \lambda a_{13}' \quad a_{14} + \lambda a_{14}'\ |, \text{ etc.},$$

where $k_x \equiv |\ a_x \ a \ a'\ |; \ k_x' \equiv |\ a_x' \ a \ a'\ |$, and

$$\xi_1 = |\ \Phi_1{}^{A\lambda^2} \quad a_{12} + \lambda a_{12}' \quad a_{13} + \lambda a_{13}' \quad a_{14} + \lambda a_{14}'\ |, \text{ etc.},$$

respectively.

§ 32. Let us consider the correspondence on c_4 cut by the degenerate conics. The method used is also capable of application to the loci of the preceding paragraph. Let $\xi\eta$ be two points on c_4, and let us express that $\xi + \mu\eta$ lies on the quadric and cubic surface given respectively by

$$\Sigma\alpha_1 (a a_x'^2 - a' a_x^2) = 0,$$
$$\Sigma\alpha_1 (a_x a_x'^2 - a_x' a_x^2) = 0,$$

---

* Cf. the remarks on the analogous form in § 18, eqn. 24.

we have

$$\Sigma\alpha_1\left\{(a_\xi a_\eta)\,a'-(a_\xi'a_\eta')\,a\right\}=\Sigma\alpha_1\left\{(a_\xi a_\eta)\,a_\eta'-(a_\xi'a_\eta')\,a_\eta\right\}$$
$$=\Sigma\alpha_1\left\{(a_\xi a_\eta)\,a_\xi'-(a_\xi'a_\eta')\,a_\xi\right\}=0$$

whence

$$\left|\left\{(a_\xi a_\eta)\,a'-(a_\xi'a_\eta')\,a\right\}\;\left\{(a_\xi a_\eta)\,a_\eta'-(a_\xi'a_\eta')\,a_\eta\right\}\;\left\{(a_\xi a_\eta)\,a_\xi-(a_\xi'a_\eta')\,a_\xi\right\}\right|$$
$$=0,$$

which expresses the correspondence between points of c_4 such that the joining chord forms part of a degenerate conic of the system. It can be expanded into

$$(L_1\ L_2\ L_3\ L_4 \backslash\overline{a_\xi a_\eta}\ \overline{a_\xi'a_\eta'})^3=0,\ \ldots\ldots\ldots\ldots(42),$$

where the L's are independent forms linear in ξ and η. It is well known that we may put

$$\xi_1=\operatorname{sn}u,\ \xi_2=\operatorname{cn}u,\ \xi_3=\operatorname{dn}u,\ \xi_4=1\,;$$
$$\eta_1=\operatorname{sn}v,\ \eta_2=\operatorname{cn}v,\ \eta_3=\operatorname{dn}v,\ \eta_4=1,$$

for a suitable modulus. Thus, for fixed v, the function on the left in (42) is an elliptic function of u, with 4-fold poles at the poles of the Jacobian Functions. It is therefore of order 16. The correspondence expressed is plainly of valency 7, and hence is of order 9. It is symmetrical, and there[*] are 32 coincidences, giving rise to points at which a tangent to c_4 forms part of a degenerate conic of Γ_4.

§ 33. We may now turn to the consideration of certain loci determined by points and lines, etc., with reference to Γ_4.

The conic of Γ_4 through y is

$$a_x^2 a_y'^2 - a_x'^2 a_y^2 = 0$$
$$\left|\,(a_x a_y'^2 - a_x' a_y^2)\ (a a_y'^2 - a' a_y^2)\ (a_y a_y'^2 - a_y' a_y^2)\,\right|=0\Bigg\}\ \ldots(43).$$

In y the second equation represents the surface filled by conics whose planes pass through x, a surface $T^7 \equiv (c_4)^3\, c_7$. When x lies on the singular curve c_4 this becomes the locus of conics passing through x. It can be verified as before that x is 4-fold on T^6, and that through x passes 9 lines on T^7 (from § 32).

§ 34. The tangent to the conic through y at y is

$$(a_x a_y)\,a_y'^2-(a_x'a_y')\,a_y^2=0\ \ldots\ldots\ldots\ldots(44),$$
$$\left|\,a_x a_y'^2-a_x'a_y^2\ \ a a_y'^2-a'a_y^2\ \ a_y a_y'^2-a_y'a_y^2\,\right|=0\ \ldots(45).$$

These equations, in y, give the locus of points of contact of tangents from x to conics of Γ_4, namely, a hyperelliptic curve c_9 passing through x.

[*] Severi, *op. cit.* § 65, p. 222, eqn. 6.

The polar with respect to x of this second equation will plainly give the locus of the lines joining corresponding points of the involution on c_9. It is a sextic surface, having c_4 as nodal curve. This involution will have coincidences at the point x, and at the double points on the degenerate conics whose planes pass through x. Now the equation of the tangent planes through a generator λ of R^3 to the corresponding quadric involve the coordinates of the line to the second power, and the coefficients of the quadric to the third power. Hence they involve λ^3. Incidentally this shews that the class of the developable generated by the planes of degenerate conics is 9.

We can now assert* $2(p+1) = 10$, or $p = 4$. Thus c_9 is of genus 4. The order and genus of c_9 can be very simply obtained from the plane representation† of the cubic (44), and we derive the further information that c_9 meets c_4 sixteen times, and the polar line g of x with respect to the pencil of quadrics seven times. It meets c_{11} in the 9 points which occur above.

§ 35. *The conics of Γ_4 which touch a given plane $l_x = 0$.*

Exactly as for Γ_3 (§ 24) we have to express that the following planes are coaxal:

$$| \, a_x + \lambda a_x' \;\; a + \lambda a' \;\; A \, | = 0,$$
$$(a_\xi a_x) + \lambda a_\xi' a_x' = 0,$$
$$l_x = 0,$$

where $l_\xi = 0$. This gives the equation of a surface of order 9, which has c_4 as a 4-fold curve and c_{11} as a simple curve; being given by bordering the matrix of c_{11} (eqn. (41)) with the coefficients of $l_x = 0$. Its section by $l_x = 0$ is a 9-tic curve, locus of points of contact of conics of Γ_4 with this plane. It has four 4-fold points on c_4, and passes simply through‡ the points on c_{11} and c_7. The corresponding lines of contact envelope a curve of class 8.

We can conclude that of the conics through a point of c_4, four touch an arbitrary plane at the point. Also that the conics of Γ_4 which touch a plane fill a surface of order 18.

§ 36. *The conics of Γ_4 which meet a line l describe a surface of order 16 on which c_4 is sevenfold, c_{11} is double, l is simple, and the three conics having l as chord are double.* From this result, or preferably from a simple correspondence argument (in the plane of the conic in question), we deduce that 21 conics of Γ_4 meet twice

* As in § 17.

† Clebsch "Die Geometrie auf den Flächen dritter Ordnung," *Crelle's Journal*, Bd. LXV, 1863, p. 359. Taking six fundamental points 1 .. 6, we may represent g by the conic 2,3,4,5,6. Then we easily see that c_4 is represented by a quintic $1^2, 2^2, 4^2, 5^2, 6^2$, and c_9 by a sextic $1^4 2, 3, 4, 5, 6$.

‡ This involves a similar restriction on the relative situation of c_4, c_7, and c_{11} as arose in connection with P_3 (§ 24).

a conic in arbitrary position. In particular the congruence contains 21 circles. It may include ∞^1, when the circle at infinity passes through a pair of points of the involution on c_7, and it may consist entirely of circles, if c_4 includes the circle at infinity.

§ 37. The congruence Γ_4 can be generalized by the replacement of the cubic scroll by a rational scroll of any order n, giving rise to a linear congruence of class n, in general; our analysis includes the cases $n=1$, $n=2$.

For $n = 2$, the scroll to be taken is a regulus*. This occurs, for example, when $c_x' = c' = 0$. The generators of the regulus are

$$\left\| \begin{array}{ccc} a_x + \lambda a_x' & b_x + \lambda b_x' & c_x \\ a + \lambda a' & b + \lambda b' & c \end{array} \right\| = 0.$$

The singular curve is a quintic of genus 2,

$$\left\| \begin{array}{ccc} (a_x a_x'^2 - a_x' a_x^2) & (b_x a_x'^2 - b_x' a_x^2) & c_x \\ (a a_x'^2 - a' a_x^2) & (b a_x'^2 - b' a_x^2) & c \end{array} \right\| = 0.$$

In addition to the generators of the regulus there are ∞^1 singular chords, forming the conjugate regulus. The conics associated with any such singular chord fill a cubic surface passing through c_5^2 and c_4.

For $n - 1$ the scroll reduces to a plane pencil, given (say) by

$$\left\| \begin{array}{ccc} a_x + \lambda a_x' & b_x & c_x \\ a + \lambda a' & b & c \end{array} \right\| = 0.$$

The singular curve is a plain cubic. There is a star of singular chords. The class being unity, we must be dealing with a very special case of Montesano's Congruence.

§ 38. There is a second form which we may take to represent a conic by a matrix, namely

$$\left\| \begin{array}{ccc} a_x^2 & b_x & c_x \\ a_x & b & c \end{array} \right\| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(46).$$

To the same stage of development we thus get congruences

$$\Gamma_1', \quad \left\| \begin{array}{ccc} \alpha_1 a_x^2 + \alpha_2 b_x^2 + \alpha_3 c_x^2 & \alpha_1 a_x + \alpha_2 b_x + \alpha_3 c_x & \alpha_1 A_x + \alpha_2 B_x + \alpha_3 C_x \\ d_x & d & D \end{array} \right\| = 0$$

$$\ldots\ldots(47);$$

$$\Gamma_2', \quad \left\| \begin{array}{ccc} \alpha_1 a_x^2 + \alpha_2 b_x^2 + \alpha_3 c_x^2 & \alpha_1 a_x + \alpha_2 b_x + \alpha_3 c_x & d_x \\ \alpha_1 A_x^2 + \alpha_2 B_x^2 + \alpha_3 C_x^2 & \alpha_1 a + \alpha_2 b + \alpha_3 c & d \end{array} \right\| = 0$$

$$\ldots\ldots(48);$$

---

* This is Pieri's general congruence of class two §§ 9–11 *loc. cit.* in the notes, § 2. He also considers the case when the regulus reduces to a cone or to a conic envelope.

$$\Gamma_3', \left\| \begin{matrix} a_x^2 & \alpha_1 b_x + \alpha_2 c_x + \alpha_3 d_x & \alpha_1 B_x + \alpha_2 C_x + \alpha_3 D_x \\ a_x & \alpha_1 b + \alpha_2 c + \alpha_3 d & \alpha_1 B + \alpha_2 C + \alpha_3 D \end{matrix} \right\| = 0 \ldots (49).$$

Γ_1' and Γ_2' are both representations of Montesano's Congruence, and can be reduced to the simple form in (3a). Γ_1' however is expressed in a form which puts in evidence the two generating nets of quadrics, according as we take the first and second, or first and third, columns. These two nets can, as before, be replaced by any two nets whose base points form groups of the g_8^1. The conics of the system are obtained as the intersection of corresponding quadrics of collinear nets, residual to a conic of a collinear plane net of conics

$$d_x = \alpha_1 a_x^2 + \alpha_2 b_x^2 + \alpha_3 c_x^2 = 0.$$

We can obtain from this two subcases, noting first that the singular curve is

$$\left\| \begin{matrix} a_x^2 & b_x^2 & c_x^2 & d_x \\ a_x & b_x & c_x & d \\ A_x & B_x & C_x & D \end{matrix} \right\| = 0 \quad (c_7^5).$$

In the first place the net of conics may reduce to a fixed line l, together with any line in the plane $d_x = 0$. This line forms part of c_7^5 and is 4-secant to the residual sextic of genus 2. The conics of the system meet c_6^2 five times and l once. The centre of the star formed by the planes of the conics lies on c_6^2. This is to be distinguished from the similar special case which occurs in a paper by Stuyvaert[*], in which the centre lies on the quadrisecant.

The nets of quadrics each have as base group a line and four points on c_6^2. These points vary in a g_4^2, for we can add to the first column of the matrix the third multiplied by any linear form, which, equated to zero, represents a plane through the first line. There is also a series of generating nets of general character, of whose base points seven lie on the sextic.

§ 39. Secondly the net of conics may reduce to a fixed conic c_2. This forms part of c_7^5 residual to a rational quintic c_5^0, of which it is 6-secant. The conics of the system are 2-secant to c_2, and 4-secant to c_5. The centre of the star lies on c_5. The special nets have one degree of freedom, the residual base points describing a g_2^1 on c_5. There are also generating nets of general character, six of whose base points lie on c_5.

§ 40. The singular curve of Γ_3' (which is also given by the intersection of two collinear nets of quadrics with a common base curve) is

$$\left\| \begin{matrix} a_x^2 b - a_x b_x \\ a_x^2 B - a_x B_x \end{matrix} \right\| = 0 \quad \ldots \ldots \ldots \ldots \ldots (50),$$

[*] *Loc. cit.* in § 2.

a sextic of genus 2. c_2, the common conic, is 6-secant to c_6^2. The conics of Γ_3' are bisecant to c_2, and 4-secant to c_6^2. The generating nets have one degree of freedom, the base points (outside c_2) describing a fundamental g_2'.

This is in fact a congruence discovered by Pieri[*]. The conic c_2 may be replaced by any member of the congruence of 6-secant conics of c_6^2. (Another special case of Montesano's Congruence.) If $b = B = c = C = 0$ (for example), we fall back on the congruence of § 39, which is seen therefore to be a special case of both congruences.

§ 41. Returning to the general case, we may readily verify that the conics of Γ_3' lie in the planes of a series of pencils, whose axes describe a cubic scroll. The generators of this scroll join in fact corresponding points of the fundamental involution on c_6^2. These facts are well known, but they lead to interesting relations between Γ_3' and other congruences of conics. In the first place we may take without loss of generality $A = b = B = c = 0$, and then to the pencil, whose axis is the line l_λ, namely,

$$\begin{vmatrix} b_x + \lambda B_x & c_x + \lambda C_x & d_x + \lambda D_x \\ 1 & 0 & \lambda \\ \xi & 1 & 0 \end{vmatrix} = 0,$$

corresponds in the net given by $b_x + \phi B_x$, etc., the system

$$\begin{vmatrix} a_x^2 & \lambda(b_x + \phi B_x) - \lambda\xi(c_x + \phi C_x) - (d_x + \phi D_x) \\ a_x & \lambda - \phi \end{vmatrix} = 0,$$

a pencil, whose base is c_2 and the curve

$$\begin{Vmatrix} a_x^2 & \lambda(b_x + \phi B_x) - (d_x + \phi D_x) & c_x + \phi C_x \\ a_x & \lambda - \phi & 0 \end{Vmatrix} = 0.$$

This conic describes a linear congruence γ_1 of class zero, its conics lying in the planes through the line l, namely,

$$c_x = C_x = 0,$$

which is in fact the double line of the cubic scroll, and the quadrisecant of c_6^2. The conics in such a plane form a pencil, of whose base points two lie on c_2, and two on the same c_6^2. The join of these residual points is in fact the generator l_ϕ of the scroll.

The conics in the plane ϕ of the pencil are cut by quadrics of the pencil whose base consists of c_2 and

$$\begin{Vmatrix} a_x^2 & b_x + \phi B_x & d_x + \phi D_x \\ a_x & 1 & \phi \end{Vmatrix} = 0.$$

---

[*] § 5 of the paper cited in note (3). See also Stuyvaert, *loc. cit.*, in the same note. For the application to the theory of circles see Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, p. 516.

Thus the congruence γ_1 is of the general type considered with Γ_3 (§ 25). γ_1 and Γ_3' are so related that any curve of either meets ∞^1 curves of the other in two points. Thus γ_1 may be considered as a system of singular conics of Γ_3' (it is not exhaustive, as in fact we shall see), and vice versa.
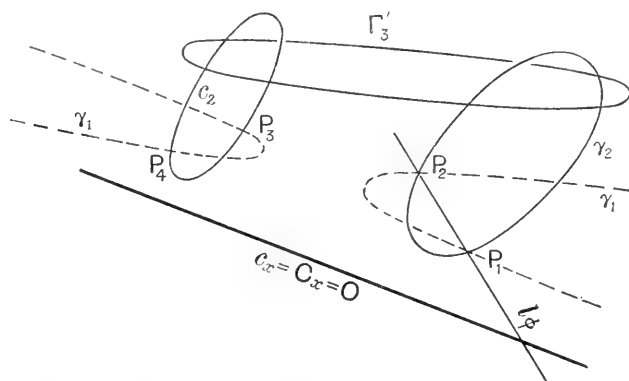
Fig. 2

§ 42. Again, it is easily seen that as residual base conic for the pencil of quadrics, cutting the conics of γ_1 in the plane $c_x + \phi C_x = 0$, may be taken the conic

$$\left\| \begin{array}{ccc} a_x{}^2 & b_x + \phi B_x + \mu(c_x + \phi C_x) & \phi_x + \phi D_x + \nu(c_x + \phi C_x) \\ a_x & 1 & \phi \end{array} \right\| = 0.$$

The system of all such conics is a *complex* γ_2, of which ∞^2 conics pass through each pair of points of the fundamental involution on the sextic. Its order and class* are easily seen to be 0, and 3, respectively. In fact a plane only contains a conic of the system if it pass through a pair of the involution (and then it contains a pencil of such conics), while the planes of the conics through an arbitrary point are the planes projecting from that point the generators of the cubic scroll. If μ and ν are connected by a linear relation,

$$\alpha\mu + \gamma\nu - \beta = 0,$$

then the corresponding set of conics is easily seen to be a special case of type Γ_4, namely, the singular quartic breaks up into two conics and the septimic in the sextic and its quadrisecant.

The second conic is

$$\left\| \begin{array}{ccc} a_x{}^2 & \alpha b_x + \beta c_x + \gamma d_x & \alpha B_x + \beta C_x + \gamma D_x \\ a_x & \alpha & \gamma \end{array} \right\| = 0.$$

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* Montesano "Una estensione del problema, etc.," *Ann. di Mat.*, Ser. III, t. I, 1898, p. 321.

In fact it is the general conic of Γ_3'. These special Γ_4's belong to the fourth group of linear congruences with a doubly-directing singular conic considered in the paper of Pieri's, so often quoted (§ 7). In them, when interpreted as circle congruences the analogue of c_{11}^5, a c_{10}^4, is to be interpreted as the locus of null circles of the congruence. (§ 31 of this paper.)

Thus any conic of Γ_3' may be taken with c_2 as the singular quartic of a special congruence of type Γ_4; the ∞^2 such congruences being associated with the same singular sextic, and their totality forming a complex of conics of order zero, and class three.

This complex must also contain the congruence γ_1, for the conic of γ_1 defined by ϕ and λ meets all the conics of Γ_3' in the planes of the pencil (l_λ). It thus belongs to ∞^1 of the Γ_4 congruences.

§ 43. Before leaving the subject we might mention that Γ_3' *contains, in general, 21 conics meeting an arbitrary conic twice; and, in particular, 21 circles.* It may consist entirely of circles. The intermediate stage is also possible, as appears from the results of the last two parameters, namely, when γ_1 contains the circle at infinity. The double directrix of the scroll lies at infinity, and c_2 is also a circle. The complex γ_2 contains in this case ∞^2 circles.

Lastly, we might also mention that the general form of Γ_3', as given by its matrix, contains also the first and second types of congruences with a singular, doubly directing, conic, as discussed by Pieri.

The second, to take them in what is here the most natural order, occurs when the nets of quadrics reduce to pencils in distinct parameters. The equation can be reduced to

$$\left\| \begin{array}{ccc} a_x^2 & \alpha_1 b_x + \alpha_2 c_x & \alpha_1 B_x + \alpha_3 D_x \\ a_x & \alpha_2 & \alpha_3 \end{array} \right\| = 0.$$

The singular curve reduces to the common base curve,

$$a_x^2 = a_x = 0,$$

and the separate residual base conics

$$b_x = a_x^2 - a_x c_x = 0;$$
$$B_x = a_x^2 - a_x D_x = 0;$$

which meet the first conic twice, but do not meet each other. There is a fourth, apparently singular, conic to be rejected as explained in the Note (§ 25). The conics of the system are bisecant to each of the three singular conics, and their planes form a star, whose centre is

$$c_x - D_x = b_x = B_x = 0.$$

Thus it is bilinear, and is therefore also a special case of Montesano's Congruence.

§ 44. The first case is the system of conics with two fundamental points, and a directrix singular conic. It is therefore the intersection of a net of quadrics with a base conic, and a pencil of planes whose axis passes through the two residual base points of the net of quadrics. It is therefore of class zero and can be represented by a matrix

$$\left\| \begin{array}{ccc} a_x^2 & \alpha_2 x_3 + \alpha_3 x_4 & a_3 x_3 + a_4 x_4 \\ a_x & \alpha_1 & a \end{array} \right\| = 0,$$

with $a_{11} = a_{22} = 0$.

§ 45. *Appendix.* As already stated we do not get for the first form of the matrix equation of a conic, and the remaining four types, given by Stuyvaert, any essentially new congruences. The same is true for the second form of matrix. As far as the parameters are concerned we may denote an element by a symbol (i, j, k) to denote the parameters which occur. Then types III— V are

$$\left\| \begin{array}{ccc} (1, 2) & (1, 2) & (1, 2) \\ (1, 3) & (1, 3) & (1, 3) \end{array} \right\| = 0 \quad \dots\dots\dots\dots(\text{III}),$$

$$\left\| \begin{array}{ccc} (1, 2, 3) & (1, 3) & (1, 3) \\ (1, 2) & 1 & 1 \end{array} \right\| = 0 \quad \dots\dots\dots\dots(\text{IV}),$$

$$\left\| \begin{array}{ccc} (1, 2, 3) & (1, 2) & (1, 2) \\ (1, 2) & 1 & 1 \end{array} \right\| = 0 \quad \dots\dots\dots\dots\dots(\text{V}).$$

Type (VI) must be written down in full, its linearity depending on certain identities among the coefficients,

$$\left\| \begin{array}{ccc} \alpha_1 a + \alpha_2 b + \alpha_3 c & \alpha_1 a' + \alpha_2 b' + \alpha_3 c' & \alpha_1 a'' + \alpha_2 b'' + \alpha_3 c'' \\ \alpha_1 d + \alpha_2 c & \alpha_1 d' + \alpha_2 c' & \alpha_1 d'' + \alpha_2 c'' \end{array} \right\| = 0 \dots(\text{VI}),$$

where the symbols a, b, etc., stand for functions of x.

In the case of the first matrix we have to superimpose on each of these the form

$$\left\| \begin{array}{ccc} a_x^2 & a_x & a \\ a_x'^2 & a_x' & a' \end{array} \right\| \dots\dots\dots\dots\dots\dots\dots\dots(51).$$

In the table a symbol $\{i, j, k\}$, where i, j, k take the values 0, 1, 2, denotes the order in which the columns of (51) are taken in connection with the various types. Taking each case in turn, we can, by linear and quadratic transformations in the α plane, reduce it to a known form, with the exception of a certain congruence of class zero. We merely state the results.

Matrix Type		*Congruence Type*
III, VI.		Montesano's bilinear congruence.
IV.	$\{2, 1, 0\}$	A congruence of class zero, with a singular c_s^7.
	$\{0, 2, 1\}$	Montesano's bilinear congruence.
	$\{1, 2, 0\}$	Γ_4; associated with a regulus (§ 37).
V.	$\{2, 1, 0\}$	As for IV $\{2, 1, 0\}$.
	$\{0, 2, 1\}$	Montesano's congruence.
	$\{1, 2, 0\}$	Γ_4; associated with the system of tangents to a conic.

With the second form of representation of a conic (§ 38) we have to superimpose

$$\left\| \begin{array}{ccc} a_x^2 & b_x & c_x \\ a_x & b & c \end{array} \right\| \quad \dots\dots\dots\dots\dots(52),$$

and proceed as before. In the symbols denoting the subdivision, $A(1)$ implies that the form in (52) is used as written; $A(2)$ that the first and second, or first and third, columns are interchanged. In $B(1)$ and $B(2)$ the rows are interchanged.

Matrix Type	*Congruence Type*
III; A or B	Γ_4; associated with a quadric.
IV; $A(1)$ or $B(1)$	A congruence of class zero, associated with a singular c_s^7.
$A(2)$	Γ_4; associated with a quadric.
$B(2)$	Γ_4; associated with a plane pencil of rays (§ 37).
V; $A(1)$ or $B(1)$	Γ_3 (§ 14).
$A(2)$	Γ_4; associated with a conic envelope.
$B(2)$	Γ_4; associated with a pencil.

Type VI is obviously not a possible form.

On the Rational Solutions of the Indeterminate Equations of the Third and Fourth Degrees. By L. J. MORDELL, Manchester College of Technology.

§ 1.   Mathematicians have been familiar with very few questions for so long a period with so little accomplished in the way of general results*, as that of finding the rational solutions†, or say for shortness, the solutions of indeterminate equations of genus unity of the forms

$$\left. \begin{aligned} \zeta^2 &= a\xi^4 + b\xi^3\eta + c\xi^2\eta^2 + d\xi\eta^3 + e\eta^4 \\ y^2 &= ax^4 + bx^3 + cx^2 + dx + e \end{aligned} \right\} \quad \ldots\ldots\ldots(1),$$

$$0 = f(x, y, z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where f is a ternary homogeneous cubic in x, y, z, including as a particular case

$$y^2 = 4x^3 - g_2 x - g_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3);$$

and there is no loss of generality in assuming that the coefficients of all equations in this paper are integers. Our present knowledge is based on three types of results, of which the first enables us in general to find an infinite number of solutions when a finite number have already been found, e.g. by trial, and has been known in principle for some centuries. For a value of x, y, z, satisfying equation (2) defines a rational point P on the cubic curve (2); and the tangent at P will meet the cubic in another rational point P_1 different in general from P. Not only can this process be repeated with P_1, but if another rational point Q is known, then the intersection of the chord PQ with the cubic gives also a rational point. This process can in general be continued indefinitely.

The analytical interpretation is obvious from equation (3). For if we know several solutions, say x_1, y_1; x_2, y_2, ..., we define arguments u_1, u_2 ... by writing

$$x_1 = \wp(u_1), \quad y_1 = \wp'(u_1), \quad \text{etc.}$$

in the usual notation of elliptic functions. The addition formula then shows that the formulae

$$x = \wp(m_1 u_1 + m_2 u_2 \ldots), \quad y = \wp'(m_1 u_1 + m_2 u_2 \ldots) \quad \ldots(4)$$

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* See, for example, vol. II. of Dickson's *History of the Theory of Numbers.*

† We may suppose that ξ, η, ζ in equation (1), and x, y, z in equation (2) are all integers.

give in general an infinite number of solutions by taking for $m_1, m_2, \ldots$ all integer values, positive, negative and zero.

The second type of result is that certain classes of equations, e.g.

$$x^3 + y^3 + pz^3 = 0$$

where $p > 2$ is a prime of the forms $9n + 2$, $9n + 5$ or the square of such a prime, has only the solution

$$x = 1, \quad y = -1, \quad z = 0.$$

The third result* is that all the solutions of equations (1), (2) can be found if we know only one solution, and all the solutions of equation (3), where g_2, g_3 are the well-known invariants of the quartic (1) [written with binomial coefficients a, $4b$, $6c$, $4d$, e] or multiples of the fundamental invariants of the cubic (2). Conversely we can solve completely equation (3) if we know the complete solution of equations (1) or (2).

I shall now prove that if any of these equations (1, 2, 3) have an infinite number of solutions, then the method of infinite descent applies, that is to say, all the solutions can be expressed rationally in terms of a finite number by means of the classic method. In other words, the solution of equation (3) is given by (4) where $u_1, u_2 \ldots$ are *finite* in number.

§ 2.   The last result mentioned and some developments therefrom will be required later, so that a very simple proof may be given here.   There is no loss of generality in considering the equation (the coefficients c, d, etc. need not be integers),

$$\zeta^2 = \xi^4 + 6c\xi^2\eta^2 + 4d\xi\eta^3 + e\eta^4,$$

of which one solution is given by $\xi = 1$, $\eta = 0$, or say

$$y^2 = x^4 + 6cx^2 + 4dx + e$$

with a known rational solution $x = \infty = 1/0$.   Put

$$2s = x^2 + c + y$$

so that s is rational if both x and y are.

By eliminating y

$$4s^2 - 4s(x^2 + c) = 4cx^2 + 4dx + e - c^2,$$

or $$4(c + s)x^2 + 4dx = 4s^2 - 4sc + c^2 - e,$$

whence $2x(s + c) = -d \pm [d^2 + (s + c)(4s^2 - 4sc + c^2 - e)]^{\frac{1}{2}},$

reducing to $$2x(s + c) = -d + t,$$

where $$t^2 = 4s^3 - g_2 s - g_3,$$

* See my paper "Indeterminate equations of the third and fourth degree."
Quarterly Journal of Pure and Applied Mathematics, vol. xlv, 1914, pages 180, 186.
    Particular cases have been given by Sylvester and other writers.

and g_2, g_3 are the well-known invariants of the quartic with binomial coefficients. Clearly x and y are rational if s and t are. The transformation is birational, and by putting

$$s = \wp(u), \quad t = \wp'(u)$$

we can establish a one to one correspondence between the points on the quartic, and a period parallelogram in the u plane. Further, the parabola

$$y = -x^2 + \lambda x + \mu$$

is changed into $\qquad 2s - c = \dfrac{\lambda}{2}\left(\dfrac{t-d}{s+c}\right) + \mu,$

or say the parabola $\qquad t = Ns^2 + Ls + M.$

This meets the cubic in four points corresponding to, say, u_1, u_2, u_3, u_4, for which

$$u_1 + u_2 + u_3 + u_4 \equiv 0 \pmod{\omega_1, \omega_2}$$

where ω_1, ω_2 are the periods of the $\wp$ function.

For the general quartic (page 180 of my *Quarterly* paper)

$$z^2 = ax^4 + 4bx^3y \ldots + ey^4 \equiv f(x, y),$$

when we know one solution x_0, y_0, the general solution is given by

$$x = 2x_0\left(tf_0^{\frac{3}{2}} + g_0\right) + \frac{\partial f_0}{\partial y_0}\left(h_0 - sf_0\right),$$

$$y = 2y_0\left(tf_0^{\frac{3}{2}} + g_0\right) - \frac{\partial f_0}{\partial x_0}\left(h_0 - sf_0\right),$$

where $f_0 = f(x_0, y_0)$, h_0 is the Hessian $(b^2 - ac)p_0^4 + \ldots$ and g_0 the sextic covariant of the quartic f_0. Also

$$t^2 = 4s^3 - g_2 s - g_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

§ 3. It will be convenient to prove now another result which will be required later. It has been shown in my *Quarterly* paper that all the integer solutions of the equation in g_1, h, a

$$g_1^2 = h^3 - G_2 ha^2 - G_3 a^3 \quad\ldots\ldots\ldots\ldots\ldots(5),$$

with a odd and prime to h, are given by taking

$$a = F(p, q) = (A, B, C, D, E)(p, q)^4,$$
$$h = H(p, q) = (B^2 - AC)p^4 + \ldots \quad\ldots\ldots\ldots\ldots(6),$$
$$2g_1 = G(p, q)$$

where $F(p, q)$ is a representative of the classes (finite in number) of binary quartics with invariants

$$g_2 = 4G_2, \quad g_3 = 4G_3,$$

$-H(p, q)$ is the Hessian and $G(p, q)$ the sextic covariant of $F(p, q)$. Further, p and q are any coprime integers for which a is odd and prime to h.

The new result is that these formulae (6) still give the solution if only a is prime to h, so that a may also be even—provided of course that p, q are any coprime integers for which a is prime to h. For the well-known syzygy of the quartic (writing G for shortness instead of $G(p, q)$, etc.) gives

$$G^2 = 4H^3 - 4G_2 HF^2 - 4G_3 F^3,$$

showing that (6) is a solution of (5).

Conversely, if g_1, h, a are given and h is prime to a, it will be shown that we can find a quartic with invariants, $4G_2, 4G_3$,

$$(a, b, c, d, e)(x, y)^4 \quad \dots\dots\dots\dots\dots(7),$$

where a, b, etc., are integers, and b and hence $h = b^2 - ac$ are both prime to a. This quartic will be equivalent to a representative of the finite number of classes of binary quartics, whence

$$a = F(p, q), \quad h = H(p, q).$$

For suppose c, d, e are given by

$$ac = b^2 - h,$$
$$a^2 d = b^3 - 3bh + 2g_1,$$
$$a^3 e = 4G_2 a^2 - 3h^2 + b^4 - 6b^2 h + 8bg_1;$$

then it is easily verified [see page 171 of my *Quarterly* paper] that the invariants of the quartic (7) are $4G_2, 4G_3$, that is

$$4G_2 = ae - 4bd + 3c^2,$$
$$4G_3 = ace + 2bcd - ad^2 - b^2 e - c^3 \quad \dots\dots\dots\dots(7).$$

It will now be shown that c, d, e are integers for a suitable value of b, namely,

$$b \equiv g_1/h \pmod{a^2}.$$

For
$$ac \equiv (g_1^2 - h^3)/h^2 \pmod{a}$$
$$\equiv 0 \pmod{a}$$

from equation (5), so that c is an integer.

Also
$$a^2 d \equiv g_1^3/h^3 - g_1 \pmod{a^2}$$
$$\equiv g_1(g_1^2 - h^3)/h^3 \pmod{a^2}$$
$$\equiv 0 \pmod{a^2}$$

from equation (5), so that d is an integer.

Also from equation (7) ae and $(ac - b^2)e$ are integers, that is, ae and he are integers. Hence, as a is prime to h, e is also an integer. This proves the result.

§ 4. Consider now equation (1) which is taken in the form

$$x^4 - px^3y - qx^2y^2 - rxy^3 - sy^4 = az^2 \ldots\ldots\ldots(8),$$

where there is no loss of generality in supposing x is prime to y, and that a is not a perfect square. Suppose also that the quartic field $K(\theta)$ is defined by the equation

$$\theta^4 - p\theta^3 - q\theta^2 - r\theta - s = 0,$$

which is supposed to have no rational linear factors in θ. Hence θ is either a root of an irreducible quartic or of an irreducible quadratic. In the latter case the field is generated by the roots of both quadratics. The left-hand side of (8) splits up into factors

$$x - \theta y \quad \text{and} \quad x^2 + (\theta - p) x^2 y + \ldots,$$

which in the field $K(\theta)$ can have only a finite number of ideal factors in common. Hence we have the equation in ideals

$$(x - \theta y) = \lambda T^2,$$

where λ is one of a finite number of ideals and T is an ideal. As the number of ideal classes is finite, we can put

$$T = uv,$$

where v is an algebraic number given by

$$nv = a + b\theta + c\theta^2 + d\theta^3 \ldots\ldots\ldots\ldots\ldots(8\mathrm{a}),$$

with a, b, c, d integers, while u and n are a pair taken from a finite number of ideals and ordinary integers respectively. Hence

$$(x - \theta y) = \lambda u^2 v^2.$$

But all the units in the field $K(\theta)$ can be written in the form $U_1 U_2^2$ where U_1, U_2 are units, for a finite number of values of U_1. Hence as λu^2 must be a principal ideal, we have an equation of the form

$$x - \theta y = \sigma v^2 / M \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where M is one of a finite number of ordinary integers, σ one of a finite number of algebraic integers, and v is an algebraic number of the form (8 a). Some of the equations (9) may supply only a finite number of values of x, y, v; but if the equation (8) has an infinite number of solutions, one of the equations (9) will also give an infinite number of values for x, y, v. Hence we have for a particular set x_0, y_0, v_0, and there are only a finite number of such sets required,

$$M (x_0 - \theta y_0) = \sigma v_0^2.$$

We may also suppose that x_0, y_0 is the smallest set satisfying this equation, reckoning the magnitude of sets from the maximum value of $|x_0|$, $|y_0|$. Hence we can deduce an equation of the form

$$M^2 (x - \theta y) (x_0 - \theta y_0) = (A + B\theta + C\theta^2 + D\theta^3)^2 \ldots(10),$$

which has an infinite number of integer values of x, y, A, B, C, D, where the integers M, x_0, y_0 are selected from a finite set. We can also deduce such an equation if (9) is satisfied by two sets of values. If (9) is satisfied by only one set, we can call it an isolated set.

The success of my investigation depends upon the fact that from equation (10), x and y can be expressed rationally in terms of a new solution z_1, x_1, y_1 of equation (8), where x_1, y_1 are practically linear functions of A, B, C, D. Moreover, by considering the three equations conjugate to equation (10), it is clear that*

$$A, B, C, D = O\left[\max |x|^{\frac{1}{2}}, |y|^{\frac{1}{2}}\right]$$

and that the same result holds for $x_1, y_1, \ldots$. From a sequence

$$|\xi_n| < \kappa |\xi_{n-1}|^{\frac{1}{2}}, \quad |\xi_{n-1}| < \kappa |\xi_{n-2}|^{\frac{1}{2}}, \ldots |\xi_1| < \kappa |\xi_0|^{\frac{1}{2}}$$

we have (if $\kappa > 1$) $\qquad |\xi_n| < \kappa^2 |\xi_0|^{1/2^n}$,

so that the method of infinite descent applies, that is, by applying the same process to $x_1 y_1$, we deduce solutions $x_2 y_2$, $x_3 y_3 \ldots$ until we come to a solution $x = x_n$, $y = y_n$, which either cannot be expressed in the form (10), or if it can, leads to a solution identical with x_n, y_n; or which will be a solution of an equation (9), that is x_n, y_n will be either a minimum set or an isolated set. Hence x, y can be expressed rationally in terms of a finite number of solutions $x_1 y_1 z_1$, $x_2 y_2 z_2$, $\ldots x_n y_n z_n$.

§ 5. To simplify the algebra, consider first the case when

$$(x - \theta y)\, \theta = (a + b\theta + c\theta^2 + d\theta^3)^2 \ldots\ldots\ldots\ldots(11),$$

and $\qquad \theta^4 = p\theta^3 + q\theta^2 + r\theta + s,$

corresponding to the solution for which $x_0 = 0$.

The square of the right-hand side is

$$a^2 + 2ab\theta + (b^2 + 2ac)\,\theta^2 + 2\,(bc + ad)\,\theta^3 + (c^2 + 2bd)\,\theta^4 + 2cd\theta^5 + d^2\theta^6.$$

Also $\qquad \theta^5 = p\theta^4 + q\theta^3 + r\theta^2 + s\theta,$

$$\theta^6 = p\,(p\theta^4 + q\theta^3 + r\theta^2 + s\theta) + q\theta^4 + r\theta^3 + s\theta^2.$$

Hence equating coefficients of θ^0, θ^3 on both sides of (11),

$$a^2 + s\,(c^2 + 2bd + 2cdp + d^2p^2 + d^2q) = 0,$$

$$2ad + 2bc + 2cdq + d^2\,(pq + r) + p\,(c^2 + 2bd + 2pcd + d^2p^2 + d^2q) = 0.$$

This result still holds when the quartic has two irreducible quadratic factors, as equation (11) is true if θ is the root of either quadratic. Eliminating b between these equations by multiplying the first equation by $c + pd$, the second by $-sd$ and adding, we have on arranging the result in powers of a,

$$(c + pd)\,a^2 - 2asd^2 + (c + pd)\,(s\,[c + pd]^2 + qsd^2)$$
$$- s\,(d^3\,[p^3 + pq] + d^3\,[pq + r] + 2cd^2\,[p^2 + q] + c^2dp) = 0.$$

Put now $c + pd = \rho$, and this becomes

$$a^2\rho - 2asd^2 + s\rho^3 - psd\rho^2 - qsd^2\rho - srd^3 = 0,$$

whence $\quad a\rho = sd^2 \pm [-s(\rho^4 - p\rho^3 d - q\rho^2 d^2 - r\rho d^3 - sd^4]^{\frac{1}{2}}.$

Also from equation (11), by changing θ into its conjugate values and multiplying the resulting equations together

$$-s(x^4 - px^3y - qx^2y^2 - rxy^3 - sy^4) = z^2 \text{ say } \ \ldots\ldots(12).$$

It is also clear that

$$a, b, c, d, \rho = O\left[\,|\,x\,|^{\frac{1}{2}},\,|\,y\,|^{\frac{1}{2}}\,\right],$$

and that as b and also x, y from (11) are rationally expressed in terms of a, ρ, d, the solution x, y of (12) is expressed in terms of another solution ρ, d of (12). Hence the method of infinite descent applies, as we can continue the process with the solution ρ, d first removing their common factors, until we come to a solution, say x_n, y_n which either cannot be expressed in the form (11), that is (10) with $x_0 = 0$, or if it can, leads to a solution

$$x_{n+1} = x_n, \ \ y_{n+1} = y_n.$$

So we now consider the case wherein we do not take $x_0 = 0$.

We turn then to equation (10), namely

$$M^2(x - \theta y)(x_0 - \theta y_0) = (A + B\theta + C\theta^2 + D\theta^3)^2 \ \ldots(10),$$

and put $\quad\quad\quad x_0 - \theta y_0 = \phi,$

where we note $y_0 = 0$ is excluded, since a in equation (8) is not a perfect square. The resulting equation takes the form

$$(X - \phi Y)\,\phi = (a_1 + b_1\phi + c_1\phi^2 + d_1\phi^3)^2 \ \ \ldots\ldots(12\,\mathrm{a}),$$

say, where now

$$\phi^4 - p_1\phi^3 - q_1\phi^2 - r_1\phi - s_1 = 0.$$

Hence X, Y and so x, y are rationally expressible in terms of

$$\rho_1, d_1 \ \text{ and } \ [-s_1(\rho_1^4 - p_1\rho_1^3 d_1 \ldots)]^{\frac{1}{2}},$$

where $\rho_1 = c_1 + p_1 d_1$.

But since

$$\left(\frac{x_0 - \phi}{y_0}\right)^4 - p\left(\frac{x_0 - \phi}{y_0}\right)^3 - \ldots = \frac{1}{y_0^4}(\phi^4 - p_1\phi^3 \ldots),$$

we have on equating terms independent of ϕ

$$az_0^2 = x_0^4 - px_0^3 y_0 - qx_0^2 y_0^2 \ldots = -s_1.$$

Replacing also ϕ by ρ_1/d_1 and putting

$$\frac{\rho}{d} = \frac{x_0 - \rho_1/d_1}{y_0} = \frac{d_1 x_0 - \rho_1}{d_1 y_0},$$

so that $$\rho = \frac{1}{\lambda}(d_1 x_0 - \rho_1), \quad d = \frac{1}{\lambda}d_1 y_0,$$

where λ is taken so that the integers ρ and d are prime to each other, we have

$$\rho^4 - p\rho^3 d - q\rho^2 d^2 \ldots = \frac{1}{y_0^4}(\rho_1^4 - p_1 \rho_1^3 d_1 \ldots)\left(\frac{d}{d_1}\right)^4.$$

Hence x, y are rationally expressible in terms of

$$\rho, \ d \ \text{and} \ [a\,[\rho^4 - p\rho^3 d \ldots]]^{\frac{1}{2}},$$

so that not only are ρ, d another solution x, y of equation (8), but as practically linear functions of d_1, ρ_1, and hence of A, B, C, D, they are

$$O\left[\max |x|^{\frac{1}{2}}, |y|^{\frac{1}{2}}\right],$$

as is clear from the equation (10) and the conjugate equations.

Hence the method of infinite descent applies, so that all the solutions x, y, z of (8) can be expressed rationally in terms of a finite number

$$x_1, y_1, z_1; \ x_2, y_2, z_2; \ldots x_n, y_n, z_n, \ldots;$$

or rather as the method of reduction leads to solutions wherein x, y, z may have a common factor, the theorem really applies to the ratios $\dfrac{x}{y}, \dfrac{z}{y}$, that is, to all the rational solutions of the equation

$$\eta^2 = a\xi^4 + b\xi^3 + c\xi^2 + d\xi + e \quad \ldots\ldots\ldots\ldots(13),$$

where the right-hand side has no rational linear factors in ξ.

§ 6. To interpret this result geometrically put $\eta = z_1/x_1^2$, $\xi = x_1/y_1$ and suppose that equation (8) takes the form (13). The solutions $x_1, y_1, z_1, \ldots x_n, y_n, z_n$, correspond to rational points $P_1, P_2, \ldots P_n$ on the curve. Then we have the theorem : all the rational points can be found by finding the intersection P_{n+1} with the curve of parabolas

$$\eta = L\xi^2 + M\xi + N \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

passing through, say, the point P_1, and having double contact with the quartic at P_2 or P_1, and then continuing the process, including now P_{n+1}, among the points $P_1, P_2, \ldots P_n$, etc.

A simple way of proving this is to start from equation (11), change θ into $1/(\theta - \kappa)$, so that now $x_0 = 1$, $y_0 = 0$,

$$x - \theta y = (a + b\theta + c\theta^2 + d\theta^3)^2,$$

and by selecting κ properly we have

$$\theta^4 = q\theta^2 + r\theta + s.$$

Expanding and equating coefficients of θ, θ^2, etc., we find

$$bd = c^2 - qd^2 \pm (c^4 - qc^2d^2 - rcd^3 - sd^4)^{\frac{1}{2}},$$

$$2ad^2 = -2c^3 - rd^3 \mp 2c(c^4 - qc^2d^2 - rcd^3 - sd^4)^{\frac{1}{2}},$$

$$\frac{x}{y} = -\frac{a^2 + s(c^2 + 2bd + d^2q)}{2ab + r(c^2 + 2bd + d^2q) + 2cds} \quad \ldots\ldots\ldots\ldots(15).$$

Put now $\xi = x/y$, take $d = 1$, and put

$$\kappa^2 = c^4 - qc^2 - rc - s.$$

We shall now show that a parabola of the form (14) drawn through the point $\xi = \infty$, $\eta = +\infty$ of the curve

$$\eta^2 = \xi^4 - q\xi^2 - r\xi - s \quad \ldots\ldots\ldots\ldots\ldots(16),$$

to have double contact with it at the point $\xi, \eta = c - \kappa$, will meet it again in a point whose ξ coordinate is given by (15).

For changing the origin to $\xi = c$, $\eta = 0$, the quartic (16) becomes

$$\eta^2 = \xi^4 + A\xi^3 + B\xi^2 + C\xi + \kappa^2,$$

where $A = 4c$, $B = 6c^2 - q$, $C = 4c^3 - 2qc - r$.

The required parabola is then

$$\eta = \xi^2 - \frac{C}{2\kappa}\xi - \kappa,$$

and its fourth point of intersection with the quartic is given by

$$A\xi + B = -\frac{C}{\kappa}\xi + \frac{C^2}{4\kappa^2} - 2\kappa,$$

or
$$\xi = \frac{C^2 - 8\kappa^3 - 4\kappa^2 B}{4\kappa(A\kappa + C)}.$$

Also the expression (15) diminished by c, say ξ_1, is equal to ξ for

$$\xi_1 = -\frac{a^2 + s(c^2 + 2b + q)}{2ab + r(c^2 + 2b + q) + 2cs} - c,$$

where
$$b = c^2 - q + \kappa,$$

$$2a = -2c^3 - r - 2c\kappa,$$

on taking $d = 1$ as the equations are homogeneous in a, b, c, d. Hence the denominator of ξ_1 becomes

$$-2c^5 + 2qc^3 + rc^2 + qr + \kappa(-4c^3 + 2cq - r) - 2c\kappa^2$$

$$+ rc^2 + qr + 2cs,$$

or
$$-2c\kappa^2 - 2c\kappa^2 + \kappa(-C),$$

or
$$-\kappa(A\kappa + C).$$

The numerator is

$$c^6 + c^3 r + \tfrac{1}{4} r^2 + (2c^4 + cr) \, \kappa$$
$$c^6 - qc^4 - rc^3 - sc^2 + qs$$
$$- sc^2$$
$$+ 2c^2 s - 2qs + 2\kappa s,$$

or $\qquad 2c^6 - qc^4 + 2sc^2 - qs + \tfrac{1}{4} r^2 + (2c^4 + cr + 2s) \, \kappa.$

Hence $\quad \xi_1 = \dfrac{\left\{ \begin{array}{l} 2c^6 - qc^4 + 2sc^2 - qs + \tfrac{1}{4} r^2 + (2c^4 + cr + 2s) \, \kappa \\ - 4c^2 (c^4 - qc^2 - rc - s) \qquad - (4c^4 - 2qc^2 - rc) \, \kappa \end{array} \right\}}{\kappa \, (A\kappa + C)}.$

Also

$$\tfrac{1}{4} (C^2 - 8\kappa^3 - 4\kappa^2 B) = -2\kappa^3 - (6c^2 - q)(c^4 - qc^2 - rc - s) + (2c^3 - qc - \tfrac{1}{2}r)^2$$
$$= -2\kappa^3 - 2c^6 + 3qc^4 + 4rc^3 + 6c^2 s - sq + \tfrac{1}{4}r^2,$$

which is the same as the numerator of ξ_1. Hence $\xi = \xi_1$ which proves the statement.

For the analytic interpretation we must turn to § 2 and consider the birational transformation between x, y, or using now ξ, η for them, and s, t, or u. Then corresponding to the point $\xi = \infty$, $\eta = +\infty$ we have $s = -c, t = d$, or, say, u_0, and to the point ξ, η, where the parabola touches the quartic, we have, say, the point u_1. Hence by the elementary properties of periodic functions

$$u_0 + u + 2u_1 \equiv 0 \pmod{\omega_1, \omega_2}.$$

Similarly operating on u_1, we have

$$v_0 + u_1 + 2u_2 \equiv 0,$$
$$w_0 + u_2 + 2u_3 \equiv 0,$$
$$\dots\dots\dots\dots\dots$$
$$k_0 + u_{n-1} + 2u_n \equiv 0,$$

and, repeating this process, we can express u in terms of the quantities $u_0, v_0, w_0, \dots$, which are finite in number, the previous work showing that the process finishes by coming to a stage where $u_n = u_0$, or v_0, or w_0, etc. Hence we have

$$u + u_0 - 2v_0 + 2^2 w_0 \dots \pm 2^n u_n = 0.$$

Now every integer can be represented in the scale 2 in the form

$$\pm 1 \pm 2 \pm 2^2 \pm 2^3 \dots,$$

the signs being independent of each other. Hence, as the $u_0, v_0, \dots$ need not be all different, we have

$$u = m_1 \xi_1 + m_2 \xi_2 + \dots m_n \xi_n,$$

where $m_1, m_2, \dots m_n$ are any integers negative, positive, or zero,

n is finite, and ξ_1, ξ_2, $\dots \xi_n$ are n given quantities corresponding to n values of u. A more symmetrical notation is

$$u = m_1 u_1 + m_2 u_2 \dots + m_n u_n.$$

§ 7. We must still discuss the case when the quartic (8) has a rational linear factor, so that a solution of equation (8) is at hand with $z = 0$. Hence from § 2, all that we require is the complete solution of the equation

$$t^2 = 4s^3 - g_2 s - g_3 \quad \dots\dots\dots\dots\dots(3),$$

supposed to have an infinite number of rational solutions.

Suppose first of all that there is a binary quartic in x, y with invariants g_2, g_3, without rational linear factors, which becomes a perfect square for one value of x, y, and hence from § 2 for an infinite number of values of x, y. From § 2 the values of x, y are expressed in terms of $s = \wp(u)$, $t = \wp'(u)$ and from § 6 the general value of u is known. Hence the general solution of equation (3) is given by

$$s = \wp(m_1 u_1 + \dots m_n u_n), \quad t = \wp'(m_1 u_1 + \dots m_n u_n),$$

where the quantities u_1, u_2, $\dots u_n$ are finite in number and m_1, m_2, $\dots m_n$ are any integers, positive, negative, or zero.

Next consider the case when no such quartic exists. Multiplying equation (3) throughout by 16, we see there is no loss of generality in putting

$$g_2 = 4G_2, \quad g_3 = 4G_3,$$

where G_2, G_3 are integers, and considering the equations

$$t^2 = s^3 - G_2 s - G_3.$$

Writing now $s = x/y$ with x prime to y, we can put

$$z^2 = y(x^3 - G_2 xy^2 - G_3 y^3).$$

Hence both factors on the right hand are perfect squares, so that* by § 3 we can put

$$x = H(x_1, y_1), \quad y = F(x_1, y_1), \quad \dots\dots\dots(16\,\mathrm{a})$$

where F is a representative of the classes of binary quartics with invariants $4G_2$, $4G_3$. By hypothesis F has a rational linear factor, so that we can suppose F is given by

$$y_1(4Bx_1^3 + 4Dx_1 y_1^2 + E_1 y_1^3).$$

---

* We have an infinite number of quartics (included among a finite number of classes) invariants $4G_2$, $4G_3$ with first coefficient y^2, but it does not seem easy to prove these have no rational linear factors.

On writing down the condition that its invariants are $4G_2$, $4G_3$, this takes the form

$$y_1 \left(4bx_1^3 - \frac{4G_2}{b} x_1 y_1^2 - \frac{4G_3}{b^2} y_1^3 \right) \quad \ldots\ldots\ldots\ldots (17),$$

where b has a finite number of integer values.

Hence $y_1 [(bx_1)^3 - G_2 (bx_1) y_1^2 - G_3 y_1^3]$

is a perfect square, and x_1, y_1 are prime to each other since x is prime to y. If b and y_1 have a common factor κ, put

$$y_1 = \kappa Y, \quad b = \kappa B,$$

then $Y [(Bx_1)^3 - G_2 (Bx_1) Y^2 - G_3 Y^3]$

is a perfect square, and now Y is prime to Bx_1. Hence

$$Bx_1 = h (x_2, y_2), \quad Y = f (x_2, y_2),$$

or $bx_1 = \kappa h (x_2, y_2), \quad y_1 = \kappa f (x_2, y_2)$

where f is again a binary quartic with invariants $4G_2$, $4G_3$, which must also be of the form (17) with not necessarily the same b.

Now[*] x_1, y_1 are $O [\max | x |^{\frac{1}{4}}, \quad | y |^{\frac{1}{4}}]$,

$$x_2, y_2 \text{ are } O [\max | x_1 |^{\frac{1}{4}}, \quad | y_1 |^{\frac{1}{4}}],$$

so that the method, etc., of infinite descent applies and we must arrive at a finite number of solutions, in terms of which all the others can be expressed. Moreover, if we put

$$x/y = \wp (u), \quad bx_1/y_1 = \wp (u_1),$$

we have $u = 2u_1$[†], showing that the same rule holds as before.

§ 8.   Finally we consider the case of the homogeneous ternary cubic of genus one, which we write as

$$f (x, y, z) = 0.$$

If this equation has an infinite number of integer solutions, we first of all apply a linear transformation

$$x, y, z = L (\xi, \eta, \zeta),$$

so that the coefficient of ζ^3 is zero in the new equation, which then becomes

$$\zeta^2 S_1 + 2\zeta S_2 + S_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots (18),$$

where S_1, S_2, S_3 are binary linear, quadratic and cubic forms in ξ, η.

Hence $\zeta S_1 = - S_2 \pm (S_2^2 - S_1 S_3)^{\frac{1}{2}}.$

---

[*] This follows from (16 a) as $x_1 - ey_1$ is the square of a quadratic function of x, y for three values of e.

[†] See my paper "The inversion of the integral, etc." *Messenger of Mathematics*, vol. 43, 1915, page 140.

The radical is now a binary quartic in ξ, η. Hence ξ, η and so ζ can be rationally expressed in terms of a finite number of solutions, say, $\xi_1, \eta_1; \xi_2, \eta_2 \dots \xi_n, \eta_n$. Hence any solution x, y, z can be rationally expressed in terms of a finite number of solutions $x_1, y_1, z_1; x_2, y_2, z_2, \dots x_n, y_n, z_n$; or, more accurately, this is the time for the ratios $x/z, y/z$ in terms of $x_1/z_1, y_1/z_1$, etc.

Moreover, the method of derivation is the classical one in § 1. For a linear transformation

$$\xi', \eta' = L_1(\xi, \eta), \quad \zeta' = \zeta + a\xi + b\eta$$

reduces the cubic (18) to the form

$$\zeta^2\eta - \zeta\xi^2 + S_3 = 0,$$

so that
$$2\zeta\eta = \xi^2 \pm (\xi^4 - S_3\eta)^{\frac{1}{2}}.$$

Another substitution

$$\xi' = \xi + k\eta, \quad \zeta' = \zeta + A\xi + B\eta$$

gives
$$2\zeta\xi = \xi^2 + c\eta^2 \pm (\xi^4 + 6c\xi^2\eta^2 + 4d\xi\eta^3 + e\eta^4)^{\frac{1}{2}},$$

which is the birational transformation considered in § 2. Hence we can take

$$\frac{\zeta}{\eta} = \wp(u), \quad \frac{\xi}{\eta} = \frac{\wp'(u) - \wp'(u_0)}{\wp(u) - \wp(u_0)},$$

where from § 6 the general value of u is given by

$$u = m_1 u_1 + m_2 u_2 \dots + m_n u_n$$

where $m_1, m_2, \dots m_n$ are any integers positive, negative or zero, and $u_1, u_2, \dots u_n$ are finite* in number. Also the parameters v_1, v_3, v_3 of three collinear points on the cubic satisfy the equation

$$v_1 + v_2 + v_3 + u_0 \equiv 0 \quad (\text{mod } \omega_1, \omega_2).$$

Hence the result follows.

In conclusion, I might note that the preceding work suggests to me the truth of the following statements concerning indeterminate equations, none of which, however, I can prove. The left-hand sides are supposed to have no squared factors in x, the curves represented by the equations are not degenerate, and the genus of the equations is supposed not less than one.

(1) The simultaneous indeterminate equations

$$ax^4 + bx^3 + cx^2 + dx + e = y^2,$$
$$a_1 x^4 + b_1 x^3 + c_1 x^2 + d_1 x + e_1 = y_1^2$$

can be satisfied by only a finite number of *rational* values of x.

---

* u_0 is included amongst them.

(2) The equation

$$ax^4 + bx^3 + cx^2 + dx + e = y^2$$

can be satisfied by only a finite number of *integral* values of x; and the same theorem holds for

$$ax^n + bx^{n-1} \dots kx + l = y^2.$$

(3) The equation

$$ax^6 + bx^5y + \dots fxy^5 + gy^6 = z^2$$

can be satisfied by only a finite number of *rational* values of x and y with the obvious extension to equations of higher degree.

(4) The same theorem holds for the equation

$$ax^4 + by^4 + cz^4 + 2fy^2z^2 + 2gz^2x^2 + 2hx^2y^2 = 0.$$

(5) The same theorem holds for any homogeneous equation of genus greater than unity, say, $f(x, y, z) = 0$.

It may be noted that if $f = 0$ represents a curve of genus unity, all of its rational points can be expressed rationally by means of a finite number of them; since Poincaré has proved[*] that $f = 0$ can be transformed into a cubic by a birational transformation with rational coefficients.

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[*] *Journal de Mathématique*, 5th series, vol. vii, 1901, page 177.

Prime Lattice Permutations. By Major P. A. MacMahon, Sc.D., F.R.S.

1. An assemblage of letters $\alpha_1^{k_1}\alpha_2^{k_2} \ldots \alpha_n^{k_n}$ in respect of the letters in the *definite order*

$$\alpha_1, \ \alpha_2, \ \ldots, \ \alpha_n$$

is said to be a lattice assemblage if

$$k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n.$$

Such an assemblage may be denoted by

$$|\ k_1 k_2 \ldots k_n\ | \ \text{ or } \ |\ \alpha_1^{k_1}\alpha_2^{k_2} \ldots \alpha_n^{k_n}\ |,$$

where in the usual notation $(k_1 k_2 \ldots k_n)$ is a partition of the number Σk. For each assemblage there is a corresponding partition.

If we take any permutation of the lattice assemblage we call it a lattice permutation if, a line being drawn between any two letters or after the last letter, the assemblage to the left of such line is a lattice assemblage. The line in question may be drawn in Σk positions so that a lattice permutation implies Σk lattice assemblages and the like number of partitions.

A lattice permutation is either prime or composite. It is composite if it be possible to draw a line between some two letters in such wise that on each side of the line a lattice permutation of some assemblage is in evidence. Thus $\alpha\beta\alpha\gamma\,|\,\alpha\beta$ is a composite lattice permutation of the assemblage $|\ \alpha^3\beta^2\gamma\ |$ because $\alpha\beta\alpha\gamma$, $\alpha\beta$ are lattice permutations of the assemblages $|\ \alpha^2\beta\gamma\ |$, $|\ \alpha\beta\ |$ respectively. On the other hand $\alpha\beta\alpha\alpha\gamma\beta$ is a prime lattice permutation of the assemblage $|\ \alpha^3\beta^2\gamma\ |$ [*].

2. We consider the lattice permutations of the assemblage $|\ k_1 k_2 \ldots k_n\ |$ and, further, the subdivision of these into prime and composite lattice permutations.

If a lattice permutation be composite, we must be able to draw one or more lines between letters dividing it into prime permutations. In correspondence therewith we can separate the whole assemblage into one or more component assemblages.

Denote by $l\ |\ k_1 k_2 \ldots k_n\ |$ any lattice permutation of $|\ k_1 k_2 \ldots k_n\ |$,

,, $L\ |\ k_1 k_2 \ldots k_n\ |$ the number of such.

,, $p\ |\ k_1 k_2 \ldots k_n\ |$ any prime lattice permutation of $|\ k_1 k_2 \ldots k_n\ |$,

,, $P\ |\ k_1 k_2 \ldots k_n\ |$ the number of such.

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* MacMahon, *Combinatory Analysis*, vol. I, Cambridge, 1915, ch. v, p. 124.

If we can separate the lattice assemblage into components in the manner

$$| k_1 k_2 \dots k_n | = | k_1' k_2' \dots k_n' | . | k_1'' k_2'' \dots k_n'' | \dots$$

we can proceed to a composite lattice permutation

$$p \, | k_1' k_2' \dots k_n' | . p \, | k_1'' k_2'' \dots k_n'' | \dots$$

in a number of ways denoted by

$$P \, | k_1' k_2' \dots k_n' | . P \, | k_1'' k_2'' \dots k_n'' | \dots .$$

Every lattice permutation

$$l \, | k_1 k_2 \dots k_n |$$

can be dealt with and we can account for the whole of them. We may therefore write

$$\begin{aligned}
L \, | k_1 k_2 \dots k_n | = {} & P \, | k_1 k_2 \dots k_n | \\
& + \Sigma P \, | k_1' k_2' \dots k_n' | . P \, | k_1'' k_2'' \dots k_n'' | \\
& + \Sigma P \, | \qquad\qquad | . P \, | \qquad\qquad | . P \, | \qquad\qquad | \\
& + \dots
\end{aligned}$$

until we obtain a numerical identity.

The succession of numbers $k_1 k_2 \dots k_n$ where $k_1 \geqslant k_2 \geqslant \dots \geqslant k_n$ denotes on the one hand a lattice assemblage of letters and on the other a partition of a number. It also denotes a multipartite number which specifies a lattice assemblage without specification of the particular letters $\alpha_1 \alpha_2 \dots \alpha_n$. There is a theory of the compositions (viz. partitions in which the order of the parts is of moment) of multipartite numbers which is involved in the present question. In general the part of a partition (or composition) of a multipartite number is or is not of the form

$$k_1' k_2' \dots k_n' \text{ when } k_1' \geqslant k_2' \geqslant \dots \geqslant k_n'.$$

If the former is the case we call it a *lattice part*, and we note that we are only concerned with compositions into lattice parts. We consider the lattice-part compositions of the multipartite number $k_1 k_2 \dots k_n$.

3. A simplification arises when

$$k_1 = k_2 = \dots = k_n = k,$$

for then the only lattice parts that present themselves are

$$(11\dots), \ (22\dots), \ (33\dots), \ \dots \ (kk\dots),$$

there being n numbers in each bracket.

Ex. gr. If the multipartite be (333) the compositions are

$$(333), \ (222, 111), \ (111, 222), \ (111, 111, 111).$$

In fact in this particular case we are concerned with the compositions of the unipartite number k.

Thus when $k = 3$, we really have to deal with the compositions (3), (21), (12), (111) of the number 3.

We are led to the relations

$$L\,|111| = P\,|111|$$
$$L\,|222| = P\,|222| + \{P\,|111|\}^2$$
$$L\,|333| = P\,|333| + 2P\,|222|\,P\,|111| + \{P\,|111|\}^3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

yielding

$$P\,|111| = L\,|111|$$
$$P\,|222| = L\,|222| - \{L\,|111|\}^2$$
$$P\,|333| = L\,|333| - 2L\,|222|\,L\,|111| + \{L\,|111|\}^3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Since we know (*loc. cit.*) that

$$L\,|kkk| = \frac{(3k)!\,2}{(k+2)!\,(k+1)!\,k!},$$

giving $\qquad L\,|111| = 1,\ \ L\,|222| = 5,\ \ L\,|333| = 42,$

we find $\qquad P\,|111| = 1,\ \ P\,|222| = 4,\ \ P\,|333| = 33.$

In general we have the two formulæ

$$L\,|kkk\ldots|_n = \Sigma\,\frac{(\Sigma\mu)!}{\mu_1!\,\mu_2!\ldots\mu_k!}\,\{P\,|111\ldots|_n\}^{\mu_1}\{P\,|222\ldots|_n\}^{\mu_2}\ldots\{P\,|kkk\ldots|_n\}^{\mu_k},$$

$$P\,|kkk\ldots|_n = \Sigma\,\frac{(-)^{\Sigma\mu+1}(\Sigma\mu)!}{\mu_1!\,\mu_2!\ldots\mu_k!}\,\{L\,|111\ldots|_n\}^{\mu_1}\{L\,|222\ldots|_n\}^{\mu_2}\ldots\{L\,|kkk\ldots|_n\}^{\mu_k},$$

where $\qquad L\,|kkk\ldots|_n = \dfrac{(nk)!\,(n-1)!}{(k+n-1)!\,(k+n-2)!\ldots k!},$

and the summations are for all partitions

$$1^{\mu_1}\,2^{\mu_2}\ldots k^{\mu_k}\ \text{of the number } k.$$

This is the solution of the problem of the enumeration of the prime lattice permutations of the assemblage

$$\alpha_1{}^k\alpha_2{}^k\ldots\alpha_n{}^k.$$

Otherwise we may assert that in regard to the equation

$$x^k - L\,|111\ldots|_n\,x^{k-1} - L\,|222\ldots|_n\,x^{k-2} - \ldots - L\,|kkk\ldots|_n = 0,$$

the homogeneous product sum, of order $s\,(\not> k)$, of the roots is

$$P\,|sss\ldots|_n.$$

In particular when $n = 2$,

$$x^k - x^{k-1} - 2x^{k-2} - 5x^{k-3} - 14x^{k-4} - \ldots - \frac{(2k)!}{(k+1)!\,k!} = 0,$$

and we find that $\qquad P\,|s+1,\,s+1| = L\,|s,\,s|^*.$

* Cf. *Netto Combinatorik*, Leipzig, 1901, § 122, p. 192 *et seq.*

4. The passage to the general case of the assemblage is now clear. Taking the assemblage

$$\alpha_1{}^{k_1}\alpha_2{}^{k_2}\ldots\alpha_n{}^{k_n}\big| \quad k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n,$$

we find the relation

$$
\begin{aligned}
&\frac{1}{1 - P\,|\,1\,|\,x_1 - P\,|\,11\,|\,x_1 x_2 - P\,|\,111\,|\,x_1 x_2 x_3 - P\,|\,1111\,|\,x_1 x_2 x_3 x_4 - \ldots} \\
&\qquad\quad - P\,|\,2\,|\,x_1{}^2 \quad - P\,|\,21\,|\,x_1{}^2 x_2 \quad - P\,|\,211\,|\,x_1{}^2 x_2 x_3 \\
&\qquad\qquad\qquad\quad - P\,|\,3\,|\,x_1{}^3 \qquad - P\,|\,22\,|\,x_1{}^2 x_2{}^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad - P\,|\,31\,|\,x_1{}^3 x_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad - P\,|\,4\,|\,x_1{}^4
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + L\,|\,1\,|\,x_1 + L\,|\,11\,|\,x_1 x_2 + L\,|\,111\,|\,x_1 x_2 x_3 + L\,|\,1111\,|\,x_1 x_2 x_3 x_4 + \ldots \\
&\quad + L\,|\,2\,|\,x_1{}^2 \quad + L\,|\,21\,|\,x_1{}^2 x_2 \quad + L\,|\,211\,|\,x_1{}^2 x_2 x_3 \\
&\qquad\qquad\qquad\quad + L\,|\,3\,|\,x_1{}^3 \qquad + L\,|\,22\,|\,x_1{}^2 x_2{}^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad + L\,|\,31\,|\,x_1{}^3 x_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad + L\,|\,4\,|\,x_1{}^4
\end{aligned}
$$

leading to the two formulæ

$$L\,|\,k_1 k_2 k_3 \ldots\,| = \Sigma\,\frac{(\Sigma\kappa)!}{\kappa_1!\,\kappa_2!\ldots}\,\{P\,|\,k_1' k_2' k_3' \ldots\,|\}^{\kappa_1}\{P\,|\,k_1'' k_2'' k_3'' \ldots\,|\}^{\kappa_2}\ldots$$

$$P\,|\,k_1 k_2 k_3 \ldots\,| = \Sigma(-)^{\Sigma\kappa+1}\,\frac{(\Sigma\kappa)!}{\kappa_1!\,\kappa_2!\ldots}\,\{L\,|\,k_1' k_2' k_3' \ldots\,|\}^{\kappa_1}\{L\,|\,k_1'' k_2'' k_3'' \ldots\,|\}^{\kappa_2}\ldots$$

the summations being in respect of every lattice-part partition

$$(k_1' k_2' k_3' \ldots)^{\kappa_1}(k_1'' k_2'' k_3'' \ldots)^{\kappa_2}\ldots$$

of the multipartite number

$$(k_1 k_2 k_3 \ldots).$$

Since the value of $L\,|\,k_1 k_2 k_3 \ldots\,|$ is known (*loc. cit.*) we thus obtain an expression for $P\,|\,k_1 k_2 k_3 \ldots\,|$.

The best way of calculating these numbers is to take the relation from which the formulæ are obtained, clear the fraction and equate coefficients of like powers of $x_1{}^{k_1} x_2{}^{k_2} x_3{}^{k_3}\ldots$.

The coefficient of $x_1{}^2 x_2$ gives

$$P\,|\,21\,| + P\,|\,11\,|\,L\,|\,1\,| + P\,|\,1\,|\,L\,|\,11\,| - L\,|\,21\,| = 0,$$

and so forth.

The calculation is simplified by noting that $P\,|\,k_1 k_2\,| = 0$ if $k_1 > k_2$ and that

$$L\,|\,k_1 k_2 k_3 \ldots\,| = L\,|\,j_1 j_2 j_3 \ldots\,|$$

if $(k_1 k_2 k_3 \ldots)$, $(j_1 j_2 j_3 \ldots)$ be conjugate partitions.

The Theory of Modular Partitions. By Major P. A. MacMahon, Sc.D., F.R.S.

The Denotation of a Partition of a Number.

1. There are two methods of denoting partitions of numbers which have been of service in researches. The one simply denotes a part by a number which gives its magnitude and places the parts in descending order of magnitude—usually in a horizontal line. It is more convenient for my present purpose to suppose the parts placed underneath one another in a vertical line—say $\frac{3}{2}$ a partition of the number 5.

The other, after Ferrers, denotes a part by a succession of nodes, equal in number to the magnitude of the part, placed in a horizontal line and successive parts placed underneath one another in numerical order so that the left-hand nodes of the several parts are in a vertical line. Thus the Euler notation $\frac{3}{2}$ for a partition becomes

in the Ferrers notation.

The connexion between these two modes was established in a previous paper* wherein it was shown that the Ferrers representation is fundamentally one which employs units instead of nodes and is most suitably denoted by, in the above special case,

$$111$$
$$11$$

and is, like the Euler notation, numerical but in two dimensions of space and composed entirely of units. This appears when we base the theory on a deeper foundation than that furnished by Euler's intuitive method. There appears, moreover, to be no advantage in working with nodes rather than units. Every consideration and transformation of the Ferrers graphs is just as simple with units— but on the other hand, from the point of view of generalization, the unit representation possesses possibilities that are not shared by the graph of nodes. The Euler and Ferrers representations have both been of great service in the theory and it appears that the subject may be regarded from the point of view which is now taken up.

* *Phil. Trans. R. S. A.*, vol. cxcii, p. 356.

Let each part of a partition be expressed in the form

$$\nu \bmod \mu$$

so that a part $p = s\mu + \nu$.

We may denote the part by a succession of s numbers equal to μ followed by a number ν which may be any of the integers 0, 1, 2, ... $\mu - 1$; these integers being placed in a horizontal line and the zero being, where it occurs, omitted. All the parts of the partition may be similarly treated and the parts placed in successive rows with the left-hand integers of each in the same column. If the parts be placed in rows of descending order of magnitude, as regards numerical content, we obtain a representation of the partition to the modulus μ.

The partition 753221 of the number 20 would be represented to various moduli as follows:

Mod 1	2	3	4	5	6	7 to ∞
1111111	2221	331	43	52	61	7
11111	221	32	41	5	5	5
111	21	3	3	3	3	3
11	2	2	2	2	2	2
11	2	2	2	2	2	2
1	1	1	1	1	1	1

and it is evident that the Ferrers and Euler moduli are 1 and ∞ respectively.

It is convenient to call the number of integers which occur in the representation of a part to a given modulus its "*range*" with regard to the modulus.

The "range" of a partition, similarly, is a convenient phrase, such range being the same as the range of the highest part in the partition.

When the modulus is unity the representation by units has the valuable property that a partition to the same modulus is reached if it be read by columns instead of by rows. In other words the partition is conjugable and this fact has been applied by Sylvester and others to obtain interesting algebraic identities. This property is not enjoyed by the complete set of partitions to any modulus differing from unity.

When the modulus is ∞ indeed only one partition of n, viz. the partition, which is n itself, enjoys the property.

For every modulus we can uniquely select a set of partitions which enjoys the property. We call this the set of conjugable partitions to the modulus μ. If we write down the conjugable partitions of 6 to the modulus 2 we find

222	221	22	21	2
	1	2	2	2
			1	2

where observe that $\frac{21}{21}$ is not a member of the set because, read by columns, $\frac{22}{11}$ is not a partition expressed to the modulus 2.

In general, for the modulus 2, any partition which, in the Euler representation, involves an uneven part more than once cannot be comprised in the set. The enumerating generating function for the conjugable partitions of n to the modulus 2 is therefore

$$\frac{(1 + q)(1 + q^3)(1 + q^5) \dots}{(1 - q^2)(1 - q^4)(1 - q^6) \dots},$$

or in Cayley's notation $\qquad \dfrac{[1 + q^{2m+1}]}{[1 - q^{2m}]}.$

Since it may be also written

$$\frac{1}{(1 - q)(1 - q^3)(1 - q^4)(1 - q^5)(1 - q^7)(1 - q^8)(1 - q^9) \dots},$$

we see that the partitions are equi-numerous with the system in which no parts occur of the form 2 mod 4.

We are already in a position to enunciate a general theorem in regard to any conjugable set of partitions to the modulus μ.

Theorem. " Of the conjugate set of partitions to the modulus μ of the number n there are as many partitions which have a range $\left(\begin{matrix}\text{equal to}\\\text{not exceeding}\end{matrix}\right)$ k and a number of parts $\left(\begin{matrix}\text{equal to}\\\text{not exceeding}\end{matrix}\right)$ i as there are partitions which have a range $\left(\begin{matrix}\text{equal to}\\\text{not exceeding}\end{matrix}\right)$ i and a number of parts $\left(\begin{matrix}\text{equal to}\\\text{not exceeding}\end{matrix}\right)$ k."

When the range is k and the greatest part has the value j,

$$k = \underset{\triangleleft}{E} \frac{j}{\mu},$$

where $\underset{\triangleleft}{E}x$ denotes the integer not less than x.

When the range is k, j must have one of the values

$$\mu k, \ \mu k - 1, \ \mu k - 2, \ \dots, \ \mu k - (\mu - 1).$$

With the modulus μ, for a partition to appertain to the conjugable set, two adjacent parts must not terminate with numbers drawn from the series

$$1, \ 2, \ \dots, \ \mu - 1,$$

or in other words the magnitudes of two adjacent parts must not be numbers drawn from the series

$$\mu r + 1, \ \mu r + 2, \ \dots, \ \mu r + \mu - 1,$$

where r is any integer (zero included).

Hence it follows that the function which enumerates the number of partitions in the conjugable set is

$$\frac{[1 + q^{\mu m+1} + q^{\mu m+2} + \ldots + q^{\mu m+\mu-1}]}{[1 - q^{\mu m}]},$$

which may be thrown into the form

$$1 + \frac{q}{1-q} \cdot \frac{1-q^{\mu-1}}{1-q^{\mu}} + \frac{q^2}{(1-q)^2} \frac{q^{\mu}(1-q^{\mu-1})^2}{(1-q^{\mu})(1-q^{2\mu})} + \frac{q^3}{(1-q)^3} \cdot \frac{q^{3\mu}(1-q^{\mu-1})^3}{(1-q^{\mu})(1-q^{2\mu})(1-q^{3\mu})} + \ldots$$

with a denominator $\qquad [1 - q^{\mu m}],$

and where the general term in the numerator is

$$\left(\frac{q}{1-q}\right)^s \frac{q^{\binom{s}{2}\mu}(1-q^{\mu-1})^s}{(1-q^{\mu})(1-q^{2\mu}) \ldots (1-q^{s\mu})}.$$

Limitation of the Number of Parts.

2. I now proceed to consider the enumerating function when the number of parts in the partitions is limited by the number i.

For *modulus* 2 this is the coefficient of a^i in the expansion of

$$Q = \frac{(1 + aq)(1 + aq^3)(1 + aq^5) \ldots}{(1-a)(1-aq^2)(1-aq^4)(1-aq^6) \ldots} = 1 + aQ_1 + a^2 Q_2 + \ldots.$$

If in Q we write aq^2 for a and then multiply Q by

$$\frac{1 + aq}{1 - a},$$

the function Q is unaltered.

Thence we readily find that

$$Q_i = \frac{(1 - q^2)(1 - q^6)(1 - q^{10}) \ldots (1 - q^{4i-2})}{(1-q)(1-q^2)(1-q^3) \ldots (1-q^{2i})},$$

the required enumerating function.

Also the function which enumerates the partitions into exactly i parts is

$$Q_i - Q_{i-1} = \frac{1+q}{1-q^{2i}} q^{2i-1} Q_{i-1}.$$

For *modulus* 3 we obtain similarly the enumerating function

$$\frac{(1 + q + q^2)(1 + q^4 + q^5) \ldots (1 + q^{3i-2} + q^{3i-1})}{(1-q^3)(1-q^6) \ldots (1-q^{3i})}$$

for the partitions into i or fewer parts; and for

Modulus μ

$$\frac{(1 + q + \ldots + q^{\mu-1})(1 + q^{\mu+1} + \ldots + q^{2\mu-1}) \ldots (1 + q^{i\mu-\mu+1} + \ldots + q^{i\mu-1})}{(1-q^{\mu})(1-q^{2\mu}) \ldots (1-q^{i\mu})}.$$

Limitation of the Part Magnitude.

Modulus 2. If the part magnitude be $2j$ the function is

$$\frac{(1 + q)\,(1 + q^3)\,\dots\,(1 + q^{2j-1})}{(1 - q^2)\,(1 - q^4)\,\dots\,(1 - q^{2j})} ,$$

and if the part magnitude be $2j + 1$ it is

$$\frac{(1 + q)\,(1 + q^3)\,\dots\,(1 + q^{2j+1})}{(1 - q^2)\,(1 - q^4)\,\dots\,(1 - q^{2j})} .$$

Modulus 3.

For part magnitude $= 3j$ function is $\dfrac{(1 + q + q^2)\,(1 + q^4 + q^5)\dots(1 + q^{3j-2} + q^{3j-1})}{(1 - q^3)\,(1 - q^6)\,\dots\,(1 - q^{3j})}$;

$$3j + 1 \quad ,, \qquad \frac{(1 + q + q^2)\,(1 + q^4 + q^5)\dots(1 + q^{3j+1} + 0)}{(1 - q^3)\,(1 - q^6)\dots(1 - q^{3j})};$$

$$3j + 2 \quad ,, \qquad \frac{(1 + q + q^2)\,(1 + q^4 + q^5)\dots(1 + q^{3j+1} + q^{3j+2})}{(1 - q^3)\,(1 - q^6)\dots(1 - q^{3j})} ;$$

and in general for the

Modulus μ.

For part magnitude

μj function is $\dfrac{(1 + q + \dots + q^{\mu-1})\,(1 + q^{\mu+1} + \dots + q^{2\mu-1})\,\dots\,(1 + q^{\mu j-\mu+1} + \dots + q^{\mu j-1})}{(1 - q^\mu)\,(1 - q^{2\mu})\,\dots\,(1 - q^{\mu j})}$;

$\mu j + 1 \quad ,, \qquad \dfrac{(1 + q + \dots + q^{\mu-1})\dots(1 + q^{\mu j-\mu+1} + \dots + q^{\mu j-1})\,(1 + q^{\mu j+1})}{(1 - q^\mu)\,(1 - q^{2\mu})\dots(1 - q^{\mu j})}$;

$\mu j + 2 \quad ,, \qquad \dfrac{(1 + q + \dots + q^{\mu-1})\dots(1 + q^{\mu j-\mu+1} + \dots + q^{\mu j-1})\,(1 + q^{\mu j+1} + q^{\mu j+2})}{(1 - q^\mu)\,(1 - q^{2\mu})\dots(1 - q^{\mu j})}$;

....

$mj + \mu - 1 \quad ,, \qquad \dfrac{(1 + q + \dots q^{\mu-1})\,(1 + q^{\mu+1} + \dots + q^{2\mu-1})\dots(1 + q^{\mu j+1} + \dots + q^{\mu j+\mu-1})}{(1 - q^\mu)\,(1 - q^{2\mu})\,\dots\,(1 - q^{\mu j})}$.

Limitation of the Range.

3. We have only to conjugate the partitions of a set to modulus μ to see that the function which enumerates partitions for a given limitation of range is identical with that which enumerates for the same limitation of the number of parts.

In fact when the range is i the part magnitude may be any one of

$$(i - 1)\,\mu + 1,\ (i - 1)\,\mu + 2,\ \dots,\ i\mu,$$

or may take μ values.

Thence we see that the part magnitude is limited not to exceed $i\mu$.

The expression obtained above, for the limitation of the part magnitude to μj, is, on writing i for j, identical with that which has

been obtained above for the limitation of the number of parts so as not to exceed the integer i.

Limitation of the Range and Number of Parts.

4. When the range is limited to k and the number of parts to i we can now see from the preceding that the enumeration is given by the coefficient of a^i in

$$R_{i,k} = \frac{\{1 + a\,(q + q^2 + \ldots + q^{\mu-1})\}\,\{1 + a\,(q^{\mu+1} + \ldots + q^{2\mu-1})\}\ldots\{1 + a\,(q^{k\mu-\mu+1} + \ldots + q^{k\mu-1})\}}{(1-a)\,(1 - aq^\mu)\,(1 - aq^{2\mu})\ldots(1 - aq^{k\mu})}$$

and the fact of conjugation shows that this is also the coefficient of a^k in

$$R_{k,i} = \frac{\{1 + a\,(q + q^2 + \ldots + q^{\mu-1})\}\,\{1 + a\,(q^{\mu+1} + \ldots + q^{2\mu-1})\}\ldots\{1 + a\,(q^{i\mu-\mu+1} + \ldots + q^{i\mu-1})\}}{(1-a)\,(1 - aq^\mu)\,(1 - aq^{2\mu})\ldots(1 - aq^{i\mu})}.$$

It is easily verified for $k = 1$, $i = 2$ when the coefficient is found to be in each case

$$\frac{1 - q^{2\mu+1}}{1 - q}.$$

The Algebraic Genesis of the Modular Partitions.

5. I have in previous papers* considered the algebraic fraction

$$\frac{1}{(1 - \lambda_1 X_1)\left(1 - \dfrac{\lambda_2}{\lambda_1} X_2\right)\left(1 - \dfrac{\lambda_3}{\lambda_2} X_3\right)\ldots\left(1 - \dfrac{1}{\lambda_{i-1}} X_i\right)},$$

in which the general term is

$$\lambda_1^{a_1 - a_2}\,\lambda_2^{a_2 - a_3}\ldots\lambda_{i-1}^{a_{i-1} - a_i}\,X_1^{a_1}\,X_2^{a_2}\ldots X_i^{a_i}.$$

If the numbers $\alpha_1, \alpha_2, \ldots, \alpha_i$ satisfy the conditions

$$\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_i,$$

the term product $\qquad X_1^{a_1}\,X_2^{a_2}\ldots X_i^{a_i}$

denotes by its exponent a partition

$$(\alpha_1 \alpha_2 \ldots \alpha_i)$$

of the number $\Sigma\alpha$.

In order to satisfy the system of Diophantine Inequalities we must, on expansion of the algebraic fraction, reject all negative powers of $\lambda_1, \lambda_2, \ldots, \lambda_i$ and we *may* afterwards put

$$\lambda_1 = \lambda_2 = \ldots = \lambda_i = 1.$$

* *Phil. Trans. R. S. A.*, vol. cxcii, p. 356 *et seq.*

Denoting the performance of both of these operations by $\underset{\geqq}{\Omega}$, we find

$$\underset{\geqq}{\Omega}\, \frac{1}{(1-\lambda_1 X_1)\left(1-\dfrac{\lambda_2}{\lambda_1}X_2\right)\left(1-\dfrac{\lambda_3}{\lambda_2}X_3\right)\ldots\left(1-\dfrac{\lambda_{i-1}}{\lambda_{i-2}}X_{i-1}\right)\left(1-\dfrac{1}{\lambda_{i-1}}X_i\right)}$$

$$= \frac{1}{(1-X_1)(1-X_1X_2)(1-X_1X_2X_3)\ldots(1-X_1X_2X_3\ldots X_i)},$$

and this a real generating function of all partitions to modulus 1 (of all numbers) whose range does not exceed i.

The generating function is real because an X product appears in correspondence with every partition. To obtain an enumerating function it is necessary to put

$$X_1 = X_2 = \ldots = X_i = x.$$

In fact the expansion exhibits the Ferrers representation.

In the next place consider the expression

$$\underset{\geqq}{\Omega}\, \frac{1}{(1-\lambda_1 X_1)\left(1-\dfrac{\lambda_2}{\lambda_1}X_1\right)\left(1-\dfrac{\lambda_3}{\lambda_2}X_2\right)\left(1-\dfrac{\lambda_4}{\lambda_3}X_2\right)\ldots\left(1-\dfrac{\lambda_{2i-1}}{\lambda_{2i-2}}X_i\right)\left(1-\dfrac{\lambda_{2i}}{\lambda_{2i-1}}X_i\right)},$$

where, for convenience only, the number of denominator factors is taken to be even. We thence reach the real generating function

$$\frac{1}{(1-X_1)(1-X_1^2)(1-X_1^2X_2)(1-X_1^2X_2^2)\ldots(1-X_1^2X_2^2\ldots X_{i-1}^2X_i)(1-X_1^2X_2^2\ldots X_i^2)}$$

of partitions to the modulus 2 and of range i.

The fundamental set of partitions

$$1,\ 2,\ 21,\ 22,\ \ldots\ 22\ldots1,\ 22\ldots2$$

is indicated.

The real generating function of the conjugate system is

$$\frac{(1+X_1)(1+X_1^2X_2)(1+X_1^2X_2^2X_3)\ldots(1+X_1^2X_2^2\ldots X_{i-1}^2X_2)}{(1-X_1^2)(1-X_1^2X_2^2)(1-X_1^2X_2^2X_3^2)\ldots(1-X_1^2X_2^2\ldots X_i^2)}.$$

Similarly for the modulus μ we take as generating function

$$\underset{\geqq}{\Omega}\, \frac{1}{D},$$

where D is the product of

μ factors $(1-\lambda_1 X_1)\left(1-\dfrac{\lambda_2}{\lambda_1}X_1\right)\ldots\left(1-\dfrac{\lambda_\mu}{\lambda_{\mu-1}}X_1\right)$,

μ factors $\left(1-\dfrac{\lambda_{\mu+1}}{\lambda_\mu}X_2\right)\left(1-\dfrac{\lambda_{\mu+2}}{\lambda_{\mu+1}}X_2\right)\ldots\left(1-\dfrac{\lambda_{2\mu}}{\lambda_{2\mu-1}}X_2\right)$,

....

$$\mu \text{ factors } \left(1 - \frac{\lambda_{(i-2)\mu+1}}{\lambda_{(i-2)\mu}} X_{i-1}\right) \left(1 - \frac{\lambda_{(i-2)\mu+2}}{\lambda_{(i-2)\mu+1}} X_{i-1}\right) \cdots \left(1 - \frac{\lambda_{(i-1)\mu}}{\lambda_{(i-1)\mu-1}} X_{i-1}\right),$$

$$\mu \text{ factors } \left(1 - \frac{\lambda_{(i-1)\mu+1}}{\lambda_{(i-1)\mu}} X_1\right) \left(1 - \frac{\lambda_{(i-1)\mu+2}}{\lambda_{(i-1)\mu+1}} X_i\right) \cdots \left(1 - \frac{\lambda_{i\mu}}{\lambda_{i\mu-1}} X_1\right),$$

where for convenience only the number of denominator factors is supposed to be of the form $\equiv 0 \bmod \mu$, and thence proceed to the real generating function

$$\frac{1}{(1 - X_1)(1 - X_1^2) \cdots (1 - X_1^\mu)(1 - X_1^\mu X_2) \cdots (1 - X_1^\mu X_2^\mu) \cdots (1 - X_1^\mu X_2^\mu \cdots X_i^\mu)}$$

and that of the conjugate system to the modulus μ and range i

$$\frac{N}{(1 - X_1^\mu)(1 - X_1^\mu X_2^\mu)(1 - X_1^\mu X_2^\mu X_3^\mu) \cdots (1 - X_1^\mu X_2^\mu \cdots X_i^\mu)},$$

where N is the product of factors

$$(1 + X_1 + X_1^2 + \ldots + X_1^{\mu-1})$$

$$(1 + X_1^\mu X_2 + X_1^\mu X_2^2 + \ldots + X_1^\mu X_2^{\mu-1})$$

$$(1 + X_1^\mu X_2^\mu X_3 + X_1^\mu X_2^\mu X_3^2 + \ldots + X_1^\mu X_2^\mu X_3^{\mu-1})$$

$$\vdots$$

$$(1 + X_1^\mu X_2^\mu \ldots X_{i-1}^\mu X_i + X_1^\mu X_2^\mu \ldots X_{i-1}^\mu X_i^2 + \ldots + X_1^\mu X_2^\mu \ldots X_{i-1}^\mu X_i^{\mu-1}).$$

On an Integral Equation. By J. E. LITTLEWOOD. With a Note by E. A. MILNE.

[*Read* 22 May, 1922.]

1. The equation with which we shall be concerned is

(1.1) $$f'(x) = \int_0^\infty \frac{f(x+t) - f(x-t)}{2t} e^{-t} dt.$$

It has its origin in a physical problem, discussed by Mr E. A. Milne in *Monthly Notices, R.A.S.*, 81, 361 (1921)[*]. It is at once evident that

(1.2) $$f(x) = ax^2 + bx + c$$

is a solution, and it is plausible, on physical and on some mathematical grounds, that there is no other; i.e. that an $f(x)$, for which (1.1) is both significant and true, must be of the form (1.2).

The present paper does not prove so much as this, but I have succeeded in finding conditions for f of some generality, if not of very elegant form, subject to which there is no solution other than (1.2). In default of a complete solution I have not generalised unduly, and have aimed chiefly at simplicity. There is one special set of restrictions on f, viz. that $f''(x)$ is continuous, and bounded as $x \to \pm \infty$, which is of particular interest. In Mr Milne's problem $f''(x)$ is the gradient of an energy density, the special assumption appears to be physically legitimate, and so far as our problem has physical interest a solution for the special case is all that is required. From the point of view of theory there is not much interest in generalising, beyond a point, the conditions of a theorem that is probably true unconditionally, and I have done no more than modify my original proof for the special case to obtain such generality as is possible without new arguments; the only simplification we should find if we allowed the maximum assumptions would be that proofs of lemmas 1 to 3 would become unnecessary.

The solution of the problem has some purely mathematical interest; a method that is successful in any part of the theory of integral equations is probably worth recording for its own sake, and the type of argument appears to be new.

2. Let us write

(2.1) $$\phi(x) = f'(x).$$

Then, provided f' is continuous, (1.1) becomes

(2.2) $$\phi(x) = \int_0^\infty \frac{\phi_1(x+t) - \phi_1(x-t)}{2t} e^{-t} dt,$$

---

[*] See the note by Mr Milne following the present paper.

where

(2.3)
$$\phi_1(x) = \int_0^x \phi(u)\, du.$$

We suppose $x \geqslant 0$, and write

(2.4)　　$M(x) = \underset{-x \leqslant u \leqslant x}{\text{Max.}} \phi(u),\ m(x) = \underset{-x \leqslant u \leqslant x}{\text{Min.}} \phi(u),$

(2.5)　　　　　$\Omega(x) = M(x) - m(x),$

(2.6)　　　　　$\Omega_1(x) = 1 + \Omega(x).$

Thus $\Omega(x)$ is the "oscillation" of $\phi(u)$ in $-x \leqslant u \leqslant x$. It is a positive increasing function of x (in the wide sense).

We shall say that ϕ *satisfies condition C if*

(2.7)　　　　$\Omega(x+1) - \Omega(x) = o\{\Omega_1(x)\},$

as $x \to \infty$. Our main theorem may now be stated as follows.

Theorem A. If $\phi(x)$ *is an even continuous solution of* (2.2), *satisfying condition C, then* $\phi(x)$ *is a constant.*

3. It is convenient to collect the inequalities we shall require concerning $\Omega(x)$, and to begin by proving these.

Lemma 1. *We have*
$$\Omega_1(x+\xi) < 2e^{\frac{1}{3}\xi}\, \Omega_1(x) \quad (x > x_0,\ \xi \geqslant 0).$$

From condition C we have, if $x > x_0(\epsilon)$,
$$\Omega(x+1) - \Omega(x) < \epsilon \Omega_1(x),$$
$$\Omega_1(x+1) < (1+\epsilon)\, \Omega_1(x),$$
$$\Omega_1(x+2) < (1+\epsilon)\, \Omega_1(x+1) < (1+\epsilon)^2\, \Omega_1(x),$$
$$\dots\dots\dots\dots\dots\dots\dots,$$
$$\Omega_1(x+n) < (1+\epsilon)^n\, \Omega_1(x).$$

From this we deduce: first that for fixed $k > 0$,

(3.1)　$\Omega(x+k) - \Omega(x) = \Omega_1(x+k) - \Omega_1(x) = o\{\Omega_1(x)\};$

and secondly, by choosing $\epsilon = e^{\frac{1}{3}} - 1$, that
$$\Omega_1(x+\xi) < (e^{\frac{1}{3}})^{\xi+1}\, \Omega_1(x) < 2e^{\frac{1}{3}\xi}\, \Omega_1(x),$$
the result of the lemma.

Lemma 2. *We have*
$$\int_0^\infty \{\Omega(x+t) - \Omega(x)\}\, e^{-t} dt = o\{\Omega_1(x)\}.$$

The left side is less than
$$\int_0^k \{\Omega(x+t) - \Omega(x)\}\, e^{-t} dt + \int_k^\infty \Omega_1(x+t)\, e^{-t} dt$$
$$< \int_0^k o\{\Omega_1(x)\}\, e^{-t} dt + \int_k^\infty 2e^{\frac{1}{3}t}\, \Omega_1(x)\, e^{-t} dt,$$

by (3.1) and lemma 1,

$$= o\left\{\Omega_1(x)\right\} + 3e^{-\frac{2}{3}k}\,\Omega_1(x).$$

Since k is arbitrary, the left side must be of the form $o\left\{\Omega_1(x)\right\}$.

Lemma 3. *We have, as* $x \to \infty$,

$$\int_\xi^\infty \left\{\Omega(x+t) - \Omega(x)\right\} e^{-t} dt \doteq o\left\{e^{-\frac{3\xi}{5}}\,\Omega_1(x)\right\},$$

uniformly in $\xi \geqslant 0$.

Denoting the left side by P, we have, on the one hand,

$$(3.2) \qquad P \leqslant \int_0^\infty \left\{\Omega(x+t) - \Omega(x)\right\} e^{-t} dt = o\left\{\Omega_1(x)\right\}$$

by lemma 2 ; and on the other,

$$P < \int_\xi^\infty \Omega_1(x+t)\, e^{-t} dt = e^{-\xi} \int_0^\infty \Omega_1(x+\xi+t)\, e^{-t} dt$$

$$= e^{-\xi} \int_0^\infty O\left(e^{\frac{1}{3}(\xi+t)}\right) \Omega_1(x)\, e^{-t}\, dt,$$

by lemma 1,

$$(3.3) \qquad\qquad\qquad = O\left(e^{-\frac{2}{3}\xi}\right) \Omega_1.(x).$$

From (3.2) and (3.3)

$$P = \left\{o\left(\Omega_1(x)\right)\right\}^{\frac{1}{10}} \left\{O\left(e^{-\frac{2}{3}\xi}\right)\Omega_1(x)\right\}^{\frac{9}{10}} = o\left(e^{-\frac{3}{5}\xi}\right)\Omega_1(x).$$

4. The equation (2.2) may be written

$$\int_0^\infty \frac{\phi_1(x+t) - \phi_1(x-t) - 2t\phi(x)}{2t}\, e^{-t}\, dt = 0,$$

or

$$(4.1) \qquad\qquad J(x) = J_1(x) + J_2(x) = 0,$$

where

$$(4.2) \qquad J_1(x) = \int_0^\infty \frac{A_1(t)}{2t}\, e^{-t} dt, \ \ J_2(x) = \int_0^\infty \frac{A_2(t)}{2t}\, dt,$$

$$(4.3) \qquad \begin{cases} A_1(t) = A_1(x,\,t) = \phi_1(x+t) - \phi_1(x) - t\phi(x), \\ A_2(t) = A_2(x,\,t) = \phi_1(x) - \phi_1(x-t) - t\phi(x). \end{cases}$$

The ideas of the proof will become clearer if the geometric meaning of A_1 and A_2 is borne in mind. If the curve in fig. 1 is $y = \phi(x)$, $A_1(t)$ and $A_2(t')$ are respectively the areas of the shaded regions to the right and left of x, regard being had to sign, and areas being reckoned negative, when (as in the figure) they lie below the parallel to the x-axis whose ordinate is $\phi(x)$.
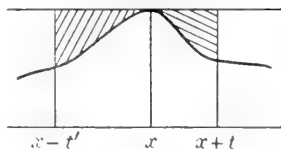
Fig. 1

Lemma 4. *If a continuous solution* ϕ *of* (2.2) *or* (4.1) *has an absolute maximum or minimum*, then* ϕ *is a constant.*

We need only observe that if x gives, say, an absolute maximum, we have $A_1(t) \leqslant 0$, $A_2(t) \leqslant 0$, for all t; and that if, in addition, $\phi(y) < \phi(x)$ for some y, greater than x, say, then (since ϕ is continuous) $A_1(t) \leqslant -\delta < 0$ for $t \geqslant y - x + \delta'$, and (4.1) is clearly false.

5. We prove Theorem A by a *reductio ad absurdum*. We take, in fact, for our hypothesis throughout that ϕ is an even, continuous, and not constant solution of (4.1), and this is to be understood as a premiss in the lemmas that follow.

Lemma 5. *There exist arbitrarily large values of* x *for which*

$$\phi(y) \leqslant \phi(x) \quad (0 \leqslant y \leqslant x);$$

and also arbitrarily large values of x *for which*

$$\phi(y) \geqslant \phi(x) \quad (0 \leqslant y \leqslant x).$$

This is immediate since ϕ is even, and has no absolute maximum or minimum.

Lemma 6. *There exists a sequence of maxima* $x = X_n$, *and minima* $x = x_n$, *with the following properties:*

(1) $0 < \ldots < X_n < x_n < X_{n+1} < x_{n+1} < \ldots$.

(2) $X_n \to \infty$, $x_n \to \infty$ *as* $n \to \infty$.

(3) $\phi(x) \leqslant \phi(X_n) \quad (0 \leqslant x \leqslant X_n)$.

(4) $\phi(x) \geqslant \phi(x_n) \quad (0 \leqslant x \leqslant x_n)$.

(5) $\phi(X_n) \geqslant \phi(x) \geqslant \phi(x_n) \quad (X_n \leqslant x \leqslant x_n)$.

(6) $\phi(x_n) \leqslant \phi(x) \leqslant \phi(X_{n+1}) \quad (x_n \leqslant x \leqslant X_{n+1})$.

There exists (by lemma 5 and our hypothesis) a ξ, such that

(5.1) $\phi(x)$ *is not constant in* $0 \leqslant x \leqslant \xi$

and $\phi(x) \leqslant \phi(\xi) \quad (0 \leqslant x \leqslant \xi)$.

Let η be the least number† greater than ξ, such that

$$\phi(x) \geqslant \phi(\eta) \quad (0 \leqslant x \leqslant \eta),$$

and ξ' the least number greater than η, such that

$$\phi(x) \leqslant \phi(\xi') \quad (0 \leqslant x \leqslant \xi').$$

Let η_1 give the absolute minimum‡ of ϕ for the range $\xi \leqslant x \leqslant \xi'$, and ξ_1 the absolute maximum‡ for $\xi \leqslant x \leqslant \eta_1$. We take $X_1 = \xi_1$,

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* In the wide sense. Thus $\phi(x)$ is an absolute maximum if $\phi(y) \leq \phi(x)$ for all values of y.

† Its existence follows from lemma 5 and (5.1).

‡ In case a set of points gives the same absolute maximum or minimum we select the least of the set (existent since, ϕ being continuous, the set is closed).

$x_1 = \eta_1$, start afresh with ξ' in the place of ξ, and continue the process indefinitely. We evidently obtain *maxima* X_n and *minima* x_n with all the six properties except perhaps (2). Finally if (2) does not hold, we have

(5.2) $$X_n \to X, \quad x_n \to X.$$

But on account of (5.1)

$$\phi(X_n) - \phi(x_n) \geqslant \phi(X_1) - \phi(x_1) > 0,$$

and this is incompatible with (5.2) and the continuity of ϕ at X.

6. We write

(6.1) $h_n = \Omega(x_n) = M(x_n) - m(x_n) = \phi(X_n) - \phi(x_n),$

(6.2) $J_1(x, \xi) = \int_\xi^\infty \dfrac{A_1(t)}{2t} e^{-t} dt, \quad J_2(x, \xi) = \int_\xi^\infty \dfrac{A_2(t)}{2t} e^{-t} dt.$

Lemma 7. *We have as* $n \to \infty$, *uniformly in* $\xi \geqslant 0$,

$$J_1(X_n, \xi) < o(h_n) e^{-\frac{3}{5}\xi}, \quad J_1(x_n, \xi) > o(h_n) e^{-\frac{3}{5}\xi}.$$

In particular (if $\xi = 0$)

$$J_1(X_n) < o(h_n), \quad J_1(x_n) > o(h_n).$$

These results remain true if J_2 *be written throughout for* J_1.

Since $\phi(X_n) = M(x_n)$, we have

$$\frac{A_1(X_n, t)}{2t} = \frac{1}{2t} \int_0^t \{\phi(X_n + t) - \phi(X_n)\} \, dt$$

$$\leqslant \frac{1}{2t} \int_0^t \{M(x_n + t) - M(x_n)\} \, dt.$$

(6.3) $\leqslant \frac{1}{2} \{M(x_n + t) - M(x_n)\} \leqslant \frac{1}{2} \{\Omega(x_n + t) - \Omega(x_n)\}.$

Hence

$$J_1(X_n, \xi) \leqslant \frac{1}{2} \int_\xi^\infty \{\Omega(x_n + t) - \Omega(x_n)\} \, e^{-t} dt$$

$$= o\{e^{-\frac{3}{5}\xi} \Omega_1(x_n)\} = o(h_n + 1) e^{-\frac{3}{5}\xi},$$

by lemma 3,

$$= o(h_n) e^{-\frac{3}{5}\xi},$$

since $h_n \geqslant h_1 > 0$.

The inequality involving x_n follows similarly from

$$A_1(x_n, t) = \frac{1}{2t} \int_0^t \{\phi(x_n + t) - \phi(x_n)\} \, dt \geqslant \frac{1}{2t} \int_0^t \{m(x_n + t) - m(x_n)\} \, dt.$$

This disposes of the results in J_1.

For the results in J_2 we observe that ϕ is even, and have regard to the geometric meaning of A_2 and the maximum property of X_n. Thus

$$A_2(X_n, t) \leqslant \begin{cases} 0 & (t \leqslant 2X_n) \\ A_1(X_n, t - 2X_n) & (t > 2X_n). \end{cases}$$

Hence, if $\xi' = \text{Max.}\,(\xi,\,2X_n)$, we have

$$J_2(X_n,\,\xi) \leqslant J_2(X_n,\,\xi') \leqslant \int_{\xi'}^{\infty} \frac{A_1(X_n,\,t-2X_n)}{2t}\,e^{-t}dt$$

$$\leqslant \int_{\xi'}^{\infty} \frac{(t-2X_n)\,\{\Omega\,(x_n+t-2X_n)-\Omega\,(x_n)\}}{2t}\,e^{-t}dt,$$

by (6.3),

$$\leqslant \tfrac{1}{2}\int_{\xi'}^{\infty} \{\Omega\,(x_n+t)-\Omega\,(x_n)\}\,e^{-t}dt$$

$$= o\,(h_n)\,e^{-\frac{3}{5}\xi'} = o\,(h_n)\,e^{-\frac{3}{5}\xi}.$$

The remaining result, involving x_n, may be proved similarly.

7. We can now proceed to the proof of Theorem A.

Let $\xi_{2n-1}=x_n-X_n,\,\xi_{2n}=X_{n+1}-x_n$. We suppose first that ξ_r does not tend to ∞ with r. Then there exists a c, such that either $\xi_{2n}<c$ for an infinity of values of n, or $\xi_{2n-1}<c$ for an infinity of values of n. The cases are interchangeable by a change of $\phi\,(x)$ to $-\phi\,(x)$; it is enough to consider the second, and we suppose that $n \to \infty$ through a set of values for which $\xi_{2n-1}<c$.
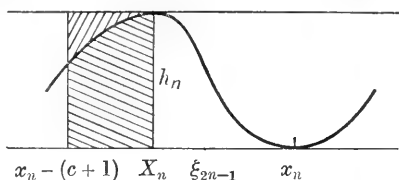
Fig. 2

Our argument is made plainer by reference to a figure. In fig. 2 the curve $y = \phi\,(x)$ lies between the horizontal lines for values of x between 0 and x_n, the base of the shaded areas is at least 1, and *one* of the shaded areas is at least $\tfrac{1}{2}h_n$.

Case (i). *The lower shaded area is at least $\tfrac{1}{2}h_n$.*
We have, by lemma 7,

$$J_1(x_n) > o\,(h_n),\ \ J_2(x_n,\,c+1) > o\,(h_n).$$

Also

$$0 = J(x_n) = J_1(x_n) + \int_0^{c+1} \frac{A_2(x_n,\,t)}{2t}\,e^{-t}dt + J_2(x_n,\,c+1).$$

Hence

(7.1) $$\int_0^{c+1} \frac{A_2(t)}{2t}\,e^{-t}dt < o\,(h_n).$$

Now in $0 \leqslant t \leqslant c+1$, we have (if $x_n > c+1$),

$$A_2(t) \geqslant 0.$$

Also

$$\frac{e^{-t}}{2t} \geqslant \frac{e^{-c-1}}{2\,(c+1)}.$$

Hence, from (7.1),

(7.2) $$\int_{c+\frac{1}{2}}^{c+1} A_2(t)\,dt < o\,(h_n).$$

On the other hand we have, by hypothesis,

$$A_2(c+1) \geqslant \tfrac{1}{2}h_n,$$

and clearly, for $c + \tfrac{1}{2} \leqslant t \leqslant c + 1$,

$$A_2(t) \geqslant A_2(c+1) - (c+1-t)h_n$$
$$\geqslant h_n\{t - (c + \tfrac{1}{2})\},$$
$$\int_{c+\frac{1}{2}}^{c+1} A_2(t)\,dt \geqslant h_n \int_{c+\frac{1}{2}}^{c+1} \{t - (c + \tfrac{1}{2})\}\,dt = \tfrac{1}{8}h_n.$$

This contradicts (7.2).

Case (ii). *The upper shaded region is at least $\tfrac{1}{2}h_n$.*
The argument is very similar. We have, writing

$$\tau = X_n - x_n + (c+1) < c + 1,$$
$$0 = J_1(X_n) + \int_0^\tau \frac{A_2(X_n,\,t)}{2t} e^{-t}dt + J_1(X_n,\,\tau),$$

(7.3) $$< o\,(h_n) + \frac{e^{-(c+1)}}{2\,(c+1)} \int_0^\tau A_2(t)\,dt,$$

since $A_2(t) \leqslant 0$ $(0 \leqslant t \leqslant \tau)$.

Also $A_2(\tau) \leqslant -\tfrac{1}{2}h_n$, and

$$A_2(t) \leqslant -\tfrac{1}{2}h_n + (t-\tau)h_n \quad (0 \leqslant t \leqslant \tau),$$

so that

(7.4) $$\int_0^\tau A_2(t)\,dt \leqslant \int_{\tau-\frac{1}{2}}^\tau A_2(t)\,dt \leqslant -\tfrac{1}{8}h_n.$$

(7.3) and (7.4) are incompatible. We have therefore disposed of the cases when ξ_r does not tend to ∞.

8. Suppose now that $\xi_r \to \infty$ as $r \to \infty$. Then there is an infinity of values of n for which $\xi_{2n} \geqslant \xi_{2n-1}$, or else an infinity for which $\xi_{2n+1} \geqslant \xi_{2n}$. The second alternative reduces to the first by a change of ϕ into $-\phi$, and we may suppose that $n \to \infty$ through a set of values for which $\xi_{2n} \geqslant \xi_{2n-1}$.

In fig. 3 the curve lies below the upper horizontal line for $0 \leqslant x \leqslant x_n$, and above the lower one for $0 \leqslant x \leqslant X_{n+1}$; further, one of the shaded areas is at least $\tfrac{1}{2}h_n \xi_{2n-1}$.
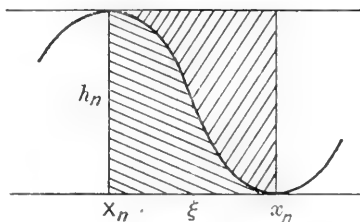
Fig. 3

Case (i). *The lower area is at least* $\frac{1}{2}h_n\xi_{2n-1}$.

We write h, ξ for h_n, ξ_{2n-1}. We have

(8.1)

$$0 = \int_0^\xi \frac{A_1(x_n, t)}{2t} e^{-t} dt + J_1(x_n, \xi) + \int_0^\xi \frac{A_2(x_n, t)}{2t} e^{-t} dt + J_2(x_n, \xi).$$

Now $A_1(t) = A_1(x_n, t) \geqslant 0$ for $0 \leqslant t \leqslant \xi_{2n}$, and *a fortiori* for $0 \leqslant t \leqslant \xi$. Also, by lemma 7,

$$J_1(x_n, \xi) + J_2(x_n, \xi) > o(h) e^{-\frac{3}{5}\xi}.$$

Hence, from (8.1),

$$\int_0^\xi \frac{A_2(t)}{2t} e^{-t} dt < o(h) e^{-\frac{3}{5}\xi},$$

and, since $A_2(t) \geqslant 0$ $(0 \leqslant t \leqslant \xi)$, we obtain

(8.2) $$\frac{1}{2\xi} \int_{\frac{1}{2}\xi}^\xi A_2(t) e^{-t} dt \leqslant \int_0^\xi \frac{A_2(t)}{2t} e^{-t} dt < o(h) e^{-\frac{3}{5}\xi}.$$

On the other hand

$$A_2(\xi) \geqslant \tfrac{1}{2}h\xi,$$

$$A_2(t) \geqslant A_2(\xi) - h(\xi - t) \geqslant h(t - \tfrac{1}{2}\xi) \quad (\tfrac{1}{2}\xi \leqslant t \leqslant \xi),$$

and so $$\frac{1}{2\xi} \int_{\frac{1}{2}\xi}^\xi A_2(t) e^{-t} dt \geqslant \frac{1}{2\xi} \int_{\frac{1}{2}\xi}^\xi h(t - \tfrac{1}{2}\xi) e^{-t} dt$$

$$= \frac{h}{2\xi} e^{-\frac{1}{2}\xi} \int_0^{\frac{1}{2}\xi} u e^{-u} du$$

(8.3) $$= \frac{h}{2\xi} e^{-\frac{1}{2}\xi} \{1 + o(1)\},$$

since $\xi \to \infty$. The results (8.2) and (8.3) are incompatible.

Case (ii). *The upper area in fig.* 3 *is at least* $\frac{1}{2}\xi h$.

We take $x = X_n$, and have, still writing $\xi = \xi_{2n-1}$, $h_n = h$,

$$0 = \int_0^\xi \frac{A_1(X_n, t)}{2t} e^{-t} dt + J_1(X_n, \xi) + \int_0^\xi \frac{A_2(X_n, t)}{2t} e^{-t} dt + J_2(X_n, \xi).$$

Hence we obtain

$$\frac{1}{2\xi} \int_{\frac{1}{2}\xi}^\xi A_2(t) e^{-t} dt \geqslant \int_0^\xi \frac{A_2(t)}{2t} e^{-t} dt \geqslant o(h) e^{-\frac{3}{5}\xi},$$

on the one hand; and on the other, by obvious modifications of the argument for case (i),

$$\frac{1}{2\xi} \int_{\frac{1}{2}\xi}^\xi A_2(t) e^{-t} dt \leqslant -\frac{h}{2\xi} e^{-\frac{1}{2}\xi} \{1 + o(1)\},$$

and these results are incompatible.

Thus our original assumption leads in all cases to a contradiction, and Theorem A is proved.

9. Some condition of the type of C appears essential to our argument. Our actual condition places little limitation on the order of magnitude of the oscillation $\Omega(x)$: its effect is to secure that $\Omega(x)$ does not increase slowly in some ranges of x, and very rapidly in others*. Unfortunately, however, it is not universally true that the sum of two functions satisfying condition C itself satisfies C, and as a result of this it is not possible to remove the restriction in Theorem A that ϕ is even, without adding others which, although allowing wide latitude to ϕ, are cumbrous in form. We may briefly indicate some of the possibilities.

It is easily verified that if ϕ is a solution of (4.1), then so is $\psi(x) = \phi(x) + \phi(-x)$; and $\phi'(x)$ is a solution subject to the possibility of formal differentiation in (4.1)†. Suppose then that $\psi(x)$ and $\phi'(x)$ satisfy C, and that ϕ is a solution. Then ψ is an even solution, and must be a constant. Hence $\phi(x) = a + \chi(x)$, where χ is odd, and $\phi'(x)$ is an even solution, and therefore a constant. In these circumstances, then, $\phi(x)$ must be of the form $ax + b$, and the $f(x)$ of § 1 must be a quadratic function of x.

An alternative hypothesis gives us:

Theorem B. If $\phi(x)$ is a solution of (4.1), and
$$\psi(x, \alpha) = \phi(\alpha + x) + \phi(\alpha - x),$$
qua function of x, satisfies condition C for every value of α_τ^+, then $\phi(x)$ is of the form $ax + b$.

$\psi(x, \alpha)$ is an even solution satisfying C, therefore a constant $c(\alpha)$. Thus

$$(9.1) \qquad \phi(\alpha + x) + \phi(\alpha - x) = c(\alpha) = \phi(2\alpha) + \phi(0).$$

Here α and x are arbitrary. If we write
$$\alpha + x = \xi, \ \alpha - x = \eta, \ \phi(t) = \phi(0) + \chi(t),$$
(9.1) becomes
$$\chi(\xi) + \chi(\eta) = \chi(\xi + \eta).$$
It is well known that the only *continuous*§ solution of this is $\chi(x) = ax$, and our theorem follows.

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* It must not increase by a finite proportion of itself in a finite range of x (when x is large).

† We note in passing that the integral of an odd solution is a solution.

‡ The o of condition C, of course, is not supposed to be uniform in α.

§ The existence of discontinuous solutions is an unsolved problem in the theory of sets of points, and is connected with the multiplicative axiom and the existence of non-measurable sets.

It is perhaps worth adding, for completeness, the simple proof of the continuous case. Evidently $\chi(0) = 0$, $\chi(-x) = \chi(0) - \chi(x) = -\chi(x)$; and if n, p, q are integral
$$\chi(n) = n\chi(1), \ \chi(1) = n\chi\left(\frac{1}{n}\right),$$
and so
$$\chi\left(\frac{p}{q}\right) = p\chi\left(\frac{1}{q}\right) = \frac{p}{q}\chi(1).$$
By continuity
$$\chi(x) = x\chi(1).$$

10. From Theorem B we can deduce at once :

Theorem C. If $f(x)$ is a solution of (1.1) *for which $f''(x)$ is continuous and bounded, then $f(x)$ is a quadratic function of x.*

Since we may differentiate (1.1) formally, $\phi(x) = f''(x)$ is a continuous and bounded solution of (4.1). We have only to verify that $\psi(x, \alpha)$ satisfies C. But since $\psi(x)$, and so also the corresponding $\Omega(x)$, is bounded, $\Omega(x)$, being an increasing function, tends to a limit, and $\Omega(x+1) - \Omega(x) = o(1)$.

Note. By E. A. MILNE.

The physical problem referred to by Mr Littlewood in § 1 of the preceding paper is that of the radiative equilibrium of an infinite mass of material stratified in parallel planes, under the assumption that there is no internal generation of energy. Each element radiates as much energy as it absorbs, but across any plane of stratification more energy is radiated in one direction than in the opposite direction, the difference being constant from plane to plane. There is thus a steady transfer of energy. The question is to determine the distribution of temperature implied by this state of affairs. The writer has reduced the problem to the solution of equation (1.1) above, in which $f'(x)$ is the intensity of black body radiation for the temperature existing at the point x; thus $f'(x) = (\sigma/\pi) T^4$, where T is the temperature at x and σ is Stefan's constant. Here the coordinate x is the optical thickness, measured from some reference point. Mr Littlewood's *Theorem C* asserts that if the gradient of the energy-density taken with respect to the optical thickness is continuous and bounded, then $f''(x)$ (or T^4) is a linear function of x. It is a simple deduction from this that if $I(x, \theta)$ is the actual intensity of radiation at x in a direction making θ with the axis of stratification, then

(10.1) $$I(x, \theta) = f'(x) + f''(x) \cos \theta.$$

The importance of this result lies in its application to the interior of a star. A star is stratified not in parallel planes but in concentric spheres, but save near the centre the curvature for this purpose can be neglected. Now it can be shown that the determination of the distribution of temperature is equivalent to that of the distribution of $I(x, \theta)$ in θ, and a plausible method of attack is to assume $I(x, \theta)$ expansible in a series of Legendre functions of $\cos \theta$, of which (10.1) gives the first two terms. This was the method used by Eddington, who first investigated the question, and he showed by considering the numerical magnitudes of the quantities involved that for an actual star all the terms after the

first two must be negligible. (*Monthly Notices*, **77**, 16 (1916). See also his paper in *Zeitschrift für Physik*, **7**, 351 (1921), where the argument is given in more detail.) Jeans has written on similar lines (*Monthly Notices*, **78**, 28 (1917)). From this it follows that the error in taking $f'(x)$ to be a linear function of x must be negligible. But since the energy-density gradient in a star may certainly be assumed to be bounded and continuous*, Mr Littlewood's result shows that the further terms in the expansion in Legendre functions are not merely negligible but zero. It should be mentioned, however, that Eddington's argument demonstrates at the same time the negligibility of the effect of curvature, besides being adapted to take account of the internal generation of energy.

In the idealised plane case, if the internal generation of energy instead of being zero is of amount $4\pi\epsilon$ per unit mass, the generalisation of (1.1) is

$$f'(x) - \frac{\epsilon}{k} = \int_0^\infty \frac{f(x+t) - f(x-t)}{2t} \, e^{-t} dt,$$

k being the absorption coefficient. Solutions of this may be sought for ϵ/k any given function of x, or better perhaps (from a physical point of view) of f. If ϵ/k is taken to be a polynomial in x of degree n, then the equation can be satisfied by taking for $f'(x)$ a polynomial of degree $n+2$. Mr Littlewood's methods could probably be extended to show that under analogous conditions this polynomial is the only solution. The important case for a star is $n = 0$, in which case T^4 is a quadratic function of x.

* If the energy-density gradient were not bounded (or discontinuous), this would practically imply that the net flux of energy and hence the force on a thin slab of the material due to radiation pressure were not bounded (or discontinuous).

The projective generation of curves and surfaces in space of four dimensions. By F. P. WHITE, M.A., St John's College.

[*Read* 15 May, *received* 29 July, 1922.]

In a paper "Über die durch collineare Grundgebilde erzeugten Curven und Flächen*," Friedrich Schur investigated systematically curves and surfaces which can be generated from the intersections of corresponding planes in two or more systems (pencils, sheaves, etc.) between which a projective relation has been established. The simplest cases, conic and ruled quadrics, are familiar enough; twisted cubics and cubic surfaces were investigated in this way by Reye†. Schur summarises Reye's results and proceeds to further cases, the twisted sextic of genus three and the quartic surface of the type known as "determinant surfaces."

The present paper contains an examination of some similar cases in four dimensional space. Veronese, in his classical "Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Scheidens‡," treats the general theory of projective generation in space of any number of dimensions and enumerates a few cases in four dimensions, but without entering into details; and Castelnuovo§ examines the cubic congruence of the third order and sixth class which is generated by three projective sheaves of (threefold) hyperplanes, with the cubic variety on which the lines of the congruence lie; but these are the only papers which I have found.

1. Consider in a four-dimensional space, S_4, two sheaves of hyperplanes, each consisting of the double infinity of hyperplanes passing through a line. Establishing a projectivity between them we may write their equations

$$\lambda p_x + \mu q_x + \nu r_x = 0, \quad \lambda p_x' + \mu q_x' + \nu r_x' = 0, \quad ...(1\cdot1)$$

where $p_x \equiv p_1 x_1 + ... + p_5 x_5$, etc.

Corresponding hyperplanes meet in a plane, and thus we get a doubly infinite system of planes. Through an arbitrary point of S_4 there passes in general *one* such plane; for equations 1·1, given x, determine one set of values of the ratios $\lambda : \mu : \nu$. An exception occurs if all the determinants of the second order of the matrix

$$\left\| \begin{array}{ccc} p_x & q_x & r_x \\ p_x' & q_x' & r_x' \end{array} \right\|$$

* Schur, *Math. Ann*, 18, 1881, 1–32.
† Reye, *Geometrie der Lage*, Abt. 2.
‡ Veronese, *Math. Ann.* 19, 1882, 215–234.
§ Castelnuovo, *Atti Istituto Veneto*, (6), 5, 1887, 1249.

vanish, i.e. if the point lie on the two dimensional cubic surface F_3 whose equations are

$$\frac{p_x}{p_x{}'} = \frac{q_x}{q_x{}'} = \frac{r_x}{r_x{}'} \qquad \ldots\ldots(1\cdot2).$$

If y is such a point, a single infinity of the planes pass through it, forming the conical quadric variety

$$\begin{vmatrix} p_x & q_x & r_x \\ p_x{}' & q_x{}' & r_x{}' \\ p_y & q_y & r_y \end{vmatrix} = 0 \qquad \ldots\ldots(1\cdot3).$$

Any plane of the system meets the quadric variety

$$\begin{vmatrix} p_x & q_x \\ p_x{}' & q_x{}' \end{vmatrix} = 0$$

in a conic which lies on

$$\begin{vmatrix} q_x & r_x \\ q_x{}' & r_x{}' \end{vmatrix} = 0$$

and on

$$\begin{vmatrix} p_x & r_x \\ p_x{}' & r_x{}' \end{vmatrix} = 0,$$

i.e. on the cubic surface. Hence the planes are secant-planes of the cubic surface F_3, each meeting it in a conic.

The surface F_3 may also be obtained as the locus of the lines of intersection of corresponding hyperplanes of the three collinear pencils

$$\alpha p_x + \beta p_x{}' = 0, \ \ \alpha q_x + \beta q_x{}' = 0, \ \ \alpha r_x + \beta r_x{}' = 0 \quad \ldots(1\cdot4).$$

It is thus a ruled surface.

The plane bases of these pencils belong to the system of secant-planes, and since the same surface is generated from any three pencils of the sheaf of pencils

$$\lambda\,(\alpha p_x + \beta p_x{}') + \mu\,(\alpha q_x + \beta q_x{}') + \nu\,(\alpha r_x + \beta r_x{}') = 0 \quad \ldots(1\cdot5)$$

it is clear that any three secant-planes may be taken as bases of generating pencils, and that any two of the lines 1·4 may be taken as bases of generating sheaves in place of 1·1.

The surface F_3, as was pointed out by Veronese*, contains a further line not included in the system of generators 1·4. For consider any three generators: they cannot meet in a point, for equations 1·4, for a point of the surface, determine one ratio $\alpha : \beta$ and thus through a point of the surface passes only one generator. Also they cannot lie in a hyperplane, or the hyperboloid determined by their common transversals would form part of the surface. Hence they have a unique transversal, which, meeting three

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* Veronese, *l.c.* p. 230.

generators, lies on each quadric variety containing the surface and therefore on the surface itself. It thus meets all the other generators, and is a *directrix*.

The plane representation of the surface is also considered by Veronese*. Every plane through a generator meets the surface in one further point, and thus, projecting from any generator on to a plane, we get a correspondence between the points of the surface and the points of the plane. The directrix meets the generator, so that one plane contains both and the directrix projects into a fundamental point. Two generators lie in a hyperplane; hence any generator projects into a line of the plane through the fundamental point.

The hyperplane sections of the surface (twisted cubics) meet the directrix once and each generator once; they therefore project into conics through the fundamental point.

A secant-plane meets the generator in one point and they together lie in a hyperplane which meets the surface in the generator and a conic. In this case the conic in the plane breaks up into a line through the fundamental point and another line not through it. Thus to the conics of the surface correspond the lines of the plane.

If we now project from an arbitrary point P upon an arbitrary hyperplane, the cubic surface will project into a ruled cubic surface in an ordinary space of three dimensions. The hyperplane joining a secant-plane to P meets the surface in the four-dimensional space in a line and a conic, and thus the plane into which it projects will be a plane through a generator, meeting the ruled cubic again in a conic. One particular secant-plane passes through P; this plane will project into a *line*, which will be the *double line* of the ruled cubic. Every secant-plane meets every generator in one point; hence the double line meets every generator, i.e. is a directrix. Any two secant-planes meet in one point which must lie on the surface, i.e. any two conics on the surface have one point in common. Projecting, the double line meets every conic on the ruled cubic in one point.

The directrix projects into a directrix.

2. Three collinear sheaves of hyperplanes

$$\lambda p_x + \mu q_x + \nu r_x = 0, \quad \lambda p_x' + \mu q_x' + \nu r_x' = 0,$$
$$\lambda p_x'' + \mu q_x'' + \nu r_x'' = 0 \quad \ldots(2\cdot1)$$

generate, by the intersection of corresponding hyperplanes, a congruence (a double infinity) of lines, which has been considered in detail by Castelnuovo †.

* Veronese, *l.c.* p. 230.
† Castelnuovo, *Atti Istituto Veneto*, (6), 5, 1887, 1249.

The lines of the congruence form the cubic variety V_3

$$\begin{vmatrix} p_x & q_x & r_x \\ p_x' & q_x' & r_x' \\ p_x'' & q_x'' & r_x'' \end{vmatrix} = 0 \qquad \dots\dots(2\cdot2).$$

The variety V_3 also contains another doubly infinite set of lines, the intersections of corresponding hyperplanes of the collinear sheaves

$$\alpha p_x + \beta p_x' + \gamma p_x'' = 0, \quad \alpha q_x + \beta q_x' + \gamma q_x'' = 0,$$
$$\alpha r_x + \beta r_x' + \gamma r_x'' = 0 \quad \dots(2\cdot3).$$

Equations 2·1 establish a correspondence between generators of the first system and the points (λ, μ, ν) of a plane; to every point of the plane corresponds a hyperplane of each sheaf and thus a generator of V_3.

To any straight line of the plane

$$A\lambda + B\mu + C\nu = 0$$

correspond three pencils of hyperplanes, and thus, by § 1, the generators of a cubic surface lying on the variety.

Its equations will be

$$\begin{Vmatrix} p_x & q_x & r_x \\ p_x' & q_x' & r_x' \\ p_x'' & q_x'' & r_x'' \\ A & B & C \end{Vmatrix} = 0 \qquad \dots\dots(2\cdot4).$$

Thus we get, corresponding to the lines of the plane, a doubly-infinite set of cubic surfaces, any two meeting in one generator, and two generators of the same set of V_3 determining one surface of the set.

Any secant-plane of one of these cubic surfaces meets the variety V_3 in a conic (on the cubic surface) and in a generator of the second system; and conversely, any plane through a generator of the second system meets V_3 again in a conic and is a secant-plane of one cubic surface.

There is a second system of cubic surfaces on V_3, given by

$$\begin{Vmatrix} p_x & q_x & r_x & l \\ p_x' & q_x' & r_x' & m \\ p_x'' & q_x'' & r_x'' & n \end{Vmatrix} = 0 \qquad \dots\dots(2\cdot5),$$

and any two cubic surfaces, one of each system, lie on a quadric variety

$$\begin{vmatrix} p_x & q_x & r_x & l \\ p_x' & q_x' & r_x' & m \\ p_x'' & q_x'' & r_x'' & n \\ A & B & C & 0 \end{vmatrix} = 0 \qquad \dots\dots(2\cdot6).$$

Castelnuovo further shows that the cubic variety V_3 has six singular points, through each of which passes a single infinity of generators. Every cubic surface of each system passes through each singular point.

3. Two collinear systems of hyperplanes, each consisting of the triple infinity of hyperplanes through a point and thus represented by the equations

$$\alpha p_x + \beta q_x + \gamma r_x + \delta s_x = 0,$$
$$\alpha p_x' + \beta q_x' + \gamma r_x' + \delta s_x' = 0 \quad ...(3\cdot1),$$

generate by the intersection of corresponding hyperplanes a triply infinite system of planes. Through any point y there pass a single infinity of such planes, forming a three-dimensional quadric cone

$$\begin{vmatrix} p_x & q_x & r_x & s_x \\ p_x' & q_x' & r_x' & s_x' \\ p_y & q_y & r_y & s_y \\ p_y' & q_y' & r_y' & s_y' \end{vmatrix} = 0 \qquad \dots\dots(3\cdot2).$$

But if the point lie on the curve

$$\frac{p_x}{p_x'} = \frac{q_x}{q_x'} = \frac{r_x}{r_x'} = \frac{s_x}{s_x'} \qquad \dots\dots(3\cdot3)$$

a double infinity of the planes will pass through it.

This curve is a rational curve of order four; the normal curve of the four dimensional space.

It is also obtained as the locus of the points of intersection of corresponding hyperplanes of the four collinear pencils

$$\lambda p_x + \mu p_x' = 0, \quad \lambda q_x + \mu q_x' = 0, \quad \lambda r_x + \mu r_x' = 0,$$
$$\lambda s_x + \mu s_x' = 0 \quad \dots\dots(3\cdot4).$$

The planes are clearly trisecant planes of the curve.

Projecting from an arbitrary point upon an arbitrary hyperplane, we get a rational quartic curve in ordinary space and its single infinity of trisecant lines which are generators of the quadric containing the curve.

Also cutting by an arbitrary hyperplane we get the four self-corresponding points of a collineation between the points of a three-dimensional space.

4. Four collinear sheaves of hyperplanes

$$\lambda p_x + \mu q_x + \nu r_x = 0, \quad \lambda p_x' + \mu q_x' + \nu r_x' = 0,$$
$$\lambda p_x'' + \mu q_x'' + \nu r_x'' = 0, \quad \lambda p_x''' + \mu q_x''' + \nu r_x''' = 0 \quad ...(4\cdot1)$$

give, by the intersection of corresponding hyperplanes, the points of a surface F_6

$$\begin{Vmatrix} p_x & q_x & r_x \\ p_x{}' & q_x{}' & r_x{}' \\ p_x{}'' & q_x{}'' & r_x{}'' \\ p_x{}''' & q_x{}''' & r_x{}''' \end{Vmatrix} = 0 \qquad \dots\dots(4\cdot2)$$

which, being the residual intersection of two cubic varieties with a common cubic surface, is of order six.

To the points of F_6 correspond the points (λ, μ, ν) of a plane.

Any three of the sheaves 4·1 generate a cubic variety; the two cubic varieties obtained by omitting the first sheaf and the second in turn meet in F_6 and also in the cubic surface F_{34}

$$\begin{Vmatrix} p_x{}'' & q_x{}'' & r_x{}'' \\ p_x{}''' & q_x{}''' & r_x{}''' \end{Vmatrix} = 0 \qquad \dots\dots(4\cdot3).$$

Every sheaf of the pencil

$$\alpha\,[\lambda p_x + \mu q_x + \nu r_x] + \beta\,[\lambda p_x{}' + \mu q_x{}' + \nu r_x{}'] = 0$$

determines with the third and fourth sheaves of 4·1 a cubic variety passing through F_6 and F_{34}. Through any point P of S_4 there passes one such variety; for P determines $(\lambda_0 : \mu_0 : \nu_0)$ uniquely from

$$\lambda p_x{}'' + \mu q_x{}'' + \nu r_x{}'' = 0, \quad \lambda p_x{}''' + \mu q_x{}''' + \nu r_x{}''' = 0,$$

and then $\alpha : \beta$ is determined uniquely from

$$\alpha\,[\lambda_0 p_x + \mu_0 q_x + \nu_0 r_x] + \beta\,[\lambda_0 p_x{}' + \mu_0 q_x{}' + \nu_0 r_x{}'] = 0.$$

If, however, the plane base of this last pencil contains P, the solution is not unique; in this case P lies on F_6.

The conjugate system, consisting of three collinear triply-infinite systems of hyperplanes

$$\alpha p_x + \beta p_x{}' + \gamma p_x{}'' + \delta p_x{}''' = 0, \quad \alpha q_x + \beta q_x{}' + \gamma q_x{}'' + \delta q_x{}''' = 0,$$
$$\alpha r_x + \beta r_x{}' + \gamma r_x{}'' + \delta r_x{}''' = 0 \quad \dots\dots(4\cdot4)$$

generates a triply-infinite set of lines. Through any point P of S_4 passes in general *one* such line. A line of the set meets the cubic variety

$$\begin{vmatrix} p_x & q_x & r_x \\ p_x{}' & q_x{}' & r_x{}' \\ p_x{}'' & q_x{}'' & r_x{}'' \end{vmatrix} = 0$$

in three points which lie on the surface F_6; i.e. the lines are tri-secant lines of the surface.

If the point P (y) lie on F_6 there pass through it a single infinity of trisecant lines, forming the conical cubic surface

$$\begin{Vmatrix} p_x & p_x{'} & p_x{''} & p_x{'''} \\ q_x & q_x{'} & q_x{''} & q_x{'''} \\ r_x & r_x{'} & r_x{''} & r_x{'''} \\ p_y & p_y{'} & p_y{''} & p_y{'''} \\ q_y & q_y{'} & q_y{''} & q_y{'''} \end{Vmatrix} = 0 \qquad \qquad \dots\dots(4{\cdot}5).$$

Consider now the plane representation of the surface. To a line in the plane, $A\lambda + B\mu + C\nu = 0$, correspond the points obtained from four pencils of hyperplanes, i.e. a rational curve of order four, its equations being

$$\begin{Vmatrix} p_x & q_x & r_x \\ p_x{'} & q_x{'} & r_x{'} \\ p_x{''} & q_x{''} & r_x{''} \\ p_x{'''} & q_x{'''} & r_x{'''} \\ A & B & C \end{Vmatrix} = 0 \qquad \qquad \dots\dots(4{\cdot}6).$$

Thus through any two points of F_6 passes one such curve of the surface, and any two such curves meet in one point.

If we take a trisecant line of F_6, given by values α_1, β_1, γ_1, δ_1 in equations 4·4, and any plane through it, say

$$\lambda_1 \left(\alpha_1 p_x + \dots + \delta_1 p_x{'''}\right) + \mu_1 \left(\alpha_1 q_x + \dots + \delta_1 q_x{'''}\right) \\ + \nu_1 \left(\alpha_1 r_x + \dots + \delta_1 r_x{'''}\right) = 0,$$
$$\lambda_2 \left(\alpha_1 p_x + \dots + \delta_1 p_x{'''}\right) + \mu_2 \left(\alpha_1 q_x + \dots + \delta_1 q_x{'''}\right) \\ + \nu_2 \left(\alpha_1 r_x + \dots + \delta_1 r_x{'''}\right) = 0,$$

it will meet the surface again in three points. Also this plane is clearly a trisecant plane of the quartic curve generated by the four pencils

$$\lambda_1 p_x + \mu_1 q_x + \nu_1 r_x + \theta \left(\lambda_2 p_x + \mu_2 q_x + \nu_2 r_x\right) = 0, \text{ etc.}$$

which is the quartic determined on F_6 by the points corresponding to $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_2, \mu_2, \nu_2)$.

Hence if we take the pencil of quartics through $P, (\lambda_1, \mu_1, \nu_1)$, and draw to each the trisecant plane through a fixed trisecant g, they will all lie in a hyperplane which passes through g and through P.

Conversely, if we take any hyperplane α, which without loss of generality we may suppose to belong to the first sheaf of 4·1, it will meet the corresponding hyperplanes of the other three sheaves in a point P of F_6; any quartic through P is generated by four pencils of hyperplanes belonging to the four sheaves, and the base of the first pencil will lie in α, will be a trisecant plane of the quartic and will contain a fixed trisecant, the base of the first sheaf.

5. It may happen that four corresponding hyperplanes of the sheaves 4·1 meet in a line instead of a point; in this case the line will lie on F_6. Analytically, if $p_x = p_1 x_1 + ... + p_5 x_5$, etc., the condition is

$$\left\| \begin{array}{cccc} \lambda p_1 + \mu q_1 + \nu r_1 & \lambda p_1' + \mu q_1' + \nu r_1' & \lambda p_1'' + \mu q_1'' + \nu r_1'' & \lambda p_1''' + \mu q_1''' + \nu r_1''' \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ \lambda p_5 + \mu q_5 + \nu r_5 & \cdots\cdots\cdots & \cdots\cdots\cdots & \lambda p_5''' + \mu q_5''' + \nu r_5''' \end{array} \right\| = 0$$

the number of solutions of which would appear to be

$$4^2 - 3^2 + 2^2 - 1 = 10.$$

That there are in fact ten lines on F_6 may be seen as follows.

Consider four corresponding pencils of the sheaves; they generate a quartic curve which is met by any hyperplane in four points. Thus through any plane of each sheaf there pass four hyperplanes which meet the corresponding hyperplanes in points of this arbitrarily taken hyperplane. Hence in each sheaf the hyperplanes which by their intersections with the corresponding hyperplanes generate a hyperplane section of the surface envelope a cone of the second kind of class four. Now consider two arbitrary hyperplanes e_1 and e_2; the corresponding cones have 16 common tangent hyperplanes; 6 of these meet the corresponding hyperplanes where the plane $e_1 e_2$ meets F_6, the other ten meet the corresponding hyperplanes in points of e_1 and also in points of e_2 and therefore in their join; hence ten sets of corresponding hyperplanes meet in a line instead of a point.

In the plane representation of the surface there are thus ten fundamental points, representing these lines.

Any hyperplane meets any quartic curve in four points and meets each line once; hence corresponding to the hyperplane sections of the surface are quartic curves of the plane passing through the ten fundamental points.

To the join of two fundamental points corresponds a conic on F_6, which meets the two corresponding lines of F_6; there are 45 such conics.

A quartic through the ten fundamental points may degenerate into:

(1) A conic through 5 and a conic through the other 5. Each meets a general quartic through the 10 in three further points, so that to it corresponds on F_6 a twisted cubic meeting five of the lines. There are 630 pairs of such twisted cubics, each pair making up a complete hyperplane section of F_6 and meeting in four points.

(2) A cubic through 9 and a straight line through the remaining one. Corresponding thereto on F_6 are a plane cubic (of genus 1) and a twisted cubic. Thus on F_6 there are ten plane cubics such

that every hyperplane of the pencil having the plane of a cubic as base meets the surface again in a twisted cubic.

(3) A cubic through 8 and the join of the other two. Corresponding thereto on F_6 are a twisted quartic of genus 1 and one of the 45 conics. Thus every hyperplane of the pencil having the plane of a conic as base meets the surface again in a twisted quartic and, since all plane cubics through eight points have a ninth point in common, all quartics of a pencil meet in a point of F_6.

A hyperplane section of the surface through a line is either:

(1) A quintic of genus 2, meeting the line twice and the other nine lines once each; represented on the plane by a quartic curve with a double point at one fundamental point and passing through the others.

(2) Another line and a quartic of genus 1 which meets the two lines twice each and the other eight lines once; represented in the plane by a quartic curve with two double points at fundamental points and passing through the others. There are 45 such quartics on F_6.

(3) A conic meeting the line once and one other line once, together with a plane cubic meeting once every line except this other one; in this case the quartic in the plane consists of the cubic through nine fundamental points and the join of two.

There are two sets of rational quartics on the surface besides those corresponding to the lines of the plane. They are represented by:

(1) Conics through four fundamental points; there being 210 singly infinite families of such.

(2) Cubics with a double point at one fundamental point and passing through six others; there are 840 such quartics, each meeting one line twice and six others once.

6. We have seen that from a point P of F_6 there can be drawn a single infinity of trisecant lines, lying upon a cubic conical surface. The points of intersection of the trisecants with F_6 form a curve on F_6. The tangent plane to F_6 at the point P meets F_6 here in four coincident points and meets it again in two points, so that each of two trisecants has two of its intersections with F_6 coinciding at P, which must therefore be a double point of the curve. Also any hyperplane through P meets the cubic cone in three lines and hence the curve in six further points. The curve is thus of order 8, with a double point at P. The plane joining P to a line of F_6 meets F_6 again in two points, as is easily seen from the plane representation, and the joins of these points to P are trisecants, so that the curve meets each line twice.

Its representation on the plane is thus a curve with double points at the ten fundamental points and a double point at the

point P' corresponding to P; it meets every quartic through the ten fundamental points in eight variable points. The order of the representative curve must thus be 7. This curve contains an involution of pairs of points Q', R' corresponding to points Q, R on F_6 lying on the same trisecant through P. It is thus hyper-elliptic.

If we take the plane through two trisecants which meet in P, say PQ_1R_1 and PQ_2R_2, it will meet F_6 in one further point K. From the result at the end of § 4, this plane is the common tri-secant plane of the quartic curves determined respectively by Q_1R_1 and Q_2R_2, and K is their point of intersection.

Hence, if we take the tangent planes to the cubic cone, they will meet F_6 again in points K which are the intersections of two consecutive quartics of this family. These tangent planes form a three-dimensional variety, obtained by joining the tangent lines of the curve of order 8 on F_6 to the point P, or by joining the lines of the tangent developable of the cubic curve of intersection of the cone with a hyperplane to the point P. It is thus of order 4. Its intersection with F_6, a curve of order 24, contains the curve of order 8 counted twice; the remainder, the locus of K, is thus of order 8.

The corresponding curve in the plane is what remains after taking twice the curve of order 7 with double points at the funda-mental points from a curve of order 16 with multiple points of order 4 at the fundamental points; it is thus a conic. This conic is the envelope of the joins of corresponding points $Q'R'$ of the curve of order 7.

That this envelope is a conic can be seen as follows. We have to show that through any point K of the surface can be drawn two quartic curves each of which meets one generator of the cone vertex P in two points. Now any trisecant plane of a quartic on the surface meets the surface again in three points lying on a tri-secant. Hence the plane joining K to a generator must meet the cone again in another generator. One such plane can be drawn, just as one chord can be drawn to a twisted cubic curve in three-dimensional space from an external point. If this plane contain the generators PQ_1R_1, PQ_2R_2, then the two quartic curves in question are those determined by Q_1R_1 and Q_2R_2.

7. From our configuration in S_4 we can obtain a configuration in three-dimensional space by taking the section by an arbitrary hyperplane. We thus get at once a sextic curve of genus 3 and its trisecants, three of which pass through any point of the curve. This would seem to be the most natural way of treating the curve; in Schur's generation of it in three dimensions from four collinear sheaves of planes, as four corresponding planes do not as a rule

meet in a point, the planes taken have to be restricted, the argument is thereby complicated and the existence of the trisecants does not follow immediately, as above*.

The quartic curves on the surface F_6 give rise to the double infinity of quadruplets of points on the curve C'_6; one such quadruplet being determined by two of its points. Also our proposition (§ 4) about the trisecant planes of the quartics through a point of F_6 gives Schur's proposition†:

" The planes of the triplets which form quadruplets with a fixed point P of C'_6 all pass through a fixed trisecant; and, conversely, the planes through a fixed trisecant meet the curve again in triplets forming quadruplets with a fixed point P of the curve."

We have thus a (1·1) correspondence between the points of C'_6 and its trisecants. To find the point corresponding to a trisecant PQ_1R_1, let PQ_2R_2, PQ_3R_3 be the other trisecants through P to C'_6, i.e. lying in the given hyperplane. If the plane $PQ_2R_2Q_3R_3$ meet F_6 again in K, then K lies on the quartic through Q_2R_2 and also on the quartic through Q_3R_3, the planes $PQ_1R_1Q_2R_2$, $PQ_1R_1Q_3R_3$ are thus trisecant planes of two quartics through K, which must therefore correspond to their intersection PQ_1R_1.

The curves of order 8 on the surface each with a double point at a point P give a doubly infinite system of sets of eight points on the sextic curve; as each of the curves lies on a cubic cone, each set of eight points lies on a twisted cubic. This system includes the sets of points arising when P is on the sextic curve in question. The cubic curve then consists of the three trisecants from P, and the eight points are made up of P counted twice and the six further intersections of these trisecants with the curve. A quadric variety can be put through one of these curves of order 8 and any one of the quartic curves; hence the quadric surfaces through one set of eight points on the curve give the doubly infinite system of quadruplets. Conversely, the quadrics through one quadruplet give sets of eight points which include all the sets above mentioned, but they also include others, such quadrics forming a system of dimension 5.

8. If we now project from an arbitrary point of S_4 upon a hyperplane we get a rational sextic surface; as one trisecant passes through the point of projection the sextic surface in S_3 will have a triplanar point (a triple point at which the tangent cone consists of three planes). The section of the projection by a plane is a sextic curve of genus 3, i.e. has seven double points. Hence the sextic surface in S_3 has a double curve of order 7. This curve is the projection of a curve on F_6 of order 14. It is repre-

sented on the plane by a curve of order 11, which has a triple point at each of the ten fundamental points.

The triple infinity of trisecants projects into a complex in S_3; the order of this complex is 8, since a hyperplane section of F_6 is a sextic curve whose trisecants form a ruled surface of order 8, and thus eight of the lines of the complex which lie in a plane pass through a given point thereof.

If, however, we project from a point lying on the surface itself, we get a quintic surface in S_3. The cubic cone of trisecants through the point of projection gives rise to a double twisted cubic curve on the quintic surface. The tangent plane at the point of projection gives a line lying on the quintic*, which has thus eleven lines in all. This quintic surface, whose plane sections are represented on a plane by quartic curves through *eleven* fundamental points, has been dealt with by Clebsch† and by Sturm‡.

By taking special positions of the point P on the surface, we may obtain special cases of the quintic surface. For instance, consider the conic on F_6 represented by the join of 9 and 10. Any hyperplane containing its plane meets F_6 further in a quartic (of genus 1) represented by a cubic through 1, 2 ... 8. All such cubics pass through a ninth point P', corresponding to a point P on F_6 which must lie in the plane of the conic. P is also the intersection of the planes of the two cubics represented by the cubics 1 ... 8, 9 and 1 ... 8, 10. Now project from P. The conic gives a double line, and the two plane cubics through P each give a double line. The cubic cone of trisecants from P in fact breaks up into three planes, the curve of order 8 into the conic and the two cubics, and the curve of order 7 in the plane into two cubics and a straight line. The quintic surface thus has, instead of a double twisted cubic, three double straight lines, of which two do not meet and the third meets both.

If P' be taken elsewhere on the plane cubic represented by the cubic through 1 ... 8, 9, the cubic cone will contain the plane of this cubic, and the remainder of the curve of order 8 on F_6 will consist of a quintic curve (of genus 2), represented by a quartic curve through 1 ... 8, 9 with a double point at 10, and must lie on a quadric cone vertex P. The projection will thus be a quintic surface with a double line and a double conic.

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* For the general case of projection in any number of dimensions, see Del Pezzo, *Rend. Napoli*, 25, 1886, 205.

† Clebsch, *Math. Ann.* 1, 1869, 284.

‡ R. Sturm, *Die Lehre von den geometrischen Verwandtschaften*, iv, 311.

An Alignment Chart for Thermodynamical Problems. By C. R. G. Cosens, B.A., King's College. (Communicated by Prof. Inglis.)

(Plate II.)

[*Read* 2 May, 1921.]

The chart shown is intended for the rapid solution of problems on the expansion of so-called "perfect gases," it is particularly adjusted for air.

The principles of its use are as follows:

A straight-edge placed across the chart in any direction cuts the four vertical scales at points which are marked with corresponding values of:

V = the volume of 1 lb. dry air, in cubic feet.

ϕ = the entropy of 1 lb. dry air.

T = the temperature above absolute zero Centigrade.

P = the pressure in lbs./sq. inch absolute.

Such a transverse line may be called a "*state-line*," since to any given state of the gas corresponds a definite transverse line, which may be at once drawn, given the values of any two of the above variables.

A vertical line passing through a point marked with a particular value n' on the horizontal scales (two of which are provided, so that by joining corresponding scale-divisions a vertical line may be drawn without the use of set-squares) is called an "*index-line.*"

If two state-lines intersect one another on the index-line n', the corresponding states are connected by the equation

$$PV^{n'} = \text{constant} \qquad \ldots\ldots(1).$$

The method of using the chart follows from the above description, and specific examples would appear unnecessary. It may, however, be pointed out that by drawing the state-line for the beginning of an expansion, and inserting a pin where it intersects the proper index-line, the straight-edge may be revolved, keeping it always in contact with the pin, so that corresponding values of P and V, or of T and ϕ, may be read off for all points on the expansion-curve.

It may also be remarked that, while the use of all the five scales together, P, V, T, ϕ and n, is restricted to the case of 1 lb. dry air, we may use the P, V and n scales together for any working substance (*e.g.* for steam); to take a particular problem, if we know two pairs of corresponding values of P and V at two points of an

expansion, we can use the chart to find n, where an equation of the form $PV^n = C$ is assumed as an approximation to the expansion-curve.

The principle on which the chart is constructed is as follows:

A square of 18″ side is taken*, and on the vertical sides logarithmically divided scales are drawn, running, for convenience, from 1 to 100 for the V scale, and from 3 to 300 for the P scale†. On the horizontal sides are drawn two identical "segmental scales‡," marked n, that is, scales such that if A and B be their extremities, and C a point marked n',

$$\frac{AC}{CB} = n' \qquad \qquad \ldots\ldots(2).$$

The points on these horizontal scales marked 1, and those marked 1·4, are joined, these lines carrying the scales marked T and ϕ, respectively.

Since the outer scales are logarithmic, the distances between points marked P' and P'' are proportional to $\log (P'/P'')$, and similarly for the points V' and V'' on the V scale; hence, from the properties of the similar triangles formed by the state-lines $P'V'$ and $P''V''$, and the property (2) of the n scale,

$$\log (P'/P'') = n \log (V''/V');$$
or $$PV^n = \text{constant}.$$

If $n = 1$, $PV = \text{constant}$, and the expansion is *isothermal*, hence the scale of temperatures must lie along the index-line $n = 1$; and since $PV = RT$, $\log T$ must be a linear function of $\log P$ and $\log V$, so the T scale must also be a logarithmic scale, and it is found that the necessary scale is related to the P and V scales in the same way as the top to the bottom scale of an ordinary slide-rule. That is to say, the T scale has 100 divisions in the same space as 10 divisions of the outer scales. The zero of the T scale is fixed by considering the volume of 1 lb. of dry air at $T = 273$ (zero Centigrade) and $P = 14\cdot7$ lbs./sq. in. (760 mm. of mercury).

Finally, since

$$\phi = K_v \log_e (P/P_0) + K_p \log_e (V/V_0),$$

ϕ is a linear function of $\log P$ and $\log V$, whence the scale of ϕ on the chart is an evenly divided scale. It clearly lies along the index-line $n = 1\cdot4$, since the adiabatic equation for air is $PV^{1\cdot4} = \text{constant}$; and two points on the scale can be fixed by calculation from the formula for ϕ, so that, the scale being an evenly divided one, it can now be easily completed.

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* The plate has been reduced from the original.

† The logarithmic scales to any modulus supplied as a supplement to Lipka's *Graphical and Mechanical Computation* (Wiley and Sons) were used.

‡ *Echelle Segmentaire*, M. D'Ocagne, *Nomographie* (Gauthier-Villars), p. 12.

The chart reproduced is on a reduced scale, but the original chart, on a square of 18″ side, will give an accuracy about equal to that obtainable with a 10″ slide-rule, using the top scales for T and ϕ, and the bottom scales for P and V. The accuracy of the upper part of the T scale could be increased by making a perspective projection of the whole chart, the upper part of the T scale being in the foreground, the new chart so obtained being used in exactly the same way as the chart shown*.

* M. D'Ocagne, *loc. cit.* pp. 135–140, *Transformation Homographique.*

Since this paper was communicated to the Society a very full discussion of projective transformations of alignment charts has been published by R. Soreau, *Nomographie, ou Traité des Abaques* (Etienne Chiron), Tom. II, chap. XVI, pp. 111–128.
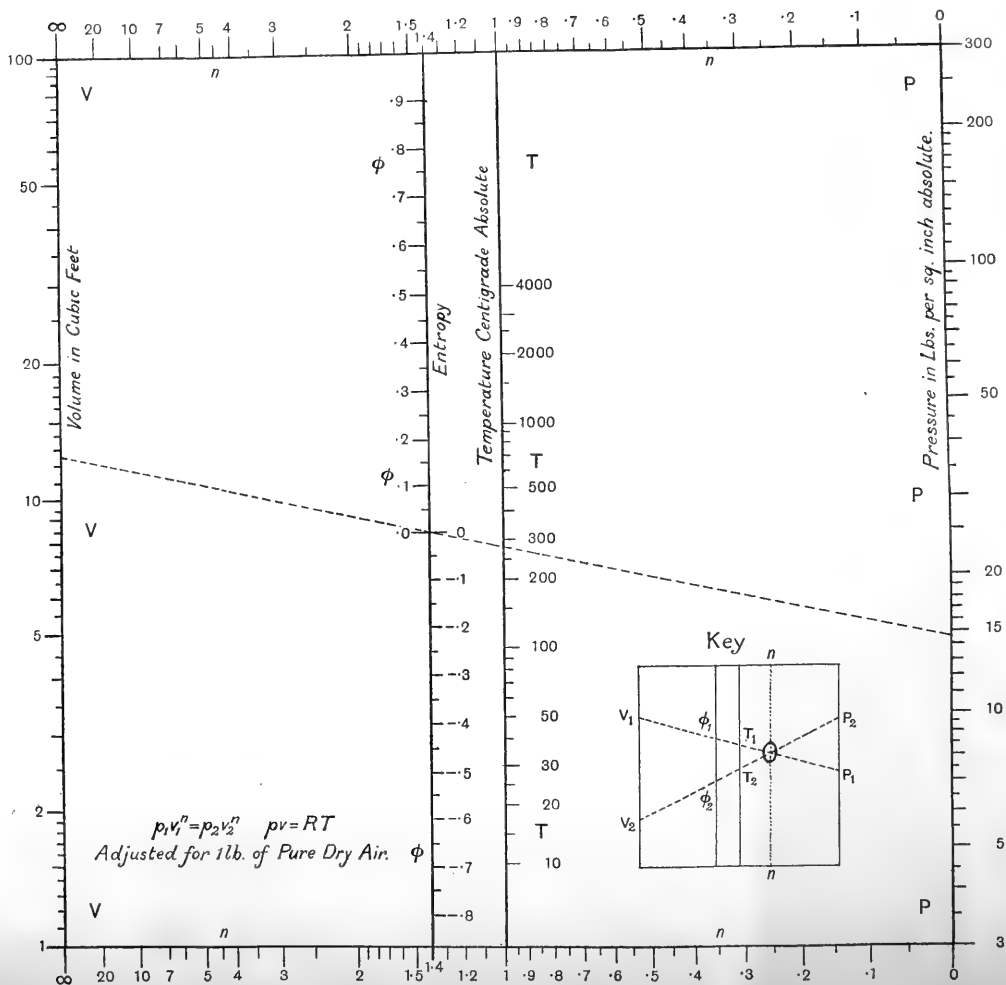
Plate II.

$p_1 v_1^n = p_2 v_2^n \qquad pv = RT$

Adjusted for 1lb. of Pure Dry Air.

The Automatic Synchronization of Triode Oscillators. By E. V. APPLETON, M.A., St John's College.

[*Received* 24 August, 1922.]

(1) The interaction of two triode oscillators presents certain problems which do not arise in the corresponding case of damped vibrators. These arise from the fact that the effective decrements of the oscillatory circuits are not constant, and thus the principle of superposition does not hold. A typical phenomenon of this kind is that recently described by Dr Vincent*, whose experiments showed that there must be certain definite phase relations between two interacting triode oscillators which are so adjusted as to be very nearly in resonance. In Dr Vincent's paper no theoretical explanation of the experimental results was attempted, but in the subsequent discussion of the paper Prof. O. W. Richardson suggested that the theory might be analogous to the theory of anomalous dispersion.

(2) In repeating Dr Vincent's experiment and various others suggested by it, certain new phenomena have been observed which will be described below. An attempt is also made to account for the phenomena in terms of a theory of the triode oscillator which takes into account the variations in the triode conductances. Briefly, the main result of the work is that when the difference of the frequencies of two interacting triodes is less than a certain amount, automatic synchronization takes place, and the two oscillators vibrate with one frequency. The critical difference of frequencies is a perfectly definite and calculable amount.

Examples of automatic synchronization have been previously met with in other branches of physics. Thus some early experiments made by the late Lord Rayleigh† on the interaction of two electrically maintained tuning forks fed by the same battery showed that, when the forks were nearly in unison, it was not possible to reduce the beat frequency indefinitely; in other words, automatic synchronization took place. A similar phenomenon‡ was noticed with two nearly unisonant organ pipes driven from the same wind chest. The Lissajous' figure representing the phase relation between the two forks was usually found to be an ellipse, but the fact that the resultant sound from the two organ pipes was practically nil indicated a phase difference of π.

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* Vincent, *Proc. Phys. Soc.* vol. XXXII, Part 2, p. 84.
† Rayleigh, *Collected Papers*, vol. V, p. 369.
‡ Rayleigh, *ibid.* vol. I, p. 409.

The experiment of Dr Vincent is thus really the triode analogue of the experiments of Lord Rayleigh, and his results are only to be explained by assuming that, if two oscillating triode systems are nearly in resonance, they vibrate with one frequency. In his experiments a third circuit, with crystal detector and galvanometer, was used to indicate the effect of bringing the two triode oscillators into resonance. It is difficult, however, to correlate the galvano-meter deflections with the phase relations of the two triodes, especially when the amplitudes of the latter are unequal, and thus in repeating this experiment I have used a cathode-ray oscillo-graph. The whole problem of the determination of the phase relations thus becomes a simple matter. For observations on two feebly coupled generators two arrangements were used. In one a special tube with two pairs of deflecting plates was used so that each triode activated a pair of plates. In the other arrangement electrostatic deflections were used for one triode and magnetic for the other. For experiments with strongly coupled generators magnetic deflections were often used for both circuits.

(3) The most striking result of the oscillograph experiments is the definite proof of the fact of automatic synchronization. Thus, if the frequency of one of the interacting oscillators is gradually varied through the state of resonance, the beats indicated by the rapidly moving trace of the oscillograph stop before the point of resonance is reached, the pattern on the screen becoming quite steady and indicating a common frequency. Such automatic synchronization is always found between two mutually coupled triode oscillators, but from the point of view of mathematical analysis it is simpler to approach the problem by considering first the case in which the action between the two oscillators is in one direction only. Such an arrangement is represented by a high-power continuous-wave wireless station acting on a low power receiver in its vicinity. In such a case it is found that there is a very large "silent interval" of the receiver condenser between the two regions of audible combination tones. Thus Mr R. Cole has found for me that it is not possible to reduce the combination tone frequency to less than about 800 per second at a small receiving station three miles from the Croydon transmitter. Laboratory ex-periments made with the cathode-ray oscillograph, and with the power of the transmitter and the distance between the sets re-duced, have shown that definite phase relations exist between the currents and voltages of the two sets within the region of synchro-nization. These agree with the relations predicted from the non-linear theory, which will now be given.

(4) We shall reduce the weaker oscillator to its simplest terms, and consider it as equivalent to an oscillatory circuit, the capacity

of which is shunted by a conductor for which the relation between current i, and potential difference v, may be written

$$i = \psi(v) = -\alpha v + \beta v^2 + \gamma v^{3*}.$$
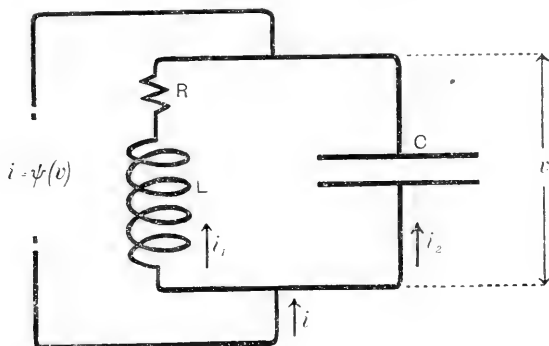
Fig. 1

Let us suppose that the electromotive force impressed on the coil (see Fig. 1) by the more powerful oscillator is V, where

$$V = E_0 \sin \omega_1 t \qquad \ldots\ldots(1).$$

As the expression of Kirchhoff's laws for the circuit we have

$$L \frac{di_1}{dt} + Ri_1 - V = \int \frac{i_2}{C} dt = -v.$$

$$i = i_1 + i_2 = \psi(v).$$

These equations lead to

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{C}\frac{d}{dt}\psi(v) + \frac{1}{CL}(v + Ri) = \frac{1}{CL} V \quad \ldots(2).$$

Now in a practical case Ri is very small compared with v, and thus (2) reduces to

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{C}\frac{d}{dt}\psi(v) + \omega_0^2 v = E_0\omega_0^2 \sin \omega_1 t \quad \ldots(3),$$

where ω_0^2 has been written for $\dfrac{1}{CL}$, and a substitution for V made from (1). We know from the nature of the problem that there will be, in general, a forced vibration of angular frequency ω_1, and also a free vibration of an unknown angular frequency ω. We thus adopt as trial solution $v = a \sin \omega t + b \sin (\omega_1 t - \phi)$, where both a and b are regarded as functions of the time t, but such that

$$\omega a >> \frac{da}{dt}, \quad \omega_1 b >> \frac{db}{dt},$$

* Cf. Appleton and van der Pol, *Phil. Mag.* vol. XLIII, Jan. 1922.

and that $\dfrac{d^2a}{dt^2}$ and $\dfrac{d^2b}{dt^2}$ are negligible. ϕ is an arbitrary phase constant.

In considering terms involving v^2 and v^3 we shall neglect higher harmonics and combination tones, and only retain terms involving the fundamental angular frequencies ω and ω_1.

We thus have

$$
\left.
\begin{aligned}
v &= a \sin \omega t + b \sin (\omega_1 t - \phi) \\
\dot{v} &= \dot{a} \sin \omega t + a\omega \cos \omega t + \dot{b} \sin (\omega_1 t - \phi) \\
&\qquad\qquad\qquad\qquad + b\omega_1 \cos (\omega_1 t - \phi) \\
\ddot{v} &= \ddot{a} \sin \omega t + 2\dot{a}\omega \cos \omega t - a\omega^2 \sin \omega t + \ddot{b} \sin (\omega_1 t - \phi) \\
&\qquad + 2\dot{b}\omega_1 \cos (\omega_1 t - \phi) - b\omega_1^2 \sin (\omega_1 t - \phi)
\end{aligned}
\right\} \quad \ldots(4).
$$

Also, retaining only terms involving angular frequencies ω_1 and ω, we may write

$$
\begin{aligned}
\psi(v) &= -\alpha a \sin \omega t - \alpha b \sin (\omega_1 t - \phi) \\
&+ \tfrac{3}{4}\gamma (a^3 + 2ab^2) \sin \omega t + \tfrac{3}{4}\gamma (b^3 + 2ba^2) \sin (\omega_1 t - \phi) \ldots(5).
\end{aligned}
$$

On substituting for v, $\dot{v}$, $\ddot{v}$, and $\psi(v)$ in (3), and equating the coefficients of $\sin \omega t$, $\cos \omega t$, $\sin (\omega_1 t - \phi)$, $\cos (\omega_1 t - \phi)$ separately to zero, we have, for stationary conditions

$$
\left.
\begin{aligned}
a (\omega_0^2 - \omega^2) &= 0 & \text{(A)} \\
\left(\frac{R}{L} - \frac{\alpha}{C}\right) a + \frac{3}{4}\frac{\gamma}{C}(a^3 + 2ab^2) &= 0 & \text{(B)} \\
b (\omega_0^2 - \omega_1^2) &= E_0\omega_0^2 \cos \phi & \text{(C)} \\
\left(\frac{R}{L} - \frac{\alpha}{C}\right) b + \frac{3}{4}\frac{\gamma}{C}(b^3 + 2ba^2) &= \frac{E_0\omega_0^2}{\omega_1} \sin \phi & \text{(D)}
\end{aligned}
\right\} \quad (6).
$$

From these equations we may deduce various facts concerning automatic synchronization. From 6 A we see that when a steady state has been reached, the free vibration, if it exists, will be of angular frequency ω_0. The question as to whether it will exist or not may be decided with the aid of 6 B, which may be written

$$
\left(\frac{R}{L} - \frac{\alpha}{C} + \frac{3}{2}\frac{\gamma}{C} b^2\right) a + \frac{3}{4}\frac{\gamma}{C} a^3 = 0 \qquad \ldots\ldots(7).
$$

We see from (7) that a sustained amplitude of frequency ω_0 is only possible so long as the coefficient of a is negative. In the absence of the disturbing electromotive force due to the stronger set we know that, since $\left(\dfrac{R}{L} - \dfrac{\alpha}{C}\right)$ is negative, the final amplitude of the free vibration is given by

$$
a^2 = \frac{4}{3}\left(\frac{\dfrac{CR}{L} - \alpha}{\gamma}\right) = a_0^2 .
$$

If, however, the impressed electromotive force of angular frequency ω_1 is so large, or $(\omega_1 - \omega_0)$ so small, that b^2 increases sufficiently to make the coefficient of a in (7) positive, the oscillation of frequency ω_0 is damped, and only a sustained oscillation of frequency ω_1 is maintained. Thus we see that suppression of the free vibration must take place when

$$b^2 = \tfrac{1}{2}a_0^2 \qquad \qquad \ldots\ldots(8)$$

or when the forced oscillation reaches an amplitude equal to $\dfrac{1}{\sqrt{2}}$ times the amplitude which would exist if the disturbing electromotive force were absent.

(It is necessary to note here that we have considered the simplest type of triode oscillation characteristic, and that the theoretical results indicate that the limiting conditions for the starting and suppression of beats should be identical. But experiments[*] have shown that this simple type of characteristic is by no means always the one met with in practice. From the results of these experiments we should expect that for conditions represented by points distant from the centre of the triode characteristic a hysteresis effect would be evident. Actually such hysteresis is often found experimentally. It seems clear that such phenomena could be accounted for by considering an oscillation characteristic with the representative equation $i = \alpha v - \gamma v^3 + \epsilon v^5$. Such a theory is not attempted here, but the lines it may follow are obvious.)

Equations 6 A, 6 B, 6 C and 6 D may be written more simply as

$$\left.\begin{array}{ll} a\,(\omega_0^2 - \omega^2) = 0 & \text{(A)} \\ -\,\alpha_1 a + \gamma_1\,(a^3 + 2ab^2) = 0 & \text{(B)} \\ b\,(\omega_0^2 - \omega_1^2) = E_0\omega_0^2\cos\phi & \text{(C)} \\ -\,\alpha_1 b + \gamma_1\,(b^3 + 2a^2 b) = \dfrac{E_0\omega_0^2}{\omega_1}\sin\phi & \text{(D)} \end{array}\right\} \quad \ldots\ldots(9).$$

To illustrate these equations we may draw a graph showing the relation between a (amplitude of free vibration), b (amplitude of forced vibration), and the angular frequency ω_1 of the impressed electromotive force. This is shown in Fig. 2.

Here the limits of frequency change of ω_1 between which synchronization or engagement takes place are shown by the points A and C. Point B indicates exact resonance, and AC represents what is often termed the "silent interval."

Within the range of synchronization there is a definite phase relation between the voltages of the two oscillators. From 9 C we see at once that at resonance, when $\omega_0^2 - \omega_1^2$ is zero, ϕ is equal to $\dfrac{\pi}{2}$.

[*] Appleton and van der Pol, *loc. cit.*

A further consideration of the equations shows that within this interval ϕ changes rapidly with change of ω_1. Broadly speaking, when the mutual inductance between the coils of the two sets is positive, this is a change from $\phi = 0$ when $\omega_1 < \omega_0$ to $\phi = \pi$ when $\omega_1 > \omega_0$, passing through $\phi = \dfrac{\pi}{2}$ when $\omega_1 = \omega_0$. These changes are exactly reversed when M is negative. Actually, however, the
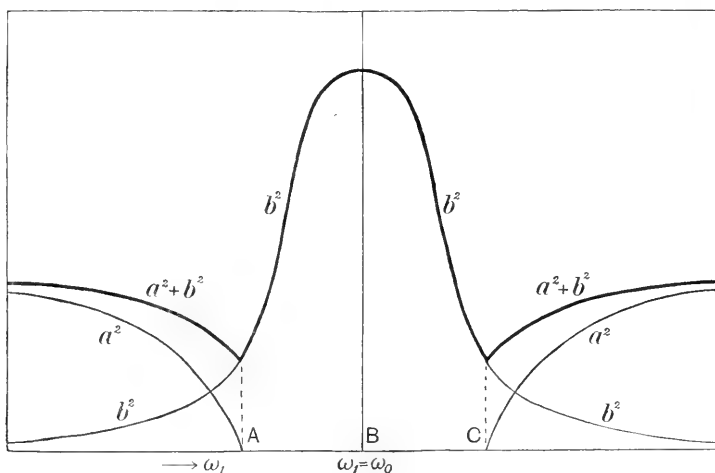
Fig. 2
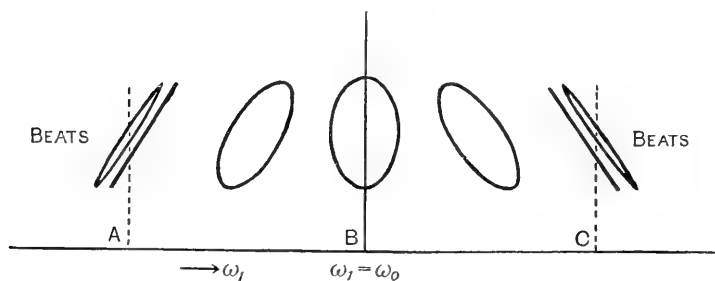
Fig. 3

theoretical results indicate that ϕ varies through a slightly greater range than this, becoming slightly less than 0 on one side of resonance and slightly larger than π on the other. When the "silent interval" is large these changes may be followed and verified with the oscillograph. Thus in a typical case the Lissajous' figure illustrating the phase relation between the synchronized oscillators varied with ω_1, as shown in Fig. 3.

In cases where the region of synchronization is small, however, the main effect observable is the change of phase from 0 to π as ω_1 passes through resonance. The extent of the region of synchronization may be calculated as follows. On squaring 9 c and 9 d and adding, we have, for conditions in which the free vibration is suppressed,

$$b^2 (\omega_0^2 - \omega_1^2)^2 + \omega_1^2 (- \alpha_1 b + \gamma_1 b^3)^2 = E_0^2 \omega_0^4.$$

On substituting for the critical value of b^2 from (8), we find that the critical value of distuning at which beats start and stop, is given by

$$\delta\omega = \omega_0 - \omega_1 = \frac{1}{2} \sqrt{\frac{2E_0^2 \omega_0^2 \gamma_1}{\alpha_1} - \frac{\alpha_1^2}{4}} = \frac{1}{\sqrt{2}} \frac{E_0 \omega_0}{a_0}, \dots (10),$$

where a_0^2 is equal to $\dfrac{\alpha_1}{\gamma_1}$, and α_1 is very small*. Thus the magnitude of the silent interval is proportional to the impressed electromotive force, and inversely proportional to the free amplitude of the weaker oscillator.

The formula (10) has been verified experimentally by Mr W. T. Gibson and the writer, and it has been found that the experimental results for a certain triode (M.O., Type L.T. 3) were in good agreement with the formula

$$\delta\omega = 0.72 \frac{E_0 \omega_0}{a_0}.$$

These experiments provide the basis of an experimental method of measuring small alternating electromotive forces such as are obtained with strong continuous-wave wireless signals. The method consists in measuring the extent of the "silent interval" in an auto-heterodyne receiver of extremely small natural amplitude (*e.g.* of the order of 0·5 volt). If $2\delta\omega$ is the range of synchronization as read off on the calibrated condenser, the signal electromotive force E_0 is given by

$$E_0 = 1.39 \frac{\delta\omega}{\omega_0} a_0 \qquad \dots\dots(11),$$

where a_0 is the amplitude of the oscillator in the absence of signals. Using this method it has been found possible to measure the E.M.F. introduced in a small antenna system at Cambridge by signals sent from the Marconi station at North Weald.

(5) In many practical cases, however, the disturbing E.M.F. is impressed (*a*) in the grid circuit of the triode, or (*b*) in the anode circuit, and not, as previously assumed, on the coil of the oscillatory

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* In all the experiments carried out the reaction coupling of the weaker oscillator was adjusted so as to be just greater than the critical value.

circuit. It is evident that synchronization can be produced in the first two mentioned cases as well as in the third. But a given E.M.F. will not be equally effective in the three cases. Thus an E.M.F. E_0, introduced in the grid circuit, will be equivalent to an E.M.F. κE_0, introduced directly in the anode circuit, where κ is the amplification factor of the triode. Also, for synchronization in cases (a) and (b) the phase relations as shown by the oscillograph indicate a change of ϕ from $-\dfrac{\pi}{2}$ to $+\dfrac{\pi}{2}$ within the region of synchronization. It will be seen that this is quite different from the case previously considered.

As an example of this type of linkage we may investigate theoretically a case in which the disturbing E.M.F. is introduced directly in the anode circuit of the triode. Thus in our equivalent circuit we may imagine it introduced by means of a coil l, the
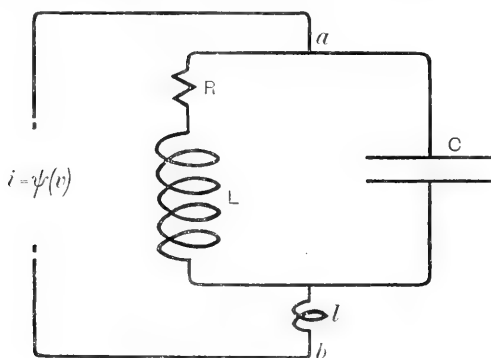
Fig. 4

reactance of which is considered small and negligible in comparison with the effective resistance of the triode (see Fig. 4).

Let the electromotive force introduced in l be $E_0 \sin \omega_1 t$, and let the potential difference between points a and b be v. Then, as before, the application of Kirchhoff's laws leads to the fundamental differential equation

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{C}\frac{d}{dt}\psi(v) + \omega_0^2 v$$

$$= E_0(\omega_0^2 - \omega_1^2)\sin \omega_1 t + E_0\frac{R\omega_1}{L}\cos \omega_1 t$$

$$= E_0\frac{R\omega_1}{L}\sqrt{1 + \left(\frac{\delta\omega}{2d}\right)^2}\cos(\omega_1 t - \chi)$$

where $\omega_0 - \omega_1 = \delta\omega,\ d = \dfrac{R}{2L}$, and $\tan \chi = \dfrac{\delta\omega}{2d}$.

Now, since in a practical case $\frac{\delta\omega}{2d} \ll 1$, we may write this as

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{C}\frac{d}{dt}\psi(v) + \omega_0^2 v = \frac{E_0 R \omega_1}{L}\cos\omega_1 t \quad \ldots(12).$$

The approximate solution of this equation proceeds on identical lines with that of (3), so that, if a solution of the form

$$v = a\sin\omega t + b\sin(\omega_1 t - \phi)$$

is assumed, the four equations* giving the relations for stationary conditions are as follows:

$$a(\omega_0^2 - \omega^2) = 0$$

$$-\alpha_1 a + \gamma_1(a^3 + 2ab^2) = 0$$

$$b(\omega_0^2 - \omega_1^2) = -\frac{E_0 R \omega_1}{L}\sin\phi$$

$$-\alpha_1 b + \gamma_1(b^3 + 2a^2 b) = \frac{E_0 R}{L}\cos\phi.$$

It will readily be seen that these equations indicate phase angle changes $-\frac{\pi}{2} \to 0 \to \frac{\pi}{2}$, as observed experimentally, when the frequency ω_1 is varied through resonance values. Also the critical value of distuning $\delta\omega$ at which beats stop is given by

$$\delta\omega = \frac{1}{\sqrt{2}}\frac{E_0 R}{L a_0}.$$

(6) We thus see that, in both of the cases considered, the critical difference of frequencies at which beats start and stop is proportional to the magnitude of the impressed electromotive force, and inversely proportional to the free undisturbed amplitude of the weaker oscillator. This result was most strikingly illustrated by an experiment in which the natural amplitude of one of the oscillators (set *A*) was varied from a very small to a very high value, while the amplitude of the other (set *B*) was maintained at a constant value. The magnitude of the silent interval $2\delta\omega$, measured as a capacity variation, was found to vary with the natural amplitude a_0 of the set *A* as indicated in the graph of Fig. 5. It will be seen that at first the region of synchronization is approximately inversely proportional to the value of a_0, so that here *B* may be regarded as driver and *A* as the driven set. When the amplitude a_0 is still further increased, so that the natural amplitude of *A* becomes larger than that of *B*, the rôles of driver and driven are reversed and the region of synchronization becomes proportional to a_0.

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* Compare equations 9 A, 9 B, 9 C and 9 D.
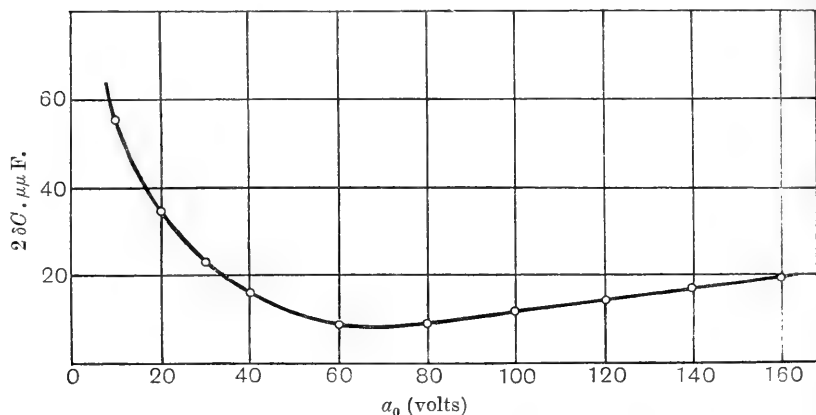
Fig. 5

(7) When the reaction from the weaker oscillator on the stronger is not negligible the problem is a little more complicated. Simple experiments, however, showed that in general the stronger oscillator caused the other to synchronize. But if the coupling between the coils of the two oscillatory circuits is greater than a certain critical value, a change is found in the phase relations between the two oscillators within the region of synchronization. These effects are to a certain extent analogous to the hysteresis phenomena observed in an ordinary triode assembly with two degrees of freedom, but
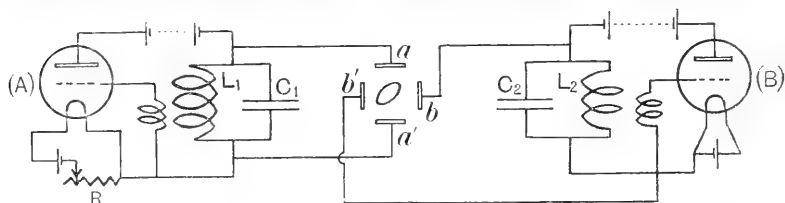
Fig. 6

in the present case the critical value of coupling is found to depend on the triode conductances as well as on the decrements of the oscillatory circuits.

The experimental arrangement used in investigating these phenomena is shown in Fig. 6, the two triode assemblies (A and B) being connected to the two sets of oscillograph deflecting plates aa' and bb'. The phase relation between the two oscillating voltages could thus be observed directly on the willemite screen of the oscillograph.

The effect of turning condenser C_1 through the position of resonance was observed for different values of the filament current

in A. The effect of varying the coupling between L_1 and L_2 was also studied. The results are summarised below.

(a) *Filament of* A *cold.*

When C was swept through resonance position the phase angle underwent the change $0 \to \frac{\pi}{2} \to \pi$, being $\frac{\pi}{2}$ exactly at resonance. As the coupling between L_1 and L_2 was increased the phase angle altered more rapidly near the point of resonance. For coupling greater than a certain critical amount irreversible discontinuities in phase and frequency were found. The critical value of coupling was found to depend only on the decrement of the oscillatory circuit $L_1 C_1$.

(b) *Filament of* A *hot, but set* A *not oscillating.*

Here the same type of phenomena was observed, but the critical value of coupling was much less.

(c) *Filament of* A *hot, and set* A *oscillating.*

When the coupling between L_1 and L_2 was small, heterodyne combination tones were observed on each side of resonance as C_1 was increased, and the usual range of synchronization observed. The phase variation $0 \to \frac{\pi}{2} \to \pi$ was now crowded within the silent region of condenser C_1.

When the coupling between L_1 and L_2 was increased the critical value of coupling was reached and the discontinuities of phase and frequency occurred within the range of synchronization. The critical value of coupling was found to be much less than in cases (a) and (b).

(8) The results of the last paragraph are readily explained if we utilise the results previously obtained. Within the range of synchronization the weaker set A may be regarded as an ordinary secondary circuit of set B, but having a decrement totally different from its natural value. Thus, within the region of synchronization we may regard the weaker triode as a simple secondary circuit of considerably reduced decrement, or, alternatively, one may regard a simple secondary circuit without triode as a synchronized circuit with infinite range of synchronization.

If we regard the weaker triode in an oscillating condition as a secondary circuit of the stronger oscillator, it is of interest to investigate how the effective decrement of this circuit varies with filament current, retroaction, anode potential, etc. This has been done by two methods, (a) and (b), but since the second method is much more reliable and accurate than the first the numerical results obtained in the latter case only will be given.

In method (a) use is made of the fact that the critical value of the coupling coefficient κ for the production of discontinuities is given by $\kappa^2 = \dfrac{4\delta^2}{\omega^2}$, where δ is the secondary damping coefficient. Thus, by increasing the coupling between sets A and B until the frequency discontinuities were heard on a third auto-heterodyne set some metres away, the variation of κ with the electrical variables of the set A was studied. The results obtained showed that the stronger the oscillation of the weaker set in the absence of any disturbing effect, the smaller the effective decrement possessed by its circuit when it had been synchronized by the stronger.

In method (b) use was made of the resonance curve obtained for a feeble triode oscillator when the frequency of a loosely coupled
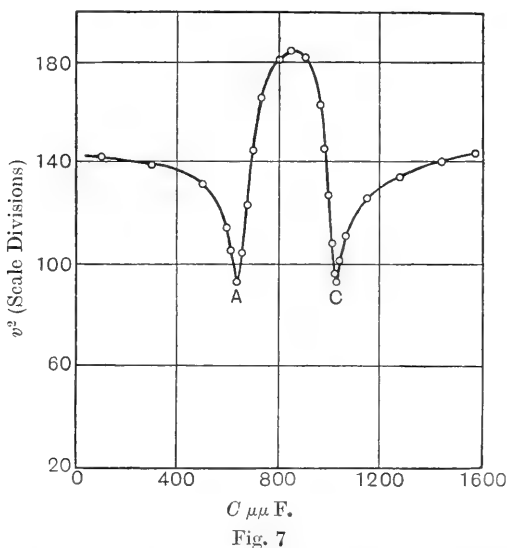
Fig. 7

powerful transmitter was varied through resonance. The amplitude of the weaker oscillator was measured with a voltmeter made up as suggested by Moullin*. The resonance curve in a typical case is shown in Fig. 7 where the square of the oscillatory potential is exhibited as a function of the transmitter condenser readings.

The region AC corresponds to the range of synchronization, and it is evident that for a given value of oscillatory E.M.F. induced in the receiver by the transmitter, the effective decrement of the former is inversely proportional to the maximum amplitude attained by it at resonance.

(It may also be noticed in passing that the non-linear theory

* *Wireless World and Radio Review*, vol. x, No. 1, April 1, 1922.

as illustrated by Fig. 2 in particular, predicts a resonance curve of the type given above.)

Experiments were carried out using an impressed E.M.F. which produced a maximum amplitude of 0·1 volt in the receiver circuit when retroaction was negligible. Maximum amplitudes were then found for different values of retroaction, the natural amplitudes of the receiver in the absence of the disturbing E.M.F. being also measured. These are tabulated below, the ratio of the effective decrement to the natural decrement being also given.

Natural amplitude a_0 in volts	Forced amplitude at resonance (volts)	Effective decrement / Natural decrement
0	0·1	1·0
0·42	0·71	0·14
0·62	0·87	0·115
0·78	0·95	0·105
0·93	1·07	0·094
1·07	1·17	0·085
1·20	1·31	0·076
1·34	1·46	0·068

These figures illustrate the fact that as the natural amplitude of the weaker oscillator is increased, the effective decrement of the circuit when synchronized by a stronger oscillator decreases, and thus the critical mutual induction at which phase discontinuities occur becomes smaller. This is in agreement with the results of method (*a*) mentioned above.

Another interesting result can be obtained from the figures of the above table. It will be seen that the increase in amplitude caused by a given E.M.F. decreases as the natural amplitude of the oscillator increases. This may be shown to be a direct consequence of the non-linear theory previously given, for if we solve equation 9 D approximately for resonance conditions we find that when $\omega_1 = \omega_0$,

$$b_{\max} = a_0 + \frac{E_0 \omega_0}{2\gamma_1 a_0^2},$$

where a_0 is the amplitude existing in the weaker oscillator in the absence of the stronger. The equation indicates that the increase in amplitude produced at resonance by an external E.M.F. is smaller the greater the natural amplitude of the oscillator.

(9) We are now in a position to explain the experiment of Vincent on the interaction of two oscillating triodes. In this experiment the frequency of one oscillator is varied through resonance, the effect being observed in a third indicator circuit acted on by

both oscillators. Remarkable alterations of amplitude are found in the indicator circuit when the two oscillators are nearly in resonance. These are a direct result of the synchronization of the two oscillators, and in particular of the special phase relations which exist between the two synchronized sets. The results in a typical case may be illustrated by Fig. 8, in which the galvanometer deflection of the detector circuit is plotted as a function of the condenser reading of one of the triodes.
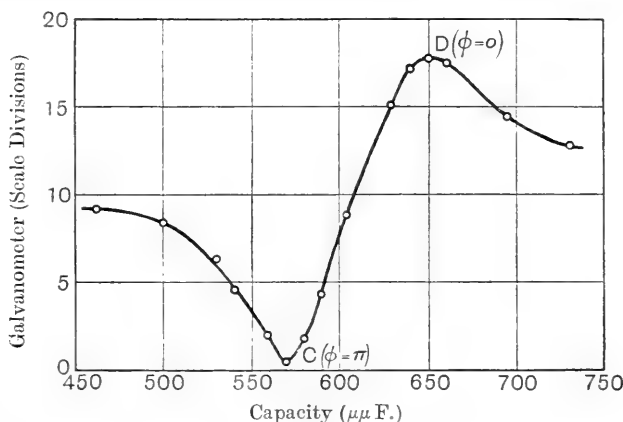
Fig. 8

The rapid change of the phase difference $\pi \to \dfrac{\pi}{2} \to 0$ within the region of synchronization is indicated by the rapid rise in the value of the detector current between the points C and D.

Since Vincent used a positive mutual inductance between the coils of the two oscillators, the phase difference of the sets should, according to the theory given above, vary from 0 to π as the frequency of the stronger set is varied through resonance. The curves shown in Vincent's paper, interpreted as suggested above, actually indicate such a variation*.

We are now in a position to state exactly the conditions necessary to obtain detector current curves of the type shown in the last figure. If the linkage between the oscillatory circuit coil of one set and the grid coil of the other is very large, we see from the preceding theory that the phase relations between the two sets within the region of synchronization, will be of the type $-\dfrac{\pi}{2} \to 0 \to \dfrac{\pi}{2}$ instead of $0 \to \dfrac{\pi}{2} \to \pi$, giving rise to quite different effects in the

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* The phase discontinuities observed with close coupling appear to have been overlooked by Vincent.

detector circuit. A comparison of the results previously obtained shows that, for the curve to be of the type obtained by Vincent, we must have $\kappa M_g << \dfrac{L\omega_0}{R} M_a$, where κ is the amplification factor of the weaker triode, and M_g and M_a the mutual inductances between the anode coil of the stronger set and the grid and anode coils respectively of the weaker set.

(10) The experiments described above were all carried out with oscillators of radio-frequency, and thus the beats outside the region of synchronization could not be followed visually on the screen of the oscillograph. Further experiments were therefore made using low-frequency oscillators, so that the phenomena near the conditions of synchronization could be studied more closely.

In the tuning-fork experiments of Rayleigh previously mentioned*, it was found that the beats of two forks with mutual influence were of a very curious type, the amplitude increasing slowly and decreasing rapidly during a beat cycle. Similar phenomena are observed with two interacting triodes of low frequency, but since in this case it is possible to control and vary the conditions more easily than with tuning-forks, it has been found possible to investigate the effect a little more in detail.

Let us first consider two triode oscillators, A and B, with no mutual influence. If n_1 and n_2 are the natural frequencies of the two systems, beats of frequency $(n_1 - n_2)$ may be detected in a third circuit very loosely coupled to both. If, however, there is appreciable mutual influence between A and B the result is quite different, in that the beat frequency may be appreciably less than $n_1 - n_2$. The effect is due to the tendency to synchronize, and, as one would expect, is most marked when the difference of frequencies is small. As we have seen previously there is a definite range of synchronization when the natural frequency of one system is varied through resonance. If the frequency of the stronger oscillator is increased, and the coupling is positive, the phase difference changes from 0 to π within the region of synchronization.

Let us consider for a moment the beats produced when the difference of frequencies is just greater than the critical difference required for synchronization. During the beat cycle the phase difference between the two oscillators varies, so that there is one part of the cycle when the conditions are favourable for synchronization. It is found, from oscillographic experiments, that the oscillators tend to maintain this particular phase relation longer than any other. Thus, in the example quoted above, when the frequency of the stronger oscillator is increased until the beats are on the verge of ceasing, the conditions are most favourable when

* Rayleigh, *Collected Papers*, vol. v, p. 369.

the sets are in phase ($\phi = 0$). Thus the part of the beat cycle in which the sets are in phase is prolonged. On the other hand, when the frequency is still further increased so that the beats just start again, the condition favourable for synchronization is that in which the sets are in anti-phase ($\phi = \pi$).

It will readily be seen that the above effects tend, in both cases, to lengthen the beat period, and so decrease the number of beats per second. We thus have, as it were, an *attraction* of frequencies which is specially marked when the difference of frequencies is small. The "attraction" is also more marked the greater the mutual linkage, and thus may be contrasted with the ordinary "repulsion" of frequencies which occurs in the corresponding case of damped vibrators when the mutual influence is increased.
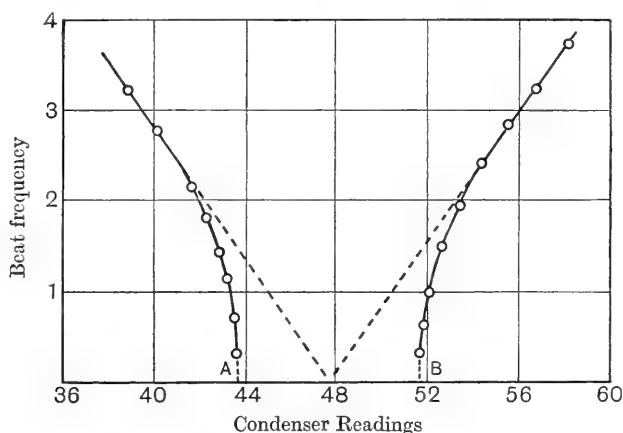
Fig. 9

(11) In investigating this particular point experiments were carried out with two low frequency oscillators with frequencies of approximately 400 per second. The beats were so slow that they could be counted by ear, and thus the beat frequency was directly measurable. The beat frequency is exhibited in Fig. 9 as a function of the readings of a small tuning condenser included in parallel with one large condenser of one of the oscillatory circuits. If no synchronization effects are contemplated the theoretical change of beat frequency with capacity is indicated by the dotted lines. The continuous lines are drawn through the experimental values.

We thus see that although the calculated and experimental values agree when the beat frequency is large the experimental values are always less than the calculated just outside the synchronization interval *AB*.

This result appears to be of importance in view of the fact that in many recent experiments the change of beat note frequency is taken as indication of capacity variation, a linear law being assumed, as suggested by the simple theory. We see that such a law is by no means justified unless the beat note frequency is high, and the conditions such that there is little or no tendency to synchronize. Thus the two oscillators should be coupled as loosely as possible.

I am indebted to Mr W. T. Gibson for most valuable assistance in the experimental work, and to Dr B. van der Pol for many helpful suggestions.

APPENDIX.

Note on the solution of equation (3).

I am indebted to Mr W. M. H. Greaves, of St John's College, for the following alternative solution of equation (3).

We may first write it as

$$\frac{d^2v}{dt^2} + (-\alpha_1 + 4\gamma_1 v^2)\frac{dv}{dt} + \omega_0^2 v = E_0\omega_0^2 \sin \omega_1 t \ ...(1\ a),$$

where, as before

$$-\alpha_1 = \left(\frac{R}{L} - \frac{\alpha}{C}\right), \text{ and } \gamma_1 = \frac{3}{4}\frac{\gamma}{C}.$$

If we neglect the small quantities α_1 and γ_1, the equation becomes

$$\frac{d^2v}{dt^2} + \omega_0^2 v = E_0\omega_0^2 \sin \omega_1 t,$$

in which case the solution is

$$v = a\cos(\omega_0 t + \beta) + \frac{E_0\omega_0^2}{\omega_0^2 - \omega_1^2}\sin \omega_1 t \quad ......(2\ a).$$

To obtain a better approximation we shall substitute for v in the term $4\gamma_1 v^2$ in $(1\ a)$. Thus, from $(2\ a)$ we have

$$v^2 = a^2\cos^2(\omega_0 t + \beta) + 2a\frac{E_0\omega_0^2}{\omega_0^2 - \omega_1^2}\sin \omega_1 t \cos(\omega_0 t + \beta)$$
$$+ \frac{E_0^2\omega_0^4}{(\omega_0^2 - \omega_1^2)^2}\sin^2 \omega_1 t \quad ......(3\ a).$$

Since $(\omega_0^2 - \omega_1^2)^2$ is small we shall retain only the term

$$\frac{E_0^2\omega_0^4}{(\omega_0^2 - \omega_1^2)^2}\sin^2 \omega_1 t$$

in the expression for v^2 and neglect the other terms which are small in comparison. We thus have

$$v^2 = \frac{1}{2}\frac{E_0^2\ \omega_0^4}{(\omega_0^2 - \omega_1^2)^2}(1 - \cos 2\omega_1 t).$$

Hence
$$-\alpha_1 + 4\gamma_1 v^2 = \lambda - \mu \cos 2\omega_1 t$$

where
$$\lambda = -\alpha_1 + \frac{2E_0{}^2\omega_0{}^4\gamma_1}{(\omega_0{}^2 - \omega_1{}^2)^2}$$

and
$$\mu = \frac{2E_0{}^2\omega_0{}^4\gamma_1}{(\omega_0{}^2 - \omega_1{}^2)^2}$$
......(4 a).

On substituting in (1 a) we have

$$\frac{d^2v}{dt^2} + (\lambda - \mu \cos 2\omega_1 t)\frac{dv}{dt} + \omega_0{}^2 v = E_0\omega_0{}^2 \sin \omega_1 t \quad ...(5\,a).$$

The solution of this equation is

$$v = Av_a + Bv_b + \frac{E_0\omega_0{}^2}{\omega_0{}^2 - \omega_1{}^2}\sin \omega_1 t + \left(-\lambda + \frac{1}{2}\mu\right)\frac{E_0\omega_0{}^2\omega_1}{(\omega_0{}^2 - \omega_1{}^2)^2}\cos \omega_1 t$$

where A and B are arbitrary constants, and where

$$v_a = \frac{1}{4}\frac{\omega_0}{\omega_1}\frac{\mu}{\omega_0{}^2 - \omega_1{}^2}e^{-\frac{1}{2}\lambda t}\left\{\frac{1}{2}(\omega_1 + \omega_0)\sin (2\omega_1 - \omega_0)\,t \right.$$
$$\left. + \frac{1}{2}(\omega_0 - \omega_1)\sin (2\omega_1 + \omega_0)\,t\right\}$$

and

$$v_b = \frac{1}{4}\frac{\omega_0}{\omega_1}\frac{\mu}{\omega_0{}^2 - \omega_1{}^2}e^{-\frac{1}{2}\lambda t}\left\{\frac{1}{2}(\omega_1 + \omega_0)\cos (2\omega_1 - \omega_0)\,t \right.$$
$$\left. - \frac{1}{2}(\omega_0 - \omega_1)\cos (2\omega_1 + \omega_0)\,t\right\}.$$

It will be seen that, according to this solution, two frequencies are present in the system only when λ is negative, so that the critical condition for the suppression of beats is that λ should be zero. When this is the case we have from (4 a)

$$\omega_0{}^2 - \omega_1{}^2 = \pm \sqrt{2}\,\frac{E_0\omega_0{}^2}{\alpha_0},$$

or
$$\omega_0 - \omega_1 = \delta\omega = \frac{1}{\sqrt{2}}\frac{E_0\omega_0}{\alpha_0},$$

which is exactly the result obtained previously (see p. 237).

On the geometrical theory of apolar quadrics. By H. G. TELLING.
(Communicated by Professor H. F. BAKER.)

[*Read* 15 May, *received* 31 July, 1922.]

The theory of apolar quadrics has already been considered in
general by Reye in his paper "Über lineare Systeme und Gewebe
von Flächen zweiten Grades*," but the treatment is only partially
geometrical, and assumes the representation of quadrics by their
equations. Some parts of the theory may be obtained more readily
by analytical methods; these, however, do not put in evidence the
details of the relations between apolar quadrics. The geometrical
theory introduces self conjugate hexads, which are defined by six
points such that the plane through any three is conjugate to the
plane through the remaining three; it also introduces self conjugate
pentads, consisting of five points such that the line joining any two
contains the pole of the plane containing the remaining three; and
self polar tetrads for which each of the four points is the pole of
the plane containing the other three. The polarity, in every case,
is taken with respect to a fundamental quadric envelope, Σ; and
a quadric surface S is said to be outpolar to Σ when a hexad can
be inscribed in S self conjugate with respect to Σ. This condition
will be shown to be equivalent to that which may be stated in
terms of a self polar tetrahedron. The quadric envelope Σ is said
to be inpolar to the quadric locus S, when S satisfies the above
conditions. Thus, the self conjugate pentads and hexads may seem
unnecessary, but they are more general than the self polar tetrads
and add to the completeness and symmetry of the theory. Such
pentads and hexads have been considered by Reye in his paper
"Über Polfünfecke und Polsechsecke räumlicher Polarsysteme†."
The self conjugate hexad is analogous to the self conjugate quad-
rangle in the theory of apolar conics: a degenerate form of quadric
outpolar with respect to Σ is a pair of planes conjugate with respect
to Σ, opposite faces of a self conjugate hexad; while, for the theory
in the plane, a degenerate conic outpolar to a given conic is deter-
mined by a pair of conjugate lines, opposite sides of a self conjugate
quadrangle.

As an intermediary step to the establishment of the properties
of quadrics passing through the points of self conjugate hexads and
the general theorem that any quadric through the intersection of
two quadrics which are outpolar to Σ, is itself outpolar to Σ, the
properties of twisted cubics passing through the vertices of self
conjugate hexads are considered. Such cubic curves are said to be

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* *Crelle*, 1874, vol. 77, pp. 269–288.   † *Ibid.*, 1877, vol. 82, § 3.

outpolar to Σ. In particular, a cubic curve, containing self polar tetrahedra with respect to Σ, will be shown to be outpolar to Σ, while the cubic developable, known to exist, containing the faces of these tetrahedra, is inpolar to Σ considered as a point locus. In this case the cubic curve may be said to be outpolar to the cubic developable. Thus the theory of apolar cubics may be included in the theory of apolar quadrics. It is shown that

(i) If a twisted cubic be such that a self conjugate hexad, with respect to Σ, may be described therein, then an infinite number of such hexads may be described therein.

(ii) If a twisted cubic be such that a self polar tetrahedron can be described therein, then an infinite number of such tetrahedra, also, of self conjugate pentads and hexads, may be described therein.

(iii) If a twisted cubic be such that a self conjugate hexad can be described therein, then self conjugate pentads and self polar tetrads may be described therein.

Similar propositions are shown to be true in the case of quadrics. For the proof of the above propositions the consideration of a certain property of a tetrahedron is necessary. As in the case of apolar conics, the theorem that given three vertices of a triangle, the lines joining each vertex to the pole of the opposite side meet in a point, is fundamental, so here, the proposition, that, in general, the lines joining the vertices of a tetrahedron to the poles of the opposite faces are generators of a regulus, is first considered. A proof is also required of the theorem, that three quartic curves, of the kind given by the intersection of two quadrics, which have seven points in common, have also an eighth. For the present purposes it is required that two of these quartic curves should each be degenerate, consisting of a twisted cubic and a chord thereof.

1. Preliminary theorems.

(a) If a tetrahedron $ABCD$ be such that no two opposite edges are conjugate, then the joins of the vertices to the poles A', B', C', D', of the faces BCD, CAD, ABD, ABC respectively, in regard to a quadric Σ, are generators of one regulus of a quadric.
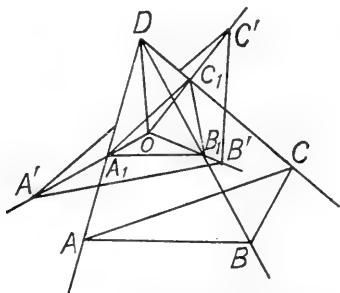
Fig. 1

For, consider a section of the figure by the plane $A'B'C'$; let this meet DA, DB, DC, respectively, in A_1, B_1, C_1; the lines B_1C_1, C_1A_1, A_1B_1 are then the poles of A', B', C', respectively, in regard to the conic section of Σ by the plane $A'B'C'$. Thus, by a theorem of plane geometry, the lines $A'A_1$, $B'B_1$, $C'C_1$ meet in a point O. Hence, since the points A', A, A_1, D are in one plane with O, DO meets AA'; similarly DO meets BB' and CC'; it also meets DD' at D. There is, similarly, a line through each of A, B, C meeting all of AA', BB', CC', DD'. These lines are therefore met by more than two transversals, and are thus generators of a quadric. These generators will be said to belong to the first system of a quadric $\phi(A, B, C, D)$, and their transversals, DO, etc., to the second system.

It may be noted, that if the line BC is conjugate to AD, so that the line $B'C'$, which is the polar of AD, meets BC, then the pole of the plane B, C, B', C', through $B'C'$, lies on AD; as this plane contains BC, this pole equally lies on $A'D'$; thus $A'D'$ meets AD and AD is conjugate to BC. Thus, of the four lines AA', BB', CC', DD', the two BB', CC' lie in a plane, and the two AA', DD' lie in a plane when BC is conjugate to AD; the quadric $\phi(A, B, C, D)$ is then replaced by these two planes.

Finally, if two pairs of edges AC, BD; AB, CD of the tetrahedron $ABCD$ be conjugate, $A'C'$ meets AC, $B'D'$ meets BD and $A'B'$
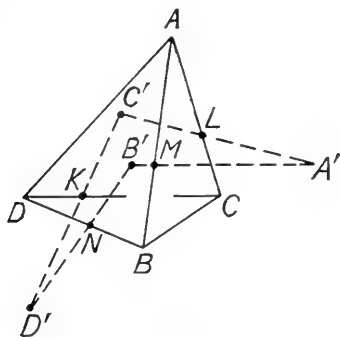
Fig. 2

meets AB, $C'D'$ meets CD, in points, L, N, M, K, respectively. The polars of the points A', B', C', with respect to the conic section of Σ in the plane $A'B'C'$, are the lines in which the planes DBC, DCA, DAB, respectively, meet the plane $A'B'C'$; the intersections of these with $B'C'$, $C'A'$, $A'B'$, respectively, thus lie on a line. That is, the lines LM, $B'C'$ meet on the plane DBC in a point, which is therefore on BC. Similarly, KN, $B'C'$ meet on BC; and in particular the joins BC and AD are conjugate. Similarly the lines LK, MN, $A'D'$ meet on AD, and the lines AA', BB', CC', DD'

meet in a point, E. The whole figure may be denoted by indices of configuration 15 ($\cdot$46) 20 (3$\cdot$3) 15 (64$\cdot$), which represent the figure of two tetrads in perspective, corresponding joins meeting in six points K, L, M, N, R, S lying on four lines forming a plane quadrilateral.

We may then state the result. If two pairs of opposite joins of four points in space are conjugate lines with respect to a quadric, the third pair also consists of two conjugate lines, and the poles of the planes containing triads of the four points are four points in perspective with the original four. The quadric ϕ (A, B, C, D) is in this case a degenerate point pair, with coincident points, determined by all the lines through the centre of perspective. A very
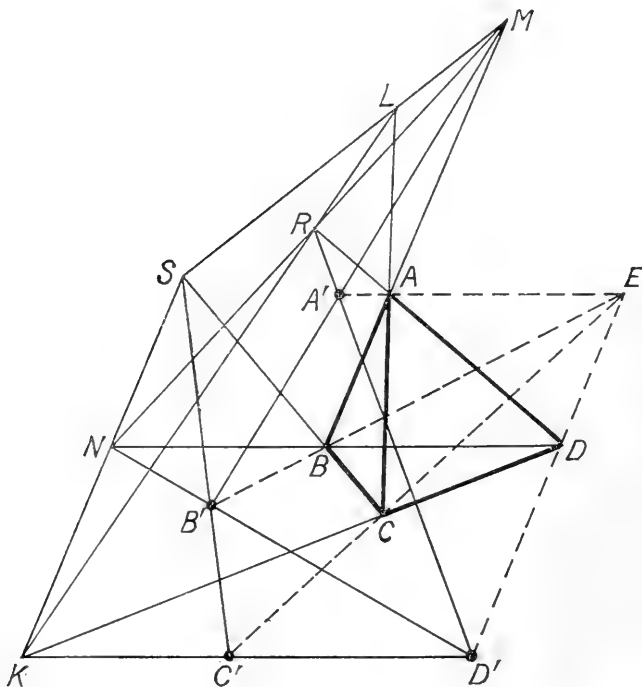
Fig. 3

particular case is that if two pairs of opposite edges of a tetrahedron be at right angles so is the third pair; such a tetrahedron has an orthocentre.

(b) If A, B, C be three points and A', B', C' three other points, not lying in the plane ABC, and such that the points, in which the polar lines of BC, CA, AB meet the plane $A'B'C'$, are in perspective with A', B', C', respectively, then the points in which the polar

lines of $B'C'$, $C'A'$, $A'B'$ meet the plane ABC are in perspective with A, B, C.

Two triads of points A, B, C; L, M, N, in one plane, are said to be in perspective when the lines AL, BM, CN meet in a point, and the points (BC, MN), (CA, NL), (AB, LM) lie in a line. Similarly, if α, β, γ and λ, μ, ν are two triads of planes all meeting in a point O, these triads are said to be in perspective when the lines $(\alpha\lambda)$, $(\beta\mu)$, $(\gamma\nu)$ lie in a plane through O, or the planes $(\beta\gamma, \mu\nu)$, $(\gamma\alpha, \nu\lambda)$, $(\alpha\beta, \lambda\mu)$ meet in a line. The section of two such triads of planes, by a plane not passing through O, is two triads of lines in perspective. For the present purpose, let O be the pole of the plane ABC. The polar lines of BC, CA, AB are three lines through O; thus, if α, β, γ be the planes joining these polar lines to A', B', C', respectively, then, by the hypothesis made, α, β, γ meet in a line
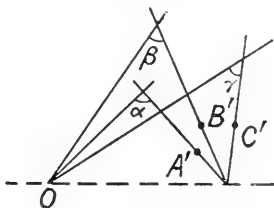
Fig. 4

through O. If the polar planes of A', B', C' meet the lines BC, CA, AB respectively in P, Q, R, these last points will be the poles of the planes joining A' to the polar line of BC, etc., that is, of the planes α, β, γ. Hence the points P, Q, R are in line. Thus the polar planes of A', B', C' meet the plane ABC in three lines which are in perspective with BC, CA, AB. The polar planes of B', C', however, intersect in the polar line of $B'C'$. The three polar lines of $B'C'$, $C'A'$, $A'B'$ thus meet the plane ABC in points which are in perspective with A, B, C.

Finally, if two points D, D' be such that the polar lines of lines DA, DB, DC, through D, be joined by planes to the point D' and the planes be in perspective with three given planes $D'B'C'$, $D'C'A'$, $D'A'B'$, through D', then the planes joining D to the polar lines of $D'A'$, $D'B'$, $D'C'$ are in perspective with the planes DBC, DCA, DAB.

(c) On a cubic curve in space, of which l, m are two chords, if A, B be two points such that the planes lA, mB are conjugate, with respect to a quadric, and also the planes lB, mA conjugate, then two points P, Q of the cubic such that the planes lP, mQ are conjugate are also such that lQ, mP are conjugate.

For, if a variable plane through l meet the curve in R, and the conjugate plane through m meet the curve in S, the range of points S is related to the range of points R; it will not then generally

happen that when R is at S, also S is at R; but if it do so happen for one corresponding pair of positions, A and B, of R and S, as we have supposed, then it will also happen for every pair, in particular for the positions P, Q, as we desired to prove. In this case the pairs of points R, S form an involution on the curve.

2. It is now possible to consider the construction, upon a quadric or upon a twisted cubic curve, of self polar tetrads, and of self conjugate pentads and hexads, with respect to the fundamental quadric envelope Σ.

First consider the construction of any self conjugate pentad. It is clear that a particular form of such a pentad consists of a self polar tetrad together with an arbitrary point E. Further, if A, B, C, D, E be any self conjugate pentad, the polar line of BC, being in the plane ADE by definition, meets AD, so that every pair of opposite joins of the four points A, B, C, D is a pair of conjugate lines; moreover, the pole of the plane ABC being in DE, and so on, E may be defined as the point of intersection of the lines AA', BB', CC', DD', using the notation of § 1. Conversely, if A, B, C, D be four points such that two, and in fact three, pairs of their opposite joins be pairs of conjugate lines, the point of meeting of the lines AA', BB', CC', DD' forms with A, B, C, D a self conjugate pentad. It is the case also that if A, B, C, D; A', B', C', D' be two tetrads in perspective from a point E, there is a quadric for which A', B', C', D' are the poles respectively of the planes BCD, CAD, ABD, ABC; $ABCDE$ then form a self conjugate pentad with respect to this quadric.

A particular form of self conjugate hexad obviously consists of a self conjugate pentad and an arbitrary point F. Conversely, if $ABCDEF$ be a self conjugate hexad such that the pole of the plane ABC lies in the line EF, say at D', then it can be shown that $ABCEF$ is a self conjugate pentad, and it then follows that $ABCD_1EF$ form a self conjugate hexad when D_1 is any point whatever. For, since EF contains the pole of ABC, the pole of EFC lies in ABC, and since $ABCDEF$ is a self conjugate hexad the pole of EFC lies in the opposite face ABD. It follows that AB contains the pole of EFC; similarly, BC, CA contain the poles of EFA, EFB respectively. Moreover, the pole of EBF lies in ACE since it lies in AC, and that of DBF in ACE for a self conjugate hexad, so that BF contains the pole of ACE. Similarly, the poles of CAF, BCF, ABF, BCE, ABE lie on the lines BE, AE, CE, AF, CF respectively. Then $ABCEF$ is a self conjugate pentad.

The construction of a general self conjugate hexad A, B, C, D, E, F will be shown to be such that four vertices A, B, C, D may be chosen arbitrarily and the fifth an arbitrary point of the quadric denoted by $\phi\,(A,\,B,\,C,\,D)$; the sixth is then determined. In this general

case it is assumed that no pair of opposite joins of A, B, C, D consists of conjugate lines with respect to Σ. Then as DEF contains D', the pole of ABC, EF meets DD'. Similarly it meets AA', BB', CC', generators of the first system of $\phi\,(A, B, C, D)$; it is thus a generator of the second system of $\phi\,(A, B, C, D)$. Thus given $ABCD$, construct $\phi\,(A, B, C, D)$, and thereon take any point E and from this draw the generator of $\phi\,(A, B, C, D)$ of the second system. A cubic can be drawn through A, B, C, D, E and meet the generators of the second system each in two points; such a cubic will lie entirely on $\phi\,(A, B, C, D)$. It will meet the generator of the second system through E in a further point F. It will be shown that $ABCDEF$ is a self conjugate hexad, and incidentally, that this is the only position for F, when A, B, C, D, E are given.

First consider the quadric $\phi\,(A, B, C, E)$ when E lies on $\phi\,(A, B, C, D)$; the quadric $\phi\,(A, B, C, E)$ will be shown to contain D, and the generator of the second system of $\phi\,(A, B, C, D)$, which passes through D'. Let A_1, B_1, C_1 be respectively the poles of the planes BCE, CAE, ABE. The three points D', A', A_1, being the poles of the planes BCA, BCD, BCE respectively, which planes meet in a line BC, are collinear, and any line through D' which meets AA' will also meet AA_1. The generator of $\phi\,(A, B, C, D)$ which passes through D' is a line meeting AA', BB', CC'; it thus meets AA_1, BB_1, CC_1; hence it is equally a generator of $\phi\,(A, B, C, E)$. To prove that the surface $\phi\,(A, B, C, E)$ contains D, note that the planes EAA', EBB', ECC' meet in a line, namely the generator through E which meets AA', BB', CC' of the surface $\phi\,(A, B, C, D)$. But $B'C'$, $C'A'$, $A'B'$ are the polar lines of DA, DB, DC respectively. Thus there are two triads of planes DBC, DCA, DAB and EBC, ECA, EAB through D, E respectively; the planes $EB'C'$, $EC'A'$, $EA'B'$ drawn from E to the polars of DA, DB, DC respectively are in perspective with the planes EBC, ECA, EAB. Therefore as a particular case of the proposition of § 1, the planes DB_1C_1, DC_1A_1, DA_1B_1 drawn from D to the polars of EA, EB, EC are in perspective with the planes DBC, DCA, DAB; that is, the planes DAA_1, DBB_1, DCC_1 meet in a line, so that a line can be drawn from D to meet AA_1, BB_1, CC_1, and D must be on the quadric $\phi\,(A, B, C, E)$.

The quadrics $\phi\,(A, B, C, D), \phi\,(A, B, C, E)$ have a generator through D' in common and therefore meet in a cubic curve in space meeting the generators of the system other than that containing DD' in two points. This cubic curve thus passes through A, B, C, D, E and meets the generator of the second system through E of $\phi\,(A, B, C, D)$ in a second point F, which is therefore also a point of the quadric $\phi\,(A, B, C, E)$, so that E and F are the points in which the line EF meets $\phi\,(A, B, C, E)$. The points A, B, C, D, E, F can now be shown to form a self conjugate hexad: for, as AA' meets EF, the point A' lies in the plane EFA, and the planes BCD, EFA are conjugate;

similarly, the planes CAD, EFB; ABD, EFC; and ABC, EFD, are pairs of conjugate planes. Moreover, since EF meets DD', it follows that FD is met by ED', which is a generator of $\phi(A, B, C, E)$, by definition of the quadric, so that FD contains three points of $\phi(A, B, C, E)$ and is therefore a generator of this quadric. Hence FD meets the lines AA_1, BB_1, CC_1. Thus A_1 is in the plane FDA so that the planes BCE, FDA are conjugate. By a similar argument the planes FBD, FCD are conjugate to the planes CAE and ABE respectively. The determination of F from E might moreover have been replaced by the determination of E from F, in which case the quadric $\phi(A, B, C, F)$ would have been considered, and since for E on $\phi(A, B, C, D)$ we have shown that D is on $\phi(A, B, C, E)$, so for F on $\phi(A, B, C, D)$, D must lie on $\phi(A, B, C, F)$, and further the generator of $\phi(A, B, C, D)$ through D' meeting AA', BB', CC' has been shown to lie on $\phi(A, B, C, E)$; it will therefore lie on $\phi(A, B, C, F)$ and the cubic common to $\phi(A, B, C, D)$, $\phi(A, B, C, F)$ passes through $ABCDF$ and, being on $\phi(A, B, C, D)$ and of the kind which meets the generators of the second system each twice, it is identified with the cubic curve common to $\phi(A, B, C, D)$, $\phi(A, B, C, E)$. Thence, as above, the planes EDA, EDB, EDC are respectively conjugate to the planes BCF, CAF, ABF. The ten conditions, which are to be satisfied if $ABCDEF$ is to be a self conjugate hexad, are now verified.

3. If a self polar tetrad of points, in regard to the fundamental quadric Σ, exists in a twisted cubic curve, other such tetrads can be constructed.

For, let E, F, G, H be such a tetrad and D an arbitrary point of the curve; let the polar plane of D, with respect to Σ, meet the curve in A, B, C. The planes joining EF and BC to G and H are such that not only is EFG conjugate to BCH, but also EFH is conjugate to BCG. Further, EFD is conjugate to BCA, whose pole is at D. Hence, by a remark (c) of § 1, the plane EFA is conjugate to BCD, so that the pole of BCD lies in the plane EFA; but since $EFGH$ is a self polar tetrad it follows similarly that the pole of BCD is in every one of the six planes joining A to the joins of $EFGH$. Thus A is the pole of BCD; similarly B is the pole of the plane CAD and C is the pole of ABD, so that $ABCD$ is a self polar tetrad.

4. If a self polar tetrad of points E, F, G, H exists on a cubic curve in space, a self conjugate hexad of points of the cubic can be found.

For, such a hexad is formed in fact by any two points of the curve together with the four given points; but such a hexad is not general in form since the quadric $\phi(E, F, G, H)$ is indeterminate. A more general self conjugate hexad can be formed by taking any

point D of the curve and by drawing from it the transversal meeting both the lines EF and GH, which is also the intersection of the planes DEF and DGH. Then take D' any point of this transversal and let the polar plane of D', with respect to Σ, meet the cubic in A, B, C. The points A, B, C, D, E, F can be shown to form a self conjugate hexad. For, as H is the pole of the plane EFG, the planes EFG, BCH are conjugate as in § 3 as G is the pole of EFH, and the planes EFH, BCG are also conjugate. The planes EFD, BCA are also conjugate since the pole D' of the latter plane lies on the plane EFD. Thus, by the remark (c) made in § 1, the planes EFA and BCD are conjugate. The pole A' of the plane BCD thus lies on the plane EFA, so that EF meets the line AA'. In a similar way, if B' and C' be the poles of the planes CAD and ABD respectively, the line EF meets the lines BB' and CC'. It appears that the quadric $\phi\,(A, B, C, D)$ does not degenerate and EF is, by what we have seen, a generator of this. Similarly, GH is also a generator of this quadric. The quadric thus contains eight points $ABCDEFGH$ of the cubic curve, and therefore contains the curve completely. The curve is therefore identified with that of the general theory, through $ABCDE$, lying on $\phi\,(A, B, C, D)$, and F may be determined from E as in that theory. Hence $ABCDEF$ is a self conjugate hexad, of points of the cubic curve.

5. If a self conjugate hexad of points exists on a cubic curve in space there exist in the cubic both self polar tetrads and other self conjugate hexads.

For, if A, B, C, D, E, F be a self conjugate hexad of points of the cubic curve it follows from the general theory that E and F must be on the quadric $\phi\,(A, B, C, D)$, and the line EF be a generator thereof; similarly, D and F must be on the quadric $\phi\,(A, B, C, E)$, the line DF being a generator of this. These quadrics, as before, intersect in a line, and in a cubic curve which passes through A, B, C, D, E, F, and this is thus identified with the given cubic curve. A further, arbitrary point, E', may be taken on this cubic, which therefore lies on $\phi\,(A, B, C, D)$, and the generator of the second system of $\phi\,(A, B, C, D)$, through E', meets the cubic again in a point F' such that $ABCDE'F'$ form a self conjugate hexad. By repetition of this process a self conjugate hexad can be found on the cubic for which five points may be chosen arbitrarily. In particular any two conjugate planes meet the cubic in points of a self conjugate hexad. From § 4, it follows that this last statement is also true when a self polar tetrad of points exists in the cubic curve.

Hence, in particular, if a self conjugate hexad exists in a cubic curve, a self polar tetrad of points of the curve can be found by taking any point D of the curve and the points in which the polar

plane of D meets the curve: for, let these points be A, B, C, then, if E, F be any two points of the curve, since the planes ABC, DEF are conjugate, the six points A, B, C, D, E, F form a self conjugate hexad, and the pole of the plane BCD lies in AEF for all positions of the points E, F so that the pole of BCD must be at A. Similarly, the poles of CAD, ABD are at B and C respectively and $ABCD$ form a self polar tetrad.

6. If a cubic curve be such that a self polar tetrad or a self conjugate hexad can be taken in it, so also can a self conjugate pentad.

For, let A, B, C be any three points of the cubic curve such that the pole D' of the plane ABC does not lie on the curve. Then a chord can be drawn, through D', meeting the curve in two points E and F. Since, by hypothesis, the curve is such that any pair of conjugate planes meet it in six points of a self conjugate hexad, it follows that any plane through EF will meet the curve again in a point G such that $ABCEFG$ is such a hexad. The pole of the plane BCE thus lies on AFG for any position of G, so that the pole of BCE must lie on AF. Similarly, the poles of BCF, CAE, CAF, ABE, ABF lie respectively on AE, BF, BE, CF, CE. Similarly, the poles of EFA, EFB, EFC lie respectively on BC, CA, AB, and by construction the pole of ABC lies on EF. Thus A, B, C, E, F form a self conjugate pentad.

Conversely, if $ABCEF$ be a self conjugate pentad of points on a cubic, any point D, of the cubic, which is taken so that no one of the line pairs DA, BC; DB, CA; DC, AB consists of conjugate lines, gives a self conjugate hexad $ABCDEF$ for which $\phi(A, B, C, D)$ does not degenerate. From the existence of this self conjugate hexad can be deduced the existence of other self conjugate hexads and self polar tetrads of points of the cubic.

7. When a cubic curve has the property that upon it can be found either a self polar tetrad or self conjugate pentad or self conjugate hexad it may be said to be outpolar to Σ, the three conditions being equivalent. The tetrahedra, in a cubic curve γ, which are self polar with respect to the quadric Σ considered as an envelope, are such that, considered reciprocally, their faces belong to the cubic developable which is the reciprocal of the given cubic curve with respect to Σ, which is self dual and is now considered as a quadric locus. Thus the cubic curve γ is outpolar to the quadric Σ considered as an envelope, while the cubic developable is inpolar to the same quadric considered as a point locus. The cubic curve γ which contains tetrahedra self polar with respect to a given quadric is then said to be outpolar to the cubic developable, the reciprocal of γ with respect to the quadric.

It is apparent that self polar tetrads are determined in a cubic, in which they can be found, by the choice of one point of the curve;

of a self conjugate pentad, three points may be taken arbitrarily upon the curve; and of a self conjugate hexad, five points may be taken arbitrarily upon the curve: the sixth is then determined.

8. Similar properties of a quadric S with respect to the fundamental quadric Σ can now be established, showing that if a self polar tetrad, or a self conjugate pentad, or a self conjugate hexad, can be inscribed therein, then so can an infinite number of such.

When a self polar tetrad of points of the quadric S exists, a twisted cubic can be drawn to lie on the quadric and pass through these four points and any other point of the quadric. On this cubic curve can be taken other self polar tetrads, of which one point lying on the curve is arbitrary, or self conjugate pentads of which three points lying on the curve are arbitrary, or self conjugate hexads of which five points lying on the curve are arbitrary. It does not, however, follow that these points, so obtained, are the most general sets which can be found in the quadric S. For instance, it can be shown that another self polar tetrad of points of the quadric can be found (when it contains one set A, B, C, D), such that two of the points are conjugate to one another with respect to Σ, but it is not the case in general that a cubic curve lying on the quadric can be drawn through $ABCD$ and these two arbitrary, conjugate points of the quadric.

If one self conjugate hexad can be taken on a quadric S, of a non-degenerate kind, then, any four points A_1, B_1, C_1, D_1 being taken on the quadric, of such generality that $\phi(A_1, B_1, C_1, D_1)$ exists, a self conjugate hexad $A_1B_1C_1D_1E_1F_1$ exists on the quadric, such that E_1 is any point of the curve of intersection of S and $\phi(A_1, B_1, C_1, D_1)$, and F_1 is the point in which S is met again by the generator of $\phi(A_1, B_1, C_1, D_1)$, of the second system through E_1. It appears that F_1 lies on the cubic curve, lying on S determined by $A_1B_1C_1D_1E_1$, and of the kind which meets each generator of the second system twice; F_1 is thus the sixth intersection on S of the curves $(S\phi)$ and the cubic in question.

Let $ABCDEF$ be a self conjugate hexad existing on S, such that $\phi(A, B, C, D)$ exists, and let E' be any point of the curve of intersection of S and ϕ, and F' a vertex of a self conjugate hexad, of which $ABCDE'$ are five vertices, so that $E'F'$ is a generator of the second system on $\phi(A, B, C, D)$. A twisted cubic γ may be described through A, B, C, D, E, F: it lies on $\phi(A, B, C, D)$ and meets each generator of the second system twice: it therefore has $E'F'$ as a chord. A second cubic γ' may be described through A, B, C, D, E', F': it lies on $\phi(A, B, C, D)$, and meets each generator of the second system twice: it therefore has EF as a chord. A pencil of quadrics can be described to contain γ and its chord $E'F'$, such a pencil being determined by seven arbitrarily chosen points of the

cubic and one arbitrarily chosen point of the chord. The quadric ϕ must belong to the pencil; let R be another quadric of the pencil. Similarly, let R' be a quadric other than ϕ which can be drawn to contain γ' and the chord EF. Then A, B, C, D, E, F, E', F' are the points of intersection of the three quadrics ϕ, R, R'; these quadrics define a net, $\phi + \lambda R + \lambda' R' = 0$, for, the quadric R does not contain $E'F'$ or it would have a composite curve of order five in common with ϕ, so that R and R' are not the same, and the equation to the net does not become an identity, or R, like ϕ and R', would contain EF. But a net may also be defined by seven general points, and one such net is defined by $ABCDEFE'$ on S, and every quadric through these points belongs to the net. The quadrics ϕ, R, R' obviously belong to it, and S therefore contains their eighth point of intersection. Hence F' lies on S, and $ABCDE'F'$ is another self conjugate hexad in S, where E' is any point of the curve of intersection of ϕ and S, and $E'F'$ is a generator of the second system on ϕ (A, B, C, D); such a generator can be drawn through any point of ϕ, and the points $E'F'$ will then be the points where this generator meets the quadric S. In particular, there is a self conjugate hexad in which $E'F'$ contains the point D', lying on ϕ, which is the pole of the plane ABC, and in this case it follows from § 1 that $ABCE'F'$ form a self conjugate pentad, the pole of the plane containing any three of these five points lying on the line joining the other two; then if D_1 be any point of the quadric S, $ABCD_1E'F'$ form a self conjugate hexad. In the general case the quadric ϕ (A, B, C, D_1) exists and the process may be repeated to obtain a further self conjugate hexad $ABCD_1E_1F_1$ of which E_1 is any point of S lying on the quadric ϕ (A, B, C, D_1). Thus starting from the original hexad A, B, C, D, E, F in S a further self conjugate hexad is obtained, $ABCD_1E_1F_1$, wherein D_1 is any point of S, and E_1 is any point of the curve of intersection of S and ϕ (A, B, C, D_1). By repetition of the process, we may now find another self conjugate hexad, in S, of which a further point, beside D_1, has an arbitrary position on the quadric S; and so on. It is thus possible to obtain a final hexad A_2, B_2, C_2, D_2, E_2, F_2, of which four of the points are quite arbitrarily chosen upon S, and the fifth is any arbitrary point of the intersection of S with a certain ϕ quadric. If the final hexad A_2, B_2, C_2, D_2, E_2, F_2 be chosen so that E_2F_2 contains the pole of the plane $A_2B_2C_2$, then A_2, B_2, C_2, E_2, F_2 is a self conjugate pentad of which three points A_2, B_2, C_2 are arbitrary points of S. If, more particularly, A_2 and B_2 be conjugate points, the polar line of A_2B_2, say E_2F_2, contains the pole of the plane $A_2B_2C_2$ whatever be the position of C_2; choosing D_2 so that ϕ (A_2, B_2, C_2, D_2) has E_2F_2 for a generator, the points $A_2B_2E_2F_2$ will form a self conjugate pentad with C_2 for any position of C_2. Hence $A_2B_2E_2F_2$ is a self polar tetrad in S,

the two points, A_2, B_2, being arbitrarily chosen conjugate points with respect to Σ.

9. Finally, if two quadrics S, S' be such that one self conjugate hexad can be taken on each, then any quadric through the intersection of S and S' has the same property.

For, let four arbitrary points A, B, C, D be taken on the curve of intersection of S and S', and let E be one of the four remaining points of intersection of ϕ (A, B, C, D) with this curve. Then in each of the quadrics the self conjugate hexads $ABCDEF_1$ and $ABCDEF_2$ can be constructed, where EF_1F_2 is a generator of ϕ (A, B, C, D), and F_1, F_2 are on S_1, S_2 respectively. Then the cubic curve on ϕ (A, B, C, D), defined by A, B, C, D, E, and meeting each generator of the second system twice, passes through F_1 and F_2, which therefore coincide in a point F, and F is on both the given quadrics. Thus any quadric through the curve of intersection of the given quadrics, which are outpolar to Σ, contains a self conjugate hexad $ABCDEF$ and is therefore, itself, outpolar with respect to Σ.

Partition Functions for temperature radiation and the internal energy of a crystalline solid. By C. G. DARWIN, F.R.S. and R. H. FOWLER.

§ 1. *Introduction.* In a series of papers which have appeared or, it is hoped, will appear shortly in the *Philosophical Magazine** the general distribution laws of an assembly of systems have been discussed at length by a special method. The essential feature of this is the calculation of *average* values with the help of certain functions which we call *partition functions*, equivalent to the Zustandsumme of Planck. In order, however, not to extend these papers unduly, the discussion of certain points was omitted to which we return here. In particular, we have used the facts that we can construct partition functions for the temperature radiation in an enclosure and also for the internal energy of a crystal. It is the principal object of this paper to construct these functions (§§ 2–4). In addition (§ 5) we make some further remarks about the construction of partition functions for systems obeying the laws of classical mechanics, functions which have to be obtained by a limiting process. These remarks are mainly mathematical in nature.

For every type of system in the assembly—free atom, Planck vibrator, rotating molecule, etc.—there exists a partition function which is typical of the system, and depends on the motion of that system alone, so long, that is, as it is not too frequently interfered with†. This function is defined originally for quantized systems, and can be extended by a limiting process to systems obeying classical laws. If the possible states of the system, according to the rules of the quantum theory, have energies ϵ_0, ϵ_1, ... and weights‡ p_0, p_1, ..., the partition function is defined as a function of a complex variable z by the equation

$$f(z) = \Sigma_r p_r z^{\epsilon_r}, \qquad \qquad \ldots \ldots (1\cdot1)$$

summed over all permissible states. The partition function for a group of N such systems is $[f(z)]^N$. It is then shown that for every group of systems in an assembly in statistical equilibrium the parameter z must have one and the same real value ϑ, which is the

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* These papers will be referred to as first, second, ... paper, etc. For the first, see *Phil. Mag.* Sept. 1922.

† The assumption of limited interference is fundamental in all statistical calculations by whatever method (first paper, § 2).

‡ The rules for assigning weights in their most general form are due in the main to Bohr and Ehrenfert. They are summarized in our first paper, § 2, and again in the second and third. The only difficulties occur for *degenerate* quantized systems.

temperature of the assembly measured on a special scale. It is connected to the absolute temperature T of the second law of Thermodynamics by the relation

$$\vartheta = e^{-1/kT}, \qquad \ldots\ldots(1\cdot11)$$

a relation which is established by perfectly general arguments*. If any system is capable of two or more different types of motion which are dynamically independent—for example, the translation and rotation of a free molecule—the partition function for the combined motion is the product of the partition functions for the separate motions and may be calculated as such.

§ 2. *The partition function for temperature radiation.* The energy of radiation in any enclosure may be thought of as the energy of a single system—the aether. The laws controlling the motion of the aether are the laws of electrodynamics, and the aether is, of course, a system described by an infinite number of coordinates. The same general principles which allow us to quantize the motion of a simple system allow us also to quantize the motion of the aether, if we can find suitable coordinates with which to describe its motion. We can suppose in the usual way† that the radiation is split up into trains of harmonic waves, each of which corresponds to one coordinate. Each such coordinate is a *normal coordinate,* and the motion in each is simple harmonic with a definite frequency ν; owing to the linear form of the general equations it is independent of all the others. If now we apply the general principles of the quantum theory to such a system described by such coordinates, it is clear that each of the coordinates must be treated exactly like a single Planck line vibrator of appropriate frequency. On these assumptions we can construct the partition function, and deduce Planck's law of temperature radiation purely as a theorem in statistical mechanics, without appeal to any other fundamental principles or to the mechanism of the processes of radiation and absorption.

The partition function for any harmonic vibrator of one degree of freedom and frequency ν has been shown‡ to be

$$(1 - z^{h\nu})^{-1}.$$

For on the general principles of the quantum theory the energy in each coordinate can only have the sequence of values $rh\nu$ ($r = 0, 1, \ldots$) and each such state has the same weight which is unity on the usual convention. In general, the aether will be a system which is heavily degenerate and there may be n coordinates each with corresponding frequency ν. But as we have shown§, the weights will then be such

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* See second paper, § 7.
† Jeans, *Dynamical Theory of Gases,* Ch. XVI.
‡ First paper, § 5.
§ First paper, § 10. The result required here is an obvious generalization.

that the partition function for all the coordinates with frequency ν is $(1 - z^{h\nu})^{-n}$. As we have to multiply the partition functions for the various independent coordinates, it therefore does not matter here if we ignore entirely all questions of degeneracy and treat the aether as a non-degenerate system. If ν_1, ν_2, ... are the possible aethereal frequencies and n_1, n_2, ... the numbers of corresponding coordinates, the partition function $R(z)$ for the whole of the radiation in the assembly is given by

$$R(z) = (1 - z^{h\nu_1})^{-n_1} (1 - z^{h\nu_2})^{-n_2} ...,$$
$$= \exp\{- \Sigma_r n_r \log(1 - z^{h\nu_r})\}. \qquad ......(2\cdot1)$$

We assume, of course, that this series or product converges, which we shall find to be the case since $|z| < 1$.

To obtain an intelligible form of $(2\cdot1)$ we must perform the usual operation of counting the various coordinates. For the group of frequencies between ν and $\nu + d\nu$ we have approximately*

$$n = \frac{8\pi V}{c^3} \nu^2 d\nu, \qquad ......(2\cdot11)$$

where V is the volume of the enclosure and c the velocity of light. Therefore

$$R(z) = \exp\left\{- \Sigma \frac{8\pi V}{c^3} \nu^2 d\nu \log(1 - z^{h\nu})\right\}.$$

On proceeding to the limit $d\nu \to 0$ we obtain (formally)

$$R(z) = \exp\left\{- \frac{8\pi V}{c^3} \int_0^\infty \nu^2 \log(1 - z^{h\nu}) d\nu\right\}. \quad ......(2\cdot2)$$

Now $\qquad - \log(1 - z^{h\nu}) = z^{h\nu} + \tfrac{1}{2}z^{2h\nu} + ... + \frac{1}{n}z^{nh\nu} + ...,$

and this series can be integrated term by term from 0 to ∞ since $|z| < 1$. We find

$$\int_0^\infty \nu^2 z^{nh\nu} d\nu = \frac{2}{n^3 h^3 (\log 1/z)^3}. \qquad ......(2\cdot21)$$

It follows that the steps of the preceding formal argument are easily justified and thus

$$R(z) = \exp\left\{\frac{16\pi V}{c^3 h^3 (\log 1/z)^3} \Sigma \frac{1}{n^4}\right\}, \quad \left(\Sigma \frac{1}{n^4} = \frac{\pi^4}{90}\right),$$
$$R(z) = \exp\left\{\frac{8\pi^5 V}{45 c^3 h^3 (\log 1/z)^3}\right\}. \qquad ......(2\cdot3)$$

With the help of $(2\cdot3)$ we can at once deduce the laws of temperature radiation. Consider for this purpose any assembly containing

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* Jeans, *loc. cit.*

radiation and N material systems of any type for which the partition function $f(z)$ is known. Then by the general arguments of the first paper the number C of weighted complexions representing the assembly with total energy E is

$$C = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{E+1}} R(z) [f(z)]^N, \qquad \ldots\ldots(2\cdot4)$$

where γ is a circle with its centre at the origin and radius less than unity. Similarly if $\bar{E}_R$ is the average radiant energy in the assembly,

$$C\bar{E}_R = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{E+1}} \left\{ z\frac{\partial}{\partial z} R(z) \right\} [f(z)]^N. \quad \ldots\ldots(2\cdot41)$$

It follows also by the arguments of the same paper that

$$\bar{E}_R = \Im \frac{\partial}{\partial \Im} \log R(\Im), \quad (\Im = e^{-1/kT}). \quad \ldots\ldots(2\cdot42)$$

Thus
$$\bar{E}_R = \frac{8\pi^5 V}{15c^3h^3 (\log 1/\Im)^4} = \frac{8}{15} \frac{\pi^5 k^4}{c^3 h^3} VT^4. \quad \ldots\ldots(2\cdot5)$$

This is the Stephan-Boltzmann law of total radiation with the usual value

$$\sigma = \frac{8}{15} \frac{\pi^5 k^4}{c^3 h^3} \qquad \ldots\ldots(2\cdot51)$$

for Stephan's constant.

To find the energy associated with a particular range of frequencies we take $R(z)$ in the form $(2\cdot1)$, and write

$$R(z) = R_1(z) R_2(z),$$

where
$$R_1(z) = \exp\left\{ -\frac{8\pi V}{c^3} \nu^2 d\nu \log(1 - z^{h\nu}) \right\}.$$

Then by the same arguments

$$C\bar{E}_{R_1} = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{E+1}} \left\{ z\frac{\partial}{\partial z} R_1(z) \right\} R_2(z) [f(z)]^N,$$

$$\cdots \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{E+1}} \left\{ z\frac{\partial}{\partial z} \log R_1(z) \right\} R(z) [f(z)]^N.$$

Thus
$$\bar{E}_{R_1} = \Im \frac{\partial}{\partial \Im} \log R_1(\Im).$$

$$= \frac{8\pi h V}{c^3} \frac{\nu^3 d\nu}{\Im^{-h\nu} - 1}. \qquad \ldots\ldots(2\cdot6)$$

With the usual notation $E_\nu d\nu$ for the energy of radiation per unit volume of frequency between ν and $\nu + d\nu$ we find

$$E_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}. \qquad \ldots\ldots(2\cdot61)$$

which is Planck's law.

§ 3. *A comparison of various deductions of the laws of temperature radiation.* The laws of temperature radiation, equations (2·5), (2·61), are here deduced by a direct statistical argument from the general principles of the quantum theory extended to systems of an infinite number of degrees of freedom. The particular point of this demonstration is that we obtain these laws without any appeal to the mechanism of exchange of energy—that is of radiation and absorption—such as forms the basis of Planck's deduction, or the far more satisfactory deduction due to Einstein*. There is further no appeal to the laws of Thermodynamics except in so far as such an appeal is essential to the definition of T. If the laws of temperature radiation are rightly to be regarded as laws of statistical equilibrium between matter and the aether, this may be claimed for a logical advantage.

Planck's deduction in its usual forms is based on a mechanism of exchange of energy according to the classical laws of electrodynamics; that is to say, the relation between the mean energy of a vibrating electron of frequency ν and the radiant energy of frequency between ν and $\nu + d\nu$ in equilibrium with it is calculated on this assumption. Einstein's deduction is based on Wien's law—that is on the laws of Thermodynamics—and on certain very simple assumptions as to the *a priori* probabilities of the emission or absorption of radiation corresponding to the change of the material system from one possible stationary state to another. The great practical advantage of Einstein's method is, however, the insight which it gives into this mechanism.

It is interesting further to note that the present deduction of the Stephan-Boltzmann law makes no assumption about the existence or magnitude of radiation pressure. That there must be a radiation pressure is of course a consequence of purely thermodynamical arguments. That its magnitude must be given by the usual formula

$$pV = \tfrac{1}{3}\bar{E}_R \qquad \qquad \ldots\ldots(3\cdot1)$$

follows from the Stephan-Boltzmann law, which is here deduced as a theorem in statistical mechanics.

It is also possible to deduce at once the entropy of the temperature radiation from the general formulae of our second paper. Ignoring the entropy-constant, any system (here the radiation) with partition function $R(\vartheta)$ contributes to the entropy

$$S = k\left\{\log R(\vartheta) + \log 1/\vartheta \cdot \vartheta\, \frac{\partial}{\partial \vartheta}\log R(\vartheta)\right\},$$

$$= \frac{32\pi^5 k^4}{45 c^3 h^3}\, VT^3, \qquad \qquad \ldots\ldots(3\cdot2)$$

$$= \tfrac{4}{3}\bar{E}_R/T, \qquad \qquad \ldots\ldots(3\cdot21)$$

in agreement with the usual formulae.

* *Phys. Zeit.* Vol. 18, p. 121 (1917).

§ 4. *The partition function for a crystal.* In a similar way we can construct the partition function for any crystal, to the same approximation as the usual expression for its internal energy. It is hardly more than a matter of a direct transcription of the argument. We shall follow the exposition of Born*. A crystal may be supposed to be built up of N congruent cells each of which contains s atoms, ions, or electrons†. The internal motion of the crystal can, Born shows, be described by $3s$ sets of normal coordinates, and the frequencies of each set ν_j ($j = 1, \ldots, 3s$) N in number are distributed uniformly through a certain phase space of 3 dimensions ϕ, χ, ψ ($-\pi \leqslant \phi$, χ, $\psi \leqslant \pi$). The three variables represent the variations of ν_j with the wave length and direction of propagation of the corresponding wave in the crystal. This analysis into normal coordinates is of course an approximation valid only so long as the internal motions are on the whole not too violent—that is, so long as the crystal is not too hot.

Any normal coordinate is a simple harmonic vibrator with partition function $(1 - z^{h\nu})^{-1}$. As in the radiation problem and for the same reason we can ignore the fact that frequencies may coincide. In any one of the $3s$ sets the number of frequencies ν_j (ϕ, ψ, χ) in an element of the phase space is

$$\frac{N}{(2\pi)^3} \, d\phi \, d\psi \, d\chi.$$

Hence the partition function for the whole set, as in § 2, is

$$\exp\left\{-\frac{N}{(2\pi)^3} \iiint_{-\pi}^{+\pi} \log\left(1 - z^{h\nu_j}\right) d\phi \, d\psi \, d\chi\right\}.$$

For the whole crystal the partition function, $K(z)$, has the form

$$K(z) = \exp\left\{-\frac{N}{(2\pi)^3} \sum_{j=1}^{3s} \iiint_{-\pi}^{+\pi} \log\left(1 - z^{h\nu_j}\right) d\phi \, d\psi \, d\chi\right\}. \quad (4\cdot1)$$

Formula ($4\cdot1$) is not of much use as it stands, and Born approximates to it. He shows that the $3s$ sets of frequencies can be divided into two groups for which approximately

$$\nu_j = c_j/\lambda + \ldots, \qquad (j = 1, 2, 3), \qquad \ldots\ldots(4\cdot21)$$

$$\nu_j = \nu_j{}^0 + c_j/\lambda + \ldots, \quad (j = 4, 5, \ldots 3s), \quad \ldots\ldots(4\cdot22)$$

where λ is the wave length of the corresponding wave in the crystal, and the c_j are definite functions‡ of the direction of the wave. We

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* *Der Dynamik der Kristallgitter*, Ch. v.

† We are not here including all the electrons in every atom. But to obtain full generality it is necessary to allow for some of the atoms being ionized, and possibly for certain electrons having an independent existence in the lattice structure.

‡ Of these c_1, c_2, c_3 are the velocities of sound in the crystal.

therefore divide up the partition function into two parts K_1 and K_2 such that

$$\log K_1 = \frac{-N}{(2\pi)^3} \sum_{j=1}^{3} \int\int\int \log{(1 - z^{h\nu_j})} \, d\phi \, d\chi \, d\psi, \quad \dots (4\cdot 23)$$

$$\log K_2 = \frac{-N}{(2\pi)^3} \sum_{j=4}^{3s} \int\int\int \log{(1 - z^{h\nu_j})} \, d\phi \, d\chi \, d\psi. \quad \dots (4\cdot 24)$$

The calculation of K_1 is closely analogous to Debye's calculation of the whole energy content. Born shows that if we change to polar coordinates for the phase-space, then

$$d\phi \, d\psi \, d\chi = \tau^2 d\tau \, d\omega, \quad 1/\lambda = \tau/2\pi\delta,$$

where δ^3 is the volume of the elementary crystal cell, and the upper limit of τ, τ_0, is a function of direction. It is sufficiently accurate, however, to give τ_0 a mean value such that the volume of the phase-space is unaltered; thus

$$(2\pi)^3 = \tfrac{4}{3}\pi\tau_0{}^3, \quad \tau_0{}^3 = 6\pi^2.$$

We thus find, with sufficient accuracy, that

$$\log K_1 = \frac{-N}{(2\pi)^3} \sum_{j=1}^{3} \int\int d\omega \int_0^{\tau_0} \tau^2 d\tau \log{(1 - z^{hc_j\tau/2\pi\delta})},$$

and by an obvious transformation that

$$\log K_1 = \frac{-V}{h^3 (\log 1/z)^3} \sum_{j=1}^{3} \int\int \frac{d\omega}{c_j{}^3} \int_0^{x_j^0} x^2 \log{(1 - e^{-x})} \, dx, \quad (4\cdot 3)$$

where V is the volume of the whole crystal and

$$x_j{}^0 = hc_j \log 1/z \left(\frac{3N}{4\pi V}\right)^{\frac{1}{3}}. \qquad \dots\dots(4\cdot 31)$$

The c_j are functions of direction. An approximation to $(4\cdot 3)$, undoubtedly valid when the temperature is not too high (z small), is obtained by replacing c_j in $(4\cdot 31)$ by a mean value $\bar{c}_j$ defined by

$$\frac{4\pi}{(\bar{c}_j)^3} = \int\int \frac{d\omega}{c_j{}^3}.$$

The upper limit of the x-integration in $(4\cdot 3)$ is then

$$h\bar{c}_j \log 1/z \left(\frac{3N}{4\pi V}\right)^{\frac{1}{3}}, \text{ or } k\Theta_j \log 1/z,$$

and after reduction

$$\log K_1 = \frac{-3N}{(\log 1/z)^3} \sum_{j=1}^{3} \frac{1}{k^3\Theta_j{}^3} \int_0^{k\Theta_j \log 1/z} x^2 \log{(1 - e^{-x})} \, dx. \quad \dots (4\cdot 4)$$

Equation $(4\cdot 4)$ gives the first part of the partition function.

It is easy to show that $\Im \dfrac{\partial}{\partial \Im} \log K_1 (\Im)$, which must represent the mean energy $\bar{E}_1$ associated with these frequencies, agrees with Born's directly calculated value. Since $\log 1/\Im = 1/kT$, we obtain for $\log K_1$ as a function of T

$$\log K_1 = -3N \sum_{j=1}^{3} \frac{T^3}{\Theta_j{}^3} \int_0^{\Theta_j/T} x^2 \log (1 - e^{-x})\, dx. \quad \text{...(4.41)}$$

If we ignore the differences between Θ_1, Θ_2, Θ_3, we obtain the same formula as we should get for the partition function by a direct calculation by Debye's method.

The rest of the partition function $\log K_2$ is easily dealt with. It is sufficiently accurate to ignore the variations of ν_j ($j = 4, 5, \ldots 3s$). Thus

$$\log K (z) = \frac{-3N}{(\log 1/z)^3} \sum_{j=1}^{3} \frac{1}{k^3 \Theta_j{}^3} \int_0^{k\Theta_j \log 1/z} x^2 \log (1 - e^{-x})\, dx$$
$$- N \sum_{j=4}^{3s} \log (1 - z^{h\nu_j{}^0}). \quad \text{......(4.5)}$$

This is the complete formula, but the last set of frequencies can again be grouped, according to Born, into two classes; $3\,(n-1)$ infra-red frequencies, not necessarily all different, and $3\,(s-n)$ ultra-violet, where n is the number of atoms (or ionized atoms) in the elementary cell and $s - n$ the number of separated electrons. We can in all practical applications ignore the electron ultra-violet frequencies and so find in all in (4.5) just $3n$ terms, where nN is the total number of atoms in the crystal. Unless in any investigation we systematically take account of the degrees of freedom of the electrons in the atoms throughout, we must systematically ignore them throughout, and may correctly ignore them here.

Equation (4.5) leads at once to the usual general formulae for the entropy, internal energy, and specific heat of a crystalline solid. Since we can construct the necessary partition functions we can suppose such crystals to form part of any of the assemblies which are discussed by the general method of our papers. This fact is made use of in connection with the vapour-pressure equation discussed in the third paper and with Nernst's Heat Theorem in the fourth.

§ 5. *Partition functions obtained by limiting processes.* It is desirable to examine somewhat more closely the validity of the use of partition functions in our general method when they are obtained by limiting processes. Such a process first occurs in § 12 of our first paper in discussing the free atoms and molecules of a perfect gas. It occurs again for systems of Planck vibrators with incommensurable frequencies and in similar cases, and also here in a somewhat

different form in obtaining the radiation and crystal partition functions.

As explained at less length in the first paper the method of procedure is in all cases as follows. We form a sequence of artificial assemblies, each of which can be exactly handled by the usual methods—that is to say, each assembly of the sequence has all the characteristics of an assembly of quantized systems only, all of whose permissible energies can be expressed as multiples of some common unit of energy. Our method can in the first instance treat such assemblies only. We choose the members of the sequence in such a way that the conditions of the artificial assemblies have as a limit the actual conditions of the real assembly. It is easy to verify that in any practical case there is no difficulty in constructing such a sequence, which can be done in infinitely many ways. We must then *prove* that the distribution laws of the sequence of artificial assemblies have a unique limit as the conditions tend to the conditions of the real assembly. We can then *assume* that this limit represents the true distribution laws of the real assembly. It must not, however, be thought that this assumption is in any way characteristic of the present method. It is probably implicit in all such discussions*. It must be justified if at all by appeal to a sort of "natural principle of uniform convergence," which underlies all discussions of natural phenomena. To obtain, however, the full benefit of the present method rather more is required than is set forth above. It is convenient not only to show that the artificial distribution laws and partition functions have a unique limit, but further that the limit of the artificial partition functions may be used throughout the discussion just as if it were an ordinary partition function, and that the same distribution laws will be obtained.

These points can best be illuminated by a more complete discussion of the partition function for free atoms, which is the most important case. The 6-dimensional phase-space of a free atom is described by coordinates $q_1, \ldots, p_3$, the three rectangular position-coordinates and their conjugated momenta. We suppose that there are P atoms of mass m and small size in the assembly, and form an artificial assembly by dividing up the phase-space into small cells $(1, 2, \ldots, t, \ldots)$ of extensions $(dq_1 \ldots dp_3)_t$. The weight of the tth cell is

$$\delta_t = (dq_1 \ldots dp_3)_t / h^3,$$

and the constant associated energy is ζ_t. Then

$$\zeta_t = \frac{1}{2m} (p_1{}^2 + p_2{}^2 + p_3{}^2)_t + Q_t; \qquad \ldots\ldots(5\cdot1)$$

* See for instance Jeans' discussions of Maxwell's distribution law. Any method which proceeds by dividing up the phase-space of a molecule into cells of finite extension, with which a constant energy is associated, makes an appeal to this principle.

Q is the potential energy of the atom in the field of any external forces, including those representing the walls, and the p's and Q are evaluated at some point in the tth cell, selected so that all the ζ_t are commensurable with each other and with any other energies that occur in the assembly.

In ordinary cases, where no limiting process is involved, it is convenient to choose a unit of energy to fit the assembly so that all the energies including the ζ_t are integers without a common factor. Here this would mean a continual change of the unit, which would obscure essential features of the limiting process. We therefore fix the unit of energy once for all, and can then assume that for all the artificial assemblies the ζ_t are chosen so that $\zeta_t = \xi_t/\varpi$, where ξ_t and ϖ are integers, and the ϖ's change from assembly to assembly. If ϖ is the smallest possible such integer, it is then unique. As the limit of the sequence is approached, $\varpi \to \infty$.

The partition function for the atoms in the artificial assembly is

$$h\,(z) = \Sigma_t \delta_t z^{\zeta_t}\,; \qquad \ldots\ldots(5{\cdot}2)$$

to develop the usual arguments we require the coefficient of z^E in expressions such as

$$[h\,(z)]^P\,[f\,(z)]^M,$$

where $[f(z)]^M$ is the partition function for the rest of the assembly. These are the same as the coefficients of $x^{E\varpi}$ in

$$[h\,(x)]^P\,[f\,(x)]^M,$$

etc., where $x = z^{1/\varpi}$, $h\,(x) = \Sigma_t\delta_t x^{\xi_t}$, etc. To obtain these coefficients we can use the ordinary process. Thus

$$C = \frac{1}{2\pi i}\int_\gamma \frac{dx}{x^{E\varpi+1}}\,[h\,(x)]^P\,[f\,(x)]^M. \qquad \ldots\ldots(5{\cdot}21)$$

On changing back to the variable z,

$$C = \frac{1}{2\pi i\varpi}\int_{\varpi\gamma} \frac{dz}{z^{E+1}}\,[h\,(z)]^P\,[f\,(z)]^M. \qquad \ldots\ldots(5{\cdot}22)$$

The contour $\varpi\gamma$ means that the integral must now be taken ϖ times round the circle γ. When we come to evaluate C and similar integrals by the usual method, we find that only one part of the multiple contour is relevant, namely, that part of the one circle on which z passes through $\mathfrak{S}$, the saddle point on the positive real axis, *with argument zero* (and not $2n\pi i$). We can therefore drop the complete contour and consider only a single description of the circle γ, with the proviso above. Further, since only the ratio of two integrals such as (5·22) is concerned in any distribution law,

we can also drop the factor* $1/\varpi$, and use throughout the discussion integrals such as

$$C = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{E+1}} [h(z)]^P [f(z)]^M, \qquad \ldots\ldots(5\cdot23)$$

in which the circle γ must be chosen to go through $\Im$ the saddle point on the positive real axis and z takes the argument zero there.

We are now free of complications due to changing units of energy, or the need for strictly one-valued integrals in (5·23), and can discuss the limiting process. The partition function for an atom in the artificial assembly, written in full, is

$$h(z) = \frac{1}{h^3} \Sigma_t \exp\left[-(\log 1/z)\left\{\frac{1}{2m}(p_1{}^2 + p_2{}^2 + p_3{}^2) + Q_t\right\}\right](dq_1 \ldots dp_3)_t.$$
$$\ldots\ldots(5\cdot3)$$

This series can only converge and have a meaning when $|z| < 1$. If the assembly is of unlimited extent (a planetary atmosphere for example), Q_t must tend to infinity sufficiently fast in the distant parts for the convergence of Σ_t. If the assembly is enclosed in a volume V, the walls are represented by intense local fields of force in which Q_t passes rapidly from zero inside the enclosure to infinity at the wall. All terms of Σ_t then vanish except those corresponding to cells inside V, and Σ_t will certainly converge as required for $|z| < 1$. Now the distribution laws for any artificial assembly are determined at once in terms of $h(\Im)$ and its differential coefficients (the first two suffice in general) by means of integrals such as (5·23). In order to make the conditions of the artificial assembly have as a limit those of the real assembly we must choose the sequence of assemblies so that all the dimensions of all the cells tend to zero. But then by the definition of an infinite integral $h(z)$ has the unique limit

$$H(z) = \frac{1}{h^3} \int^{(6)} e^{-\log 1/z\left\{\frac{1}{2m}(p_1{}^2 + p_2{}^2 + p_3{}^2) + Q\right\}} dq_1 \ldots dp_3; \quad (5\cdot4)$$

also (it is easy to verify) the differential coefficients of $h(z)$ have unique limits which are the corresponding differential coefficients of $H(z)$. We thus obtain definite limits for the sequence of distribution laws, which are taken to be the actual laws for the real assembly. This applies not only to assemblies such as are considered in the first paper, but also to the more general assemblies of the third in which dissociation and association of atoms and molecules take place. Finally, it is again easy to verify that the same distribution laws are obtained, if the limiting partition function $H(z)$ is used in (5·23), etc. throughout the calculations. Thus the limiting

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* The factor $1/\varpi$ may be relevant in some possible definitions of the entropy. We shall not refer to it again here.

partition function for a free atom, and in general for any classical system, may be used exactly as if it were an ordinary partition function of a simple quantized system.

A similar limiting process is required for assemblies of quantized systems only, if any of the frequencies or energy quanta are incommensurable. The limiting process now consists in forming a sequence of artificial assemblies in which all the frequencies are commensurable, and whose limits are the actual frequencies of the real assembly. It is easy to verify that the foregoing arguments can all be applied, and that the sequences of distribution laws have unique limits which are taken to be the true distribution laws. These true laws can further be calculated by using the limits of the partition functions in (5·23), etc. This is equivalent to ignoring the incommensurabilities.

The partition functions for temperature radiation and the internal energy of a crystal are somewhat different. We shall in general of course have to do with incommensurable frequencies, but these may be disposed of by the method of the last paragraph. This being done, the aether and the crystal are essentially quantized systems, and should need no further limiting process. It is clear in fact that formula (2·1) for $R(z)$ (the series or product being proved convergent), and a summation formula analogous to (4·1) for $K(z)$, may be used at once as ordinary partition functions. The approximations and limiting processes by which we obtained the useful forms of $R(z)$ and $K(z)$ are no longer essential in establishing the true distribution laws, and really take a secondary place. They may be applied *after* all the distribution laws have been worked out.

*A preliminary Investigation of the Intensity Distribution in the
β-ray Spectra of Radium B and C.* By J. Chadwick, Ph.D. and
C. D. Ellis, B.A.

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Two radically different views of the β-ray disintegration of
radioactive bodies have recently been put forward. Frl. Meitner*
has suggested that the disintegration electron is ejected from the
radioactive nucleus with a definite and characteristic energy, and,
taking into account the loss of energy in escaping from the nuclear
field, the electron will have a characteristic energy outside the atom.
All the disintegration electrons, however, do not escape with their
full energy; some are assumed to convert part of their energy into
γ-rays of characteristic frequency. These γ-rays will give secondary
β-ray groups in the usual way by conversion in the $K, L, M, \ldots$
levels of the radioactive atom. It is clear that on this view the
entire β-ray emission of a radioactive body consists of a series of
homogeneous groups of electrons with definite and characteristic
energies. This theory of the β-ray disintegration has been developed
and treated from a systematic quantum standpoint by Smekal†.

An earlier investigation by one‡ of us showed quite clearly that,
under the usual experimental conditions, the total β-ray emission
from radium $B + C$ consisted largely of a continuous distribution
of β-rays over a wide range of energy and that the well-known homo-
geneous groups were superimposed on this continuous spectrum.
It would appear that on the Meitner-Smekal theory this continuous
spectrum must be considered to be entirely adventitious and pro-
duced under the experimental conditions by some such agency as
scattering.

A very different view of the β-ray disintegration has been given
by one§ of us. On this theory the disintegration electrons form the
continuous spectrum, the energy of emission of the electron from
any assigned atom being variable within wide limits. The homo-
geneous groups are considered to be entirely secondary in origin
and due to the conversion of γ-rays in the electronic structure of the
radioactive atom, these γ-rays being emitted from the nucleus
during the disintegration.

* Meitner, *Zeit. f. Phys.* 9, p. 131 (1922).
† Smekal, *Zeit. f. Phys.* 10, p. 275 (1922).
‡ Chadwick, *Verh. d. D. Phys. Ges.* 16, p. 383 (1914).
§ Ellis, *Proc. Camb. Phil. Soc.* 21, p. 121 (1922).

These two theories are fundamentally distinct and involve entirely different interpretations of the continuous β-ray spectrum. On Meitner's view it is presumably held to be a fortuitous occurrence due perhaps to scattering of the homogeneous groups, whereas on our view the continuous spectrum consists of the actual disintegration electrons. The test of the independent existence of the continuous spectrum thus gives a ready means of deciding between the two theories.

In this paper we shall describe briefly some measurements on the intensity distribution in the β-ray emission of radium *B* and *C* by an ionisation method, and it will be seen that our results provide strong evidence in support of the following statements: firstly, that the continuous spectrum has a real existence which is not dependent on the experimental arrangement and that any explanation of it as due to secondary causes is untenable; and secondly, that our view of the origin of the continuous spectrum is consistent with the magnitude of the observed effects.

Experimental Arrangement.

The experimental arrangement, shown diagrammatically in Fig. 1, was identical in principle with that used by Chadwick (*loc. cit.*).
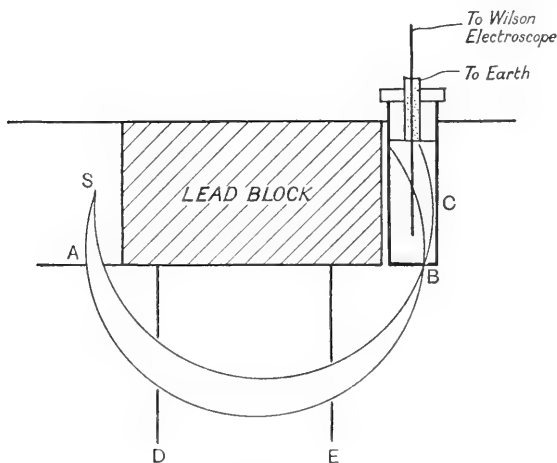

Fig. 1

The β-rays from the source *S* were focussed by a uniform magnetic field acting at right angles to the beam, and their intensity was measured by the ionisation produced in the chamber *C*, which they entered through a narrow slit *B*. By varying the magnetic field successive portions of the β-ray spectrum could be focussed

on the slit and measured in the ionisation chamber. The whole apparatus was placed in an airtight box which could be evacuated to a low pressure by means of a Gaede mercury pump. The ionisation chamber, which was at atmospheric pressure, was cut off from the rest of the apparatus by a thin sheet of mica waxed on the slit. This mica sheet weighed 1·51 mg. per sq. cm., corresponding to an air equivalent for α particles of 1·06 cm. of air. The ionisation was measured by a Wilson tilted electroscope working at a sensitivity of 32 divs. per volt, and the capacity of the whole system was about 15 cm. The slit A, defining the beam of β-rays, was 4·5 mm. wide and 10 mm. long, and distant 22 mm. from the source. The radii of curvature of the paths of the rays entering the chamber were included between 42·0 mm. and 42·7 mm. The slit in the ionisation chamber measured 1·5 mm. by 10 mm. In order to cut down stray radiation suitable screens, D and E, were inserted in the path of the beam.

The source of radium $B + C$ was a small brass plate, 10 mm. by 3 mm., made active by exposure to radium emanation. The plate was tipped so that the β-rays which passed through the slit left the plate at nearly grazing incidence. Under these conditions the focussing is comparable to that obtained with a wire of 0·5 mm. diameter. Sources of radium C were obtained by von Lerch's method on a nickel plate of the same dimensions.

Experimental Results.

When the source was in position and the box evacuated measurements were made of the ionisation produced in the chamber at different values of the magnetic field. This ionisation was due not only to the β-rays focussed on the chamber by the field but also to the γ-radiation emitted by the source and to stray β-radiation. Our results, after correction for these effects, are shown in the curves of Figs. 2 and 3.

The radium C curve was determined directly with a source of pure radium C and the ordinates give the rate of leak of the electroscope in divisions per minute for the amount of RaC in equilibrium with 1 mg. Ra. The RaB curve is also based on the amount of RaB in equilibrium with 1 mg. Ra, and was obtained by deducting the effect due to RaC from the effects observed with a source of $RaB + C$.

The effect of stray β-radiation, arising from scattering of the primary beam or produced in the screens and walls by the γ-rays, was not determined directly with the present apparatus, but with a similar one it was found to bear the same relation to the γ-ray effect as in Chadwick's experiments. The correction is small and was estimated for the RaC curve. Owing to the size of the effects obtained with RaB it was in this case of small importance to the order of accuracy of the experiment.

It should be emphasised here that no great accuracy was striven for in these preliminary experiments. Our absolute measurements have an accuracy of about 10 per cent., but the relative accuracy of points on the same curve is higher.
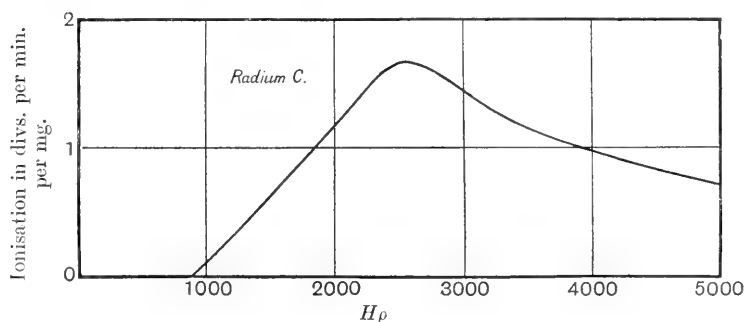
Fig. 2
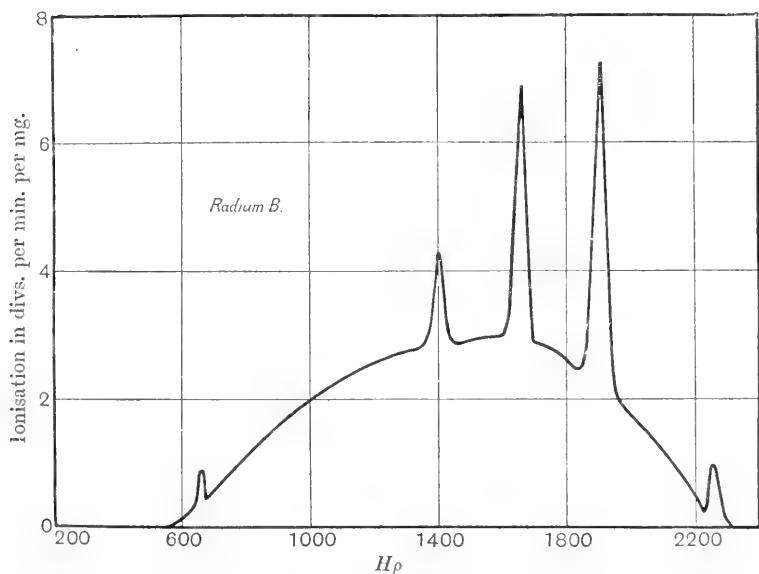
Fig. 3

It will be seen that no peaks were found in the *RaC'* curve. Chadwick's experiments showed that most of the *RaC'* lines were only of small intensity, and in our experiments the rapid decay of the source prohibited any detailed examination for peaks. We did not pursue the *RaC* curve beyond 5000*Hρ*, partly because of the

smallness of the effects and also because it was unnecessary for the deduction of the RaB curve. The experiments of Chadwick and of Varder* show that the curve ends at about $12{,}000H\rho$.

The RaB curve is shown to start at about $550H\rho$, below which point no ionisation due to primary β-rays could be detected. It should be pointed out that slower β-rays than those corresponding to $550H\rho$ may perhaps be emitted by RaB, but that we failed to detect them owing to their total absorption in the mica window of the ionisation chamber. It would appear from the general evidence, however, that the number of such slow β-rays cannot be large and consequently that their exclusion does not affect our results to any marked extent.

It can be seen from the curve that the five strong lines of the RaB β-ray spectrum were found, but that under the experimental conditions they formed only about one-fifth of the total emission, the main feature of which is the continuous spectrum†.

Discussion.

These experiments and Chadwick's earlier experiments show conclusively that the continuous spectrum is emitted from the source, that is, from the brass plate on which is deposited the $RaB + C$; but they do not prove that it arises directly from the nuclei of the disintegrating atoms. We shall now consider the evidence which leads to this conclusion.

There would appear to be only three ways in which the continuous spectrum could arise in the source. It might be supposed to consist of electrons ejected from the material of the source by the γ-rays; or it might consist of electrons which originally formed part of the homogeneous groups, but which had been rendered heterogeneous by being scattered back from the brass plate; or, lastly, the continuous spectrum might be emitted as such by the radioactive atoms.

The first possibility is ruled out at once by the magnitude of the effect.

There are two strong arguments which appear to us to decide against the second possibility. The first point is that our two curves when added together agree fairly well with Chadwick's curve for $RaB + C$, except for the difference that our peaks, owing to better focussing, are somewhat more prominent. The conditions for scattering at the source were very different in the two cases, for Chadwick's experiments were carried out with a source of radium emanation contained in an α-ray tube of 2 cm. stopping power. The second point depends on an experiment in which the active

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* Varder, *Phil. Mag.* 29, p. 725 (1915).

† The peaks are somewhat arbitrarily drawn. In order to determine the exact form of the peaks exceedingly careful measurement would be necessary.

deposit was on the under side of a thin sheet of silver, of thickness corresponding to a stopping power for α-rays of 2 cm. of air. The silver foil was 1 mm. wide and 10 mm. long, and was not tipped, but placed horizontally. In this case it is clear that there can be no scattering back of the β-rays since there is nothing immediately below the active deposit; while it is known that passage through such a thickness of matter does not greatly affect the homogeneity of the groups. The difference between results obtained in this way and those with the active deposit on the brass plate should thus give a measure of the scattering effect of the brass. Confining our attention to the *RaB* curve, we observed two effects. Firstly, in places far removed from lines, the continuous spectrum was reduced by about 20 per cent., which means that only 20 per cent. of the continuous spectrum as given in Fig. 3 can be accounted for by scattering from the brass plate. This result is in agreement with Kovarik's* measurements. He found that the amount of reflected β-radiation from brass was from 40 to 50 per cent., and in our case we are clearly taking only half the reflected rays. The other point that we found was that, although the lines were a little broadened on the low energy side, the ratio of the peaks to the continuous background was about the same as in the previous experiments. Now it is obvious that if the real emission only consists of homogeneous groups and the continuous spectrum observed under ordinary conditions is due to scattering from these lines, then in an experiment where there can be but little back scattering the homogeneous groups should be greatly increased in magnitude relative to the background. As has been stated, this effect was not observed.

In our opinion these experiments strongly support the view that the continuous spectrum is emitted by the radioactive atoms themselves, and any theory of the β-ray disintegration must take this into account. Under these circumstances our hypothesis that the continuous spectrum consists of the actual disintegration electrons seems to be the simplest way of viewing the facts. This hypothesis could be put to the test by measurements of the type described in this paper, for on this view the number of electrons emitted per second in the continuous spectrum, say of radium *B*, should be equal to the number of atoms of *RaB* disintegrating per second. Our present measurements, however, are not sufficiently accurate to test this point in a decisive way.

The curve of Fig. 3 is drawn for the amount of *RaB* in equilibrium with 1 mg. *Ra*, i.e. the number of atoms disintegrating per second is $3 \cdot 7 \times 10^7$. In order to translate the observed ionisation currents at various values of $H\rho$ into numbers of electrons at various energies we have first to correct the curve for the varying dispersion of the

* Kovarik, *Phil. Mag.* 20, p. 849 (1910).

apparatus as the magnetic field changes, and for the variation of the ionising power of the β-particle with its energy. Then, making an estimate of the number of ions which would be produced in the chamber by a β-particle of given energy, it is possible to calculate the number of electrons received per second by the ionisation chamber and hence the total number emitted per second in the continuous spectrum by the source. The number so obtained is about 3×10^7 per second, an agreement which is much better than one would expect from the rough nature of the calculation. We do not wish to lay any stress upon this agreement but regard it as showing that our hypothesis is not incompatible with the observations.

A similar test may be made on the *RaC* curve, or, what amounts to the same thing, we may compare the area of the corrected *RaC* curve with the area of the corrected *RaB* curve. Since both curves are drawn for the same number of atoms disintegrating per second it is clear that on our view the areas of the two corrected curves should be equal. Extrapolating our *RaC* curve by Chadwick's previous measurements, we find that the area of the corrected curve is about 30 per cent. greater than that of the corrected *RaB* curve. This agreement is quite satisfactory when we take into account the experimental difficulties in measuring the *RaC* curve.

We consider that the experiments described in this paper show that the continuous spectrum has a real existence and is emitted as such by the disintegrating atoms, and that its magnitude is roughly what would be expected if each disintegrating atom contributed one electron to the continuous spectrum. It appears to us that the simplest way of viewing these facts is to suppose that the continuous spectrum is formed by the actual disintegration electrons.

We hope later to be able to carry out these and similar experiments in a more accurate way and we shall therefore defer a full discussion of the points raised in this paper and of the forms of the continuous spectrum curves until greater detail is available.

On a System of Differential equations which appear in the Theory of Saturn's Rings. By W. M. H. GREAVES, Isaac Newton Student.

In a paper dealing with the influence of satellites upon the form of Saturn's Ring*, Dr G. R. Goldsbrough obtains the simultaneous differential equations

$$\left.\begin{aligned}
&\rho'' - 2\kappa\sigma' + (\Theta_{10} + \Theta_{11}\cos\phi + \Theta_{12}\cos 2\phi + \ldots + \Theta_{1r}\cos r\phi + \ldots)\rho \\
&\qquad + (\Theta_{21}\sin\phi + \Theta_{22}\sin 2\phi + \ldots + \Theta_{2r}\sin r\phi + \ldots)\sigma \\
&\qquad = 0 \\
&\sigma'' + 2\kappa\rho' + (\Theta_{41}\sin\phi + \Theta_{42}\sin 2\phi + \ldots + \Theta_{4r}\sin r\phi + \ldots)\rho \\
&\qquad + (\Theta_{50} + \Theta_{51}\cos\phi + \Theta_{52}\cos 2\phi + \ldots + \Theta_{5r}\cos r\phi + \ldots)\sigma \\
&\qquad = 0
\end{aligned}\right\}$$

$$\ldots\ldots(1)\dagger,$$

where the dashes indicate differentiations with respect to the independent variable ϕ, and κ and the Θ's are constant coefficients. All the Θ's, except Θ_{10} and Θ_{50}, are small quantities which may be regarded as being of the first order.

We shall write for all the Θ's except Θ_{10} and Θ_{50}

$$\Theta_{rs} = \mu\theta_{rs} \qquad \ldots\ldots(2),$$

where μ is small.

Goldsbrough develops a solution of these differential equations. His solution, however (as we shall see presently), is only applicable to a very special case, and even then his method will not give us the general solution, but only two distinct solutions each involving an arbitrary constant. In order to get the general solution, two more such solutions are needed.

The object of this paper is to discuss the nature of the general solution of equations (1) for different values of p, where p is a parameter which enters into the coefficients‡, so that κ and the Θ's are functions of p which we shall assume to be continuous. The discussion will be based on the general theory of equations of this type, as given for instance by Moulton in Chapter I, pp. 30 *et seq.* of his *Periodic Orbits*§. This gives us the advantage of knowing in advance that the series we obtain are convergent for sufficiently small values of μ.

* *Phil. Trans.*, A 222, pp. 101–130, 1921.
† Equations (6) of Goldsbrough's paper.
‡ In Goldsbrough's application this parameter is κ.
§ Publication No. 161 of the Carnegie Institute of Washington, 1920.

The equations (1) may be written

$$\rho'' - 2\kappa\sigma' + \Theta_{10}\rho + \mu\,(x_{11}\rho + x_{12}\sigma) = 0 \Big\}$$
$$\sigma'' + 2\kappa\rho' + \Theta_{50}\sigma + \mu\,(x_{21}\rho + x_{22}\sigma) = 0 \Big\} \qquad \ldots\ldots(3),$$

where the x's are periodic functions of ϕ with period 2π.

The equations (3) are linear differential equations with periodic coefficients and their solution is known to consist of sums of terms of the type $e^{c\phi}f(\phi)$, where $f(\phi)$ is a periodic function of ϕ of the same period as the coefficients, the period in this case being 2π.

First method of solution.

Write
$$\rho = e^{c\phi}A \Big\}$$
$$\sigma = e^{c\phi}X \Big\} \qquad \ldots\ldots(4).$$

The equations become

$$c^2A + 2cA' + A'' - 2\kappa\,(cX + X') + A\Theta_{10} + \mu\,(x_{11}A + x_{12}X) = 0 \Big\}$$
$$c^2X + 2cX' + X'' + 2\kappa\,(cA + A') + X\Theta_{50} + \mu\,(x_{21}A + x_{22}X) = 0 \Big\}$$
$$\ldots\ldots(5).$$

The solution is in general of the form[*]

$$c = c_0 + \mu c_1 + \mu^2 c_2 + \ldots + \mu^r c_r + \ldots \Big\}$$
$$A = A_0 + \mu A_1 + \mu^2 A_2 + \ldots + \mu^r A_r + \ldots \Big\} \qquad \ldots\ldots(6),$$
$$X = X_0 + \mu X_1 + \mu^2 X_2 + \ldots + \mu^r X_r + \ldots \Big\}$$

where $A_0, A_1 \ldots, X_0, X_1 \ldots$ are periodic functions of ϕ with period 2π, these series being convergent for sufficiently small values of μ.

In order to obtain the c_r, A_r and X_r we substitute the expressions (6) for c, A and X in the equations (5) and equate coefficients of like powers of μ. The terms involving μ^n will give us two simultaneous linear differential equations in A_n and X_n with *constant* coefficients which are easily solved. On imposing the condition that A_n and X_n are to be periodic with period 2π we get an equation determining c_n. The process is quite mechanical and it is unnecessary to reproduce all the actual algebra.

Proceeding in this way we get from the terms involving μ_0.

$$c_0 = n_r i \qquad \ldots\ldots(7),$$

where n_r is any root of the quartic in n,

$$(\Theta_{10} - n^2)\,(\Theta_{50} - n^2) = 4\kappa^2 n^2 \qquad \ldots\ldots(8),$$

and as usual
$$i = \sqrt{-1}.$$

We also get
$$A_0 = \alpha_r,$$
$$X_0 = \beta_r,$$

where α_r is a constant, arbitrary so far, and β_r is given by

$$2\kappa n_r i\beta_r = (\Theta_{10} - n_r{}^2)\,\alpha_r.$$

[*] Moulton, *Periodic Orbits*, p. 33.

It is convenient to determine each solution such that when $\phi = 0$, $A_0 = 1$ and $A_2 = A_3 = \ldots = 0$*.

We thus get $\qquad\qquad\qquad \alpha_r = 1$.

On proceeding to the terms involving μ we find that $c_1 = 0$ and the expressions obtained for A_1 and X_1 are of the form

$$A_1 = \alpha_{11} - \sum_{s=-\infty}^{s=+\infty} \frac{U_{1s}e^{si\phi}}{f(c_0 + si)},$$

$$X_1 = \beta_{11} - \sum_{s=-\infty}^{s=+\infty} \frac{V_{1s}e^{si\phi}}{f(c_0 + si)},$$

where α_{11} is an arbitrary constant of integration, β_{11} is another constant which is given in terms of α_{11}, U_{1s} and V_{1s} are constants and we have written

$$f(\theta) = (\theta^2 + \Theta_{10})(\theta^2 + \Theta_{50}) + 4\kappa^2\theta^2.$$

The summations extend through all integral values of s except $s = 0$. α_{11} is determined from the condition that $A_1 = 0$ when $\phi = 0$.

We then proceed to the discussion of the terms in μ^2. The process is quite straightforward and can be carried out to any stage required.

By taking in succession $c_0 = n_1 i$, $c_0 = n_2 i$, $c_0 = n_3 i$, and $c_0 = n_4 i$, where n_1, n_2, n_3, n_4 are the roots of (8), we get four solutions in this way. We thus obtain the general solution of equations (1).

We notice that the factor $f(c_0 + si)$ appears in the denominators of some of the terms in the expressions for A_1 and X_1. It is found that on proceeding to the discussion of the terms in μ^2 and higher powers of μ that higher powers of this factor will appear in the denominators of A_n, X_n and c_n.

Now $f(c_0 + si)$ vanishes if $n_r + s$ is a root of (8). In this case the method of solution outlined above will fail.

So that if there are two roots of (8), n_1 and n_2 say, such that $n_2 - n_1$ is equal to an integer, the above method will fail owing to the presence of zero factors in the denominators.

Suppose we investigate the different solutions obtained by varying the parameter p which enters into the coefficients of the original equations, and suppose that when $p = p_0$, $n_2 - n_1 = m$, where m is a positive integer.

Then when p is nearly equal to p_0, powers of a small factor will appear in the denominators of some of the terms in the series for c, A and X.

This leads us to suspect that there is a range of values of p including p_0 for which the series obtained by the method outlined above are divergent. In any case even if the series are convergent

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* Every solution thus obtained can be multiplied by an arbitrary constant.

for values of p in the neighbourhood of p_0, they would be very inconvenient for the purpose of practical applications when p has such values. For owing to the presence of the small denominators, it would be necessary to compute a large number of terms in order to obtain a reasonable approximation. What we desire in applications is a solution such that when we are dealing with small values of μ we may obtain a good approximation by neglecting the terms in μ^2 and higher powers of μ.

We are thus led to seek for some other method of development of the solutions corresponding to the roots n_1 and n_2 of (8) which shall be satisfactory for values of p in the neighbourhood of p_0.

Second method of solution.

Suppose then that $n_2 - n_1$ is nearly equal to m, where m is a positive integer, and n_1 and n_2 are two roots of (8).

Choose a_{10} such that $\nu_2 - \nu_1 = m$, where ν_1 and ν_2 are the corresponding roots of the equation

$$(a_{10} - n^2)(\Theta_{50} - n^2) = 4\kappa^2 n^2 \qquad \ldots\ldots(9).$$

Write $\Theta_{10} = a_{10} + a_{11}\mu$.

The equations (3) become

$$\left. \begin{aligned} \rho'' - 2\kappa\sigma' + a_{10}\rho + \mu\{(a_{11} + x_{11})\rho + x_{12}\sigma\} &= 0 \\ \sigma'' + 2\kappa\rho' + \theta_{50}\sigma + \mu\{x_{21}\rho + x_{22}\sigma\} &= 0 \end{aligned} \right\} \ldots(10).$$

As before we make the substitution (4) and the equations become

$$\left. \begin{aligned} c^2A + 2cA' + A'' - 2\kappa(cX + X') + Aa_{10} \\ + \mu\{(a_{11} + x_{11})A + x_{12}X\} &= 0 \\ c^2X + 2cX' + X'' + 2\kappa(cA + A') + X\Theta_{50} \\ + \mu\{x_{21}A + x_{22}X\} &= 0 \end{aligned} \right\} \ldots(11).$$

The solution is in general of the form (6)*, the A_r and X_r being as before periodic functions of ϕ with period 2π, and the series being convergent for sufficiently small values of μ.

As in the first method the A_r, X_r and c_r are determined by substituting the expressions (6) for A, X and c in (11) and equating coefficients of like powers of μ. The terms involving μ^n give us linear differential equations with constant coefficients in A_n and X_n, which can be easily solved for these quantities. By imposing on the solution thus obtained the condition that A_n and X_n are periodic in ϕ with period 2π, we determine c_n and also some of the arbitrary constants of integration which come into the solution of the differential equations which determine A_{n-1} and X_{n-1}. We shall also impose the restriction that when $\phi = 0$, $A_0 = 1$ and $A_1 = A_2 = A_3 = \ldots = 0$.

* Moulton, *Periodic Orbits*, pp. 34 and 35.

We thus obtain:

From the terms in μ^0

$$c_0 = i\nu_s \text{ for some } s,$$

where ν_1, ν_2, ν_3 and ν_4 are the roots of (9).

The solutions obtained by taking $c_0 = \nu_3 i$ and $c_0 = \nu_4 i$ will be the same as those obtained by taking $c_0 = i n_3$ and $c_0 = i n_4$ in the first method and may be obtained by that method. To obtain the other two solutions we take

$$c_0 = i\nu_1 \qquad \qquad \ldots\ldots(12),$$

and we then get

$$\left.\begin{array}{l} A_0 = \alpha_1 + \alpha_2 e^{mi\phi} \\ X_0 = \beta_1 + \beta_2 e^{mi\phi} \end{array}\right\} \qquad \ldots\ldots(13),$$

where α_1 and α_2 are arbitrary constants of integration and β_1 and β_2 are given by

$$2\kappa\nu_r\beta_r i = (a_{10} - \nu_r{}^2)\,\alpha_r.$$

From the terms in μ.

On solving the differential equations determining A_2 and X_2 which arise from these terms and imposing the condition that A_2 and X_2 are periodic in ϕ with period 2π we obtain the equations

$$\left.\begin{array}{l} \alpha_1\,(ifc_1 + g) + \alpha_2 h = 0 \\ \alpha_2\,(if'c_1 + g') + \alpha_1 h' = 0 \end{array}\right\} \qquad \ldots\ldots(14),$$

where f, g, h, f', g', h' are constant coefficients whose values work out to be as follows:

$$\left.\begin{array}{l} f = \dfrac{2}{\nu_1}\,(a_{10}\Theta_{50} - \nu_1{}^4), \qquad f' = \dfrac{2}{\nu_2}\,(a_{10}\Theta_{50} - \nu_2{}^4) \\[2mm] g = a_{11}\,(\Theta_{50} - \nu_1{}^2), \qquad g' = a_{11}\,(\Theta_{50} - \nu_2{}^2) \\[2mm] h = \tfrac{1}{2}\,(\Theta_{50} - \nu_1{}^2)\,\theta_{1m} + \dfrac{(\Theta_{50} - \nu_1{}^2)\,(a_{10} - \nu_2{}^2)}{4\kappa\nu_2}\,\theta_{2m} \\[2mm] \qquad - \kappa\nu_1\theta_{4m} + \dfrac{1}{2}\dfrac{\nu_1}{\nu_2}\,(a_{10} - \nu_2{}^2)\,\theta_{5m} \\[2mm] h' = \tfrac{1}{2}\,(\Theta_{50} - \nu_2{}^2)\,\theta_{1m} - \dfrac{(\Theta_{50} - \nu_2{}^2)\,(a_{10} - \nu_1{}^2)}{4\kappa\nu_1}\,\theta_{2m} \\[2mm] \qquad + \kappa\nu_2\theta_{4m} + \dfrac{1}{2}\dfrac{\nu_2}{\nu_1}\,(a_{10} - \nu_1{}^2)\,\theta_{5m} \end{array}\right\} \ldots(15).$$

Eliminating α_1 and α_2 from (14) we obtain

$$(ifc_1 + g)\,(if'c_1 + g') = hh' \qquad \ldots\ldots(16),$$

giving

$$c_1 = \dfrac{i\,(f'g + fg') \pm i\,\sqrt{\Delta}}{2ff'} \qquad \ldots\ldots(17),$$

where

$$\Delta = 4ff'hh' + (fg' - f'g)^2 \qquad \ldots\ldots(18).$$

We thus get in general two values of c_1. c_1 having been determined, either of equations (14) gives us a relation between α_1 and α_2. The condition that $A_0 = 1$ when $\phi = 0$ gives another relation between α_1 and α_2. These two relations determine α_1 and α_2.

The expressions obtained for A_1 and X_1 are of the form

$$\left.\begin{aligned}
A_1 &= \alpha_{11} + \alpha_{12}e^{mi\phi} - \sum_{s=-\infty}^{s=+\infty} \frac{U_{1s}e^{si\phi}}{f\,(c_0 + si)} \\
X_1 &= \beta_{11} + \beta_{12}e^{mi\phi} - \sum_{s=-\infty}^{s=+\infty} \frac{V_{1s}e^{si\phi}}{f\,(c_0 + si)}
\end{aligned}\right\} \quad \ldots..(19),$$

where U_{1s} and V_{1s} are constant coefficients, α_{11} and α_{12} are arbitrary constants of integration and β_{11} and β_{12} are given by

$$2\kappa\nu_1 i\beta_{11} = (a_{10} - \nu_1{}^2)\,\alpha_{11},$$
$$2\kappa\nu_2 i\beta_{12} = (a_{10} - \nu_2{}^2)\,\alpha_{12}.$$

The summations in (19) extend through all integral values of s except $s = 0$ and $s = m$.

The condition that $A_1 = 0$ when $\phi = 0$ gives us a relation between α_{11} and α_{12}.

We next proceed to the discussion of the terms involving μ^2. On integrating the differential equations in A_2 and X_2 which arise from these terms and imposing the condition that A_2 and X_2 are periodic in ϕ with period 2π, we obtain two equations which give us c_2 and also another relation between α_{11} and α_{12}, so that these can now be determined. The general theory in Moulton's book assures us that the process can be carried out to any stage required.

We thus obtain two solutions corresponding to the two values of c_1. These two solutions together with the two solutions corresponding to n_3 and n_4, which can be obtained in general by the first method, give us the complete solution.

It is found that when a_{11} is large, the final expressions for A_n, X_n and c_{n+1} contain terms of order $a_{11}{}^n$. Now a_{11} becomes large as p moves away from the value p_0 for which $n_2 - n_1$ is exactly an integer, and for this reason the solutions in series obtained by the second method are only suitable when p is nearly equal to p_0, and become unsatisfactory for practical purposes as p moves away from the neighbourhood of p_0. On the other hand the series for the solutions corresponding to n_1 and n_2 obtained by the first method are only suitable when p is sufficiently far removed from p_0, and become unsatisfactory as p approaches p_0.

[It should be noted that for similar reasons the method used by Goldsbrough in the paper referred to is only satisfactory when p is in the neighbourhood of p_0' where p_0' is such that when $p = p_0'$ two of the roots of equation (8) of the present paper (which is the same as equation (9) of Goldsbrough's paper) are of the form $\pm\, m$ where m is an integer. As p moves away from the neighbour-

hood of p_0', the value of $\cos 2\tau$, which is obtained from equation (17) (p. 115) of Goldsbrough's paper, will become very large. This means that large factors will appear in the numerators of the terms in Goldsbrough's solution, and this solution will be unsatisfactory. Even if it remains convergent when p moves away from p_0' we should no longer be able to obtain a good approximation by neglecting the terms involving products of the Θ's, a procedure which is in fact adopted by Goldsbrough. For the same reason, even if p be in the neighbourhood of p_0', we can only use Goldsbrough's method for the purpose of finding the two solutions corresponding to the two roots of (8) which are nearly equal to $\pm m$, and his method would not be available in practice for the purpose of finding the other two solutions.]

It is found that in the second method of solution, when $n > 1$, A_n, X_n and c_n contain terms in whose denominators powers of $\sqrt{\Delta}$ occur as factors, where Δ is given by (18). The method of solution in fact fails when $\Delta = 0$, *i.e.* when the two values of c_1 given by (17) are equal. Moulton's general theory indicates that it would be necessary in this case to assume a solution in the form of power series in $\sqrt{\mu}$.

Suppose there exists a number p_1 such that when $p = p_1$, $\Delta = 0$. Then for values of p in the immediate neighbourhood of p_1, Δ is small and the solution obtained by the second method becomes unsatisfactory.

The second method is also unsatisfactory if there exists a third root n_3 of (8) such that $n_3 - n_1$ and $n_3 - n_2$ are nearly integers. For if $n_3 - n_1$ is nearly equal to an integer s, $f(c_0 + si)$ is small, and it is found that powers of $f(c_0 + si)$ appear in the denominators of the terms in the series for A, X and c.

Let us now make the assumption that the roots of (8) are always real.

In this case c_0 is a purely imaginary quantity in each of the methods.

In the first method of solution $c_1 = 0$. Let us seek to determine the condition that must be satisfied in order that c_1 in the second method of solution should have a real part.

f, f', g, g', h and h' are now real so that the required condition is

$$\Delta < 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

where Δ has the value (18).

When $p = p_0$ so that $n_2 - n_1$ is exactly equal to an integer a_{11} is zero and (20) becomes

$$ff'hh' < 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

If when $p = p_0$ (21) is satisfied, there will be a range of values of p including p_0 for which c_1 has a real part and the extent of this range is given by (20).

Now a_{11} is a factor of g and g', and it follows that at the extremities of the range of values of p defined by $\Delta < 0$, a_{11} will in general be of zero order.

But a_{11} is a quantity of order $\dfrac{p - p_0}{\mu}$. (We are assuming that to a small change in p corresponds a change of the same order of magnitude in κ, Θ_{10}, Θ_{50} and the Θ's.)

Hence at the extremities of the range $\Delta < 0$, $p - p_0$ is of order μ; that is the extent of the range of values of p defined by (20) is of order μ.

It should be noted that the end points p_1 and p_2 of the range of values of p defined by (20) are points for which $\Delta = 0$, so that the solutions obtained by the second method become unsatisfactory as p approaches p_1 or p_2. It is clear that provided the quantity $ff'hh'$ is not small in the neighbourhood of p_0, the extents of the ranges of values of p in the neighbourhoods of p_1 and p_2 for which Δ is small so that the solutions are unsatisfactory, are small in comparison with the extent of the range (20) for which the c_1's in these solutions have real parts.

There is a very simple case in which it happens that the c_1's are partly real.

The roots of (8) are of the form $\pm a \pm b$. Suppose that when $p = p_0$, $2a = m$, where m is an integer.

Write
$$n_1 = -a,$$
$$n_2 = +a.$$

When p is nearly equal to p_0, $n_2 - n_1$ is nearly equal to m, and we apply the second method to find the solutions corresponding to the roots $\pm a$.

a_{10} is given by

$$\left(a_{10} - \frac{m^2}{4}\right)\left(\Theta_{50} - \frac{m^2}{4}\right) = \kappa^2 m^2 \qquad \ldots\ldots(22),$$

and we have
$$\nu_1 = -\tfrac{1}{2}m,$$
$$\nu_2 = +\tfrac{1}{2}m.$$

In this case we have

$$-f = f' = \frac{4}{m}\left(a_{10}\Theta_{50} - \frac{m^2}{4}\right) = F \text{ say,}$$

$$g = g' = a_{11}\left(\Theta_{50} - \frac{m^2}{4}\right) = \frac{1}{\mu}\left\{\left(\Theta_{10} - \frac{m^2}{4}\right)\left(\Theta_{50} - \frac{m^2}{4}\right) - \kappa^2 m^2\right\}$$
$$= G \text{ say,}$$

$$h = h' = \frac{1}{2}\left(\Theta_{50} - \frac{m^2}{4}\right)\theta_{1m} + \frac{\kappa m}{2}(\theta_{2m} + \theta_{4m}) - \frac{1}{2}\left(a_{10} - \frac{m^2}{4}\right)\theta_{5m}$$
$$= H \text{ say.}$$

(16) then becomes $\qquad F^2 c_1{}^2 + G^2 = H^2,$

and c_1 is real if $\qquad\qquad G^2 < H^2 \qquad\qquad\qquad$ ......(23).

If m is an even integer we have the case considered by Goldsbrough and in this case (23) is identical with Goldsbrough's condition for the reality of the exponent*.

Goldsbrough's solution now appears in the light of a particular case, and we have seen that his method does not even give us the general solution for the particular values of p in question, but only the solutions corresponding to the roots $\pm\, a$ of (8). The solutions corresponding to the roots $\,\pm\, b$ may be found by the first method of this paper unless $2b$ is nearly equal to an integer, in which case the second method must be used. But if $2a$, $2b$, $a - b$ and $a + b$ are all nearly equal to integers, both of the above methods lead to unsatisfactory solutions. It should be noted that Goldsbrough's method is unsatisfactory if both a and b are nearly integers†.

Summary. We have been discussing the solutions of the equations (1) for different values of a parameter p which enters into the coefficients in a continuous manner. We have denoted by p_0 any value of p which is such that when $p = p_0$, two roots of (8) differ by an integer. We have outlined two methods of solution. Of these the first is applicable when p is sufficiently far removed from p_0 and the second when p is in the neighbourhood of p_0.

In each case the solutions are of the type $\rho = \Sigma e^{c\phi} A$, $\sigma = \Sigma e^{c\phi} X$, where the A's and X's are periodic functions of ϕ with period 2π, and the c's are power series in μ, each c being of the form

$$c = c_0 + c_1 \mu + c_2 \mu^2 + \ldots.$$

On the assumption that the roots of (8) are always real, it is found that in each solution all the c_0's are purely imaginary quantities. In the first method of solution all the c_1's are zero, and in the second solution it may happen that there is a range of values of p including p_0 for which some of the c_1's are partly real. The extent of this range is of order μ, it being assumed that the changes in the coefficients of the original equations corresponding to a change Δp in the value of p are of order Δp. Such a range will certainly exist if p_0 is such that when $p = p_0$ two of the roots of (8) are $\pm\, \tfrac{1}{2}m$ where m is an integer. The second method of solution becomes unsatisfactory towards the extremities of the ranges of values of p for which some of the c_1's are partly real.

I am indebted to Professor H. F. Baker for the interest he has taken in the preparation of this paper, and for valuable suggestions during revision.

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* In equations (17) and (18) of Goldsbrough's paper the signs of the terms involving $\theta_{2,2n}$ have been reversed, owing to a slip amounting to a change of sign in the determinations of his $a_{2,2n}$ and $c_{2,2n}$.

† In this case some of the quantities of the type $a_{10}\Theta_{50} - n^2 (n + r)^2$ which appear in the denominators of Goldsbrough's solution are small.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1921—1922.

ANNUAL GENERAL MEETING.

October 31, 1921.

In the Cavendish Laboratory.

PROF. SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:

Prof. Seward.

Vice-Presidents:

Mr C. T. R. Wilson.
Dr E. H. Griffiths.
Prof. Newall.

Treasurer:

Mr F. A. Potts.

Secretaries:

Mr H. H. Brindley.
Prof. Baker.
Mr F. W. Aston.

Other Members of Council:

Prof. Inglis.
Mr W. H. R. Rivers.
Prof. Marr.
Mr C. T. Heycock.
Dr H. Lamb.
Prof. Hopkins.

Dr Bennett.
Dr Hartridge.
Mr H. Hamshaw Thomas.
Mr R. H. Fowler.
Mr E. Cunningham.
Mr T. C. Nicholas.

The following were elected Associates of the Society:

Miss A. V. Douglas, Newnham College.
E. Madgwick, Emmanuel College.
H. R. Mehra, Fitzwilliam Hall.
C. D. Murray, Gonville and Caius College.
Y. Nishina, Emmanuel College.
Dalip Singh, Emmanuel College.
H. de W. Smyth, Gonville and Caius College.
A. G. Thacker.
I. Tuxford, Gonville and Caius College.

The following Communications were made to the Society:

1. A new method of testing microscope objectives. By Dr HART-RIDGE.

2. (a) Determination of the coefficient of viscosity of mercury.
   (b) A Laboratory method of determining Young's Modulus for a microscopic cover slip.
By J. E. P. WAGSTAFF, M.A., St John's College. (Communicated by Prof. R. Whiddington.)

3. Some peculiarities of the Wilson ionisation tracks and a suggested explanation. By J. L. GLASSON, M.A., Gonville and Caius College. (Communicated by Prof. Sir E. Rutherford.)

4. (a) Convex Solids in Higher Space.
   (b) On certain Simply-Transitive Permutation-Groups.
By Dr W. BURNSIDE.

5. Some problems of Diophantine approximation. By Prof. G. H. HARDY and J. E. LITTLEWOOD, M.A., Trinity College.

6. On the Stability of the Steady Motion of viscous liquid contained between two rotating coaxal circular cylinders. By W. J. HARRISON, M.A., Clare College.

7. (a) Note on the Velocity of X-ray Electrons.
   (b) A Laboratory Valve-method for determining the Specific Inductive Capacities of Liquids.
By Prof. R. WHIDDINGTON.

8. On tides in the Bristol Channel. By Sir GEORGE GREENHILL.

9. On the fifth book of Euclid's Elements. By Dr M. J. M. HILL.

10. On a general infinitesimal geometry, in reference to the theory of Relativity. By W. WIRTINGER. (Communicated by Prof. H. F. Baker.)

November 14, 1921.

In the Botany School.

PROF. SEWARD, PRESIDENT, IN THE CHAIR.

The following was elected a Fellow of the Society:

J. F. Gaskell, M.D., Gonville and Caius College.

The President delivered a Lecture entitled "Greenland," which was illustrated with lantern slides.

November 28, 1921.

In the Comparative Anatomy Lecture Room.

PROF. SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

F. B. Baker, B.A., St John's College.
T. A. Brown, B.A., Trinity College.
F. H. C. Butler, B.A., St John's College.
Chas. Davison, Sc.D., Emmanuel College.
R. G. W. Norrish, B.A., Emmanuel College.
M. T. Sampson, B.A., St John's College.
S. W. P. Steen, B.A., Christ's College.
E. C. Stoner, B.A., Emmanuel College.
H. C. G. Vincent, B.A., Fitzwilliam Hall.
C. G. L. Wolf, Ph.D., Christ's College.
J. M. Wordie, M.A., St John's College.

The following was elected an Associate of the Society:

Miss M. T. Budden, Newnham College.

The following Communications were made to the Society:

1. Note on cell-division. By J. GRAY, M.A., King's College.

2. The geology of Jan Mayen (illustrated by lantern slides). By J. M. WORDIE, M.A., St John's College. (Communicated by Prof. Seward.)

3. The insect and arachnid fauna of Jan Mayen. By W. S. BRISTOWE. (Communicated by Mr H. H. Brindley.)

4. The vegetation of Jan Mayen. By J. L. CHAWORTH-MUSTERS. (Communicated by Prof. Seward.)

5. The bionomics of parasitism in certain Hymenoptera. By Miss M. D. HAVILAND. (Communicated by Mr H. H. Brindley.)

6. Note on Prof. T. H. Morgan's theory of Hen-feathering in Cocks. By M. S. PEASE, M.A., Trinity College. (Communicated by Prof. Punnett.)

7. On a problem concerning the Riemann ζ-function. By S. WIGERT. (Communicated by Prof. G. H. Hardy.)

February 6, 1922.

In the Comparative Anatomy Lecture Room.

PROF. SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

W. A. F. Balfour-Browne, M.A., Gonville and Caius College.
P. M. S. Blackett, B.A., Magdalene College.
F. L. Engledow, M.A., St John's College.
W. T. Gibson, B.A., Trinity College.
M. S. Pease, M.A., Trinity College.
J. Walton, B.A., St John's College.
W. D. Womersley, M.A., Emmanuel College.

The following were elected Associates of the Society:

> H. Burgess, Jesus College.
> S. D. Muzaffer, Fitzwilliam Hall.
> W. G. Ogg, Christ's College.
> P. Radin, Christ's College.

The following Communications were made to the Society:

1. On some new and rare Jurassic Plants from Yorkshire (V): Fertile specimens of *Dictyophyllum rugosum* L. and H. By H. HAMSHAW THOMAS, M.A., Downing College.

2. On the Food of *Teredo*, the Shipworm. By F. A. POTTS, M.A., Trinity Hall.

3. The definition of an envelope. By E. H. NEVILLE, M.A., Trinity College.

February 20, 1922.

In the Botany School.

PROF. SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Associates of the Society:

> R. E. Chapman, Emmanuel College.
> Miss E. G. Torrance, Newnham College.

Mr F. A. Potts delivered a Lecture entitled "The Marine Biology of a Tropical Island," which was illustrated with lantern slides.

February 27, 1922.

In the Cavendish Laboratory.

MR C. T. R. WILSON, VICE-PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

> W. B. Gourlay, M.A., Trinity College.
> R. A. Hayes, M.A., Trinity Hall.
> F. H. Jeffery, M.A., Trinity College.
> H. McCombie, M.A., King's College.
> E. J. Maskell, B.A., Emmanuel College.
> M. J. T. M. Needham, B.A., Gonville and Caius College
> G. W. Nicholson, M.D., Jesus College.
> L. A. Pars, B.A., Jesus College.
> L. D. Sayers, M.A., Downing College.
> A. J. Smith, B.A., Downing College.
> C. H. Spiers, M.A., Emmanuel College.
> C. West, B.A., St John's College.
> J. C. Willis, Sc.D., Gonville and Caius College.

The following was elected an Associate of the Society:

A. Lang-Brown, Christ's College.

The following Communications were made to the Society:

1. (*a*) An experiment illustrating the conservation of angular momentum.

   (*b*) A focal line method of determining the elastic constants of glass.

By Dr G. F. C. SEARLE.

2. Low Voltage Glows in Mercury Vapour. By G. STEAD, M.A., Clare College and E. C. STONER, B.A., Emmanuel College.

3. An electric wave detector. By E. V. APPLETON, M.A., St John's College.

4. An attempt to separate the Isotopes of Chlorine. By E. B. LUDLAM, B.A., Trinity College. (Communicated by Mr F. W. Aston.)

5. The measurement of Magnetic Susceptibilities at high frequencies. By M. H. BELZ. (Communicated by Prof. Sir E. Rutherford.)

6. Note on an attempt to influence the random direction of α Particle Emission. By G. H. HENDERSON. (Communicated by Prof. Sir E. Rutherford.)

7. Determination of the Coefficient of Rigidity of a thin glass beam. By J. E. P. WAGSTAFF, M.A., St John's College. (Communicated by Prof. R. Whiddington.)

March 8, 1922.

In the Botany School.

PROF. SEWARD, PRESIDENT, IN THE CHAIR.

A Report on the financial position of the Society with proposals for increasing the income was discussed.

May 1, 1922.

In the Comparative Anatomy Lecture Room.

PROF. BAKER IN THE CHAIR.

The following was elected a Fellow of the Society:

E. B. Ludlam, B.A., Trinity College.

The following Communications were made to the Society:

1. Waves of permanent type at the interface of two liquids. By Dr H. LAMB.

2. The number of primes of the form $n^2 + 1$. By Dr A. E. WESTERN.

3. The influence of electrically conducting material within the earth on various phenomena of terrestrial magnetism. By S. CHAPMAN, M.A., Trinity College and T. T. WHITEHEAD.

4. The impossibility of the coexistence of two Mathieu functions. By E. L. INCE, M.A., Trinity College.

5. A general condition for the quantisation of the conditionally periodic motions with an application for the Bohr atom. By Dr VICTOR TRKAL. (Communicated by Mr F. W. Aston.)

6. Interpretation of the β and γ ray spectra of radioactive bodies. By C. D. ELLIS, M.A., Trinity College.

May 15, 1922.

In the Comparative Anatomy Lecture Room.

MR C. T. R. WILSON, VICE-PRESIDENT, IN THE CHAIR.

The following was elected a Fellow of the Society:
J. Chadwick, Ph.D., Gonville and Caius College.

The following was elected an Associate of the Society:
P. L. Kapitza.

The following Communications were made to the Society:

1. An Experimental investigation of Soaring Flight. By Dr E. H. HANKIN.

2. The projective generation of surfaces in space of four dimensions. By F. P. WHITE, M.A., St John's College.

3. The analytical representation of the theory of congruences of conics. By C. G. F. JAMES, B.A., Trinity College. (Communicated by Prof. H. F. Baker.)

4. (a) The geometrical theory of the apolarity of quadric surfaces.
   (b) A set of fifteen quartic surfaces in space of four dimensions, and the application to the theory of cubic surfaces in ordinary space. By Miss H. G. TELLING. (Communicated by Prof. H. F. Baker.)

5. The generalisation of the theory of the circles associated with a triangle by means of the theory of plane cubic curves. By Prof. J. P. GABBATT. (Communicated by Prof. H. F. Baker.)

6. An asymptotic relation between two arithmetic sums. By B. M. WILSON, B.A., Trinity College. (Communicated by Prof. G. H. Hardy.)

May 22, 1922.

In the Anatomy School.

PROF. SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

J. E. P. Wagstaff, M.A., St John's College.
L. L. Whyte, B.A., Trinity College.

Mr J. Barcroft delivered a Lecture entitled "The Physiology of Life in the Andes," which was illustrated with lantern slides.

The following Communications were made to the Society:

1. On an integral equation. By J. E. LITTLEWOOD, M.A., Trinity College, with a note by E. A. MILNE, M.A., Trinity College.

2. On the integer solutions of the equations of the third and fourth degrees. By L. J. MORDELL.

PROCEEDINGS

On the Generalization of the Theory of Circles associated with a Triangle by means of the Theory of Plane Cubic Curves. By J. P. GABBATT. (Communicated by Prof. H. F. BAKER.)

[*Read* 15 May; *received* 19 August, 1922.]

INTRODUCTION.

The object of this paper is to shew that some of the most interesting properties of circles associated with a triangle in a euclidean plane are special cases of polar properties of cubic loci and envelopes in a projective plane. The euclidean theorems principally considered are those relating to the pedal line and circle, Feuerbach's theorem and the Wallace-Steiner properties of the quadrilateral.

The somewhat long preliminary section on properties of cubics seems to be necessary for several reasons. The writer is acquainted with no work in which are collected the whole of the known theorems required; some of those most important for the present purpose seem not previously to have been proved by synthetic methods; and most of the theorems on the relations between polar-, pole- and poloconics, on which it will appear that the euclidean theorems mainly depend, are apparently new. It is hoped, however, that the somewhat copious references to this preliminary survey, in the paragraphs which deal directly with the main subject of the paper, will enable those who are interested mainly in the generalization of the theorems of elementary geometry to dispense with a complete reading of the preliminary matter. But it should be stated that, except to a small extent as regards the treatment of quadratic transformations, no theorem is included in Part I which is not directly required for the special purpose of the paper.

The projective treatment of the geometry of the triangle has, of course, been considered by previous writers, the most systematic paper on the subject known to the author being that of Berkhan*. Except as regards the fundamental theorem† on the pedal line, little is common to Berkhan's work and the present paper.

In the bulk of previous work on the purely projective generalization of the theorems here dealt with, the circular points at infinity of euclidean geometry have been replaced by a point-pair, and generality has thereby been lost. On the other hand, the orthodox

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* "Zur projektivischen Behandlung der Dreiecksgeometrie," *Arch. d. Math. u. Phys.* (3) 11 (1907), p. 1. † *V.* (17·1) below.

non-euclidean treatment, with a Conic Absolute, does not seem to
have been particularly fruitful. This circumstance may not be un-
connected with the fact (vital in the present handling of the subject)
that the non-euclidean treatment necessarily elevates to a peculiar
position—that of the Absolute—one of three conics which, as pole-
conics of the same line* with respect to three related class-cubics,
are strictly on an equal footing. It follows that we shall discover
theorems, not generally in reciprocal pairs, but in groups of three
or multiples of three†.

An endeavour has been made to indicate the theorems which
are already known. Where proofs of such theorems are inserted, it
is hoped that they will be found simpler than those extant.
Theorems of which the enunciations are italicized are believed to be
new.

PART I.

PROPERTIES OF CUBIC CURVES

1. *Apolarity.* 1·1. Two lines which are such that each contains
the pole of the other q.‡ a given class-conic are said to be *apolar*§ to
that conic. Two points which are such that the polar of each q. a
given order-conic contains the other are said to be apolar to that
conic.

A triangle which is such that each side is the polar, q. a given
conic, of the opposite vertex is said to be apolar§ to that conic.

1·2. It is well known that, if two pairs of sides of a quadrangle
are both apolar to a given class-conic, then the third pair of sides
is also apolar to that conic: and more generally, any order-conic of
the pencil‖ determined by the vertices of the quadrangle as base-
points is apolar to the given class-conic. Further, if two order-
conics are both apolar to a given class-conic, then every order-conic
of the pencil determined by the two given order-conics is apolar to
the given class-conic; and if three order-conics, not of the same
pencil, are all apolar to a given class-conic, then all the order-conics
of the net determined by the three given order-conics are apolar to
the given class-conic.

It is unnecessary to state explicitly the corresponding theorems
on the class-conics of a range or web.

* The word *line* throughout denotes *straight line*.   † *V.*, e.g. (17·55) below.
    ‡ The symbol q. (i.e. *quoad*) throughout the paper signifies *with respect to*.
    § The word conjugate is not here used in this sense, because of its many other
uses, and because the present relation is a particular case of others ordinarily
designated apolar.
    ‖ As the terminology does not appear to be fixed in English, it may be well to
explain that here adopted. If the equations $C=0$, $C'=0$, $C''=0$ determine loci of
the same order n, and a, a', a'' be parameters; then the ∞^1 loci determined by
equations of the type $aC + a'C' = 0$ will be termed a *pencil*, and the n^2 common points
of C, C' the *base-points* of the pencil; the terms *range*, *base-line* being similarly used
of envelopes: the ∞^2 loci determined by equations of the type $aC + a'C' + a''C'' = 0$
will be termed a *net*; the term *web* being similarly used of envelopes.

1·3. Given any order-cubic T; it is well known* that there is one and only one class-cubic Θ, such that the polar conic, q. T, of every point in the plane of the two cubics is apolar to the pole conic, q. Θ, of every line in that plane; and conversely, that if any order-conic s be apolar to every pole conic, q. Θ, then s is a polar conic, q. T. White* uses the term *doubly apolar* to express the relation between T and Θ. We shall adopt the more usual terminology, and simply describe T as *apolar* to Θ, and conversely.

It is easy to shew† that the Hessian of T is the Cayleyan of Θ, and that the Cayleyan of T is the Hessian of Θ. There is thus complete duality between the properties of the two curves and of the conics, etc., associated with them. The term *duality* will be understood in this sense throughout the paper.

2. *The T-pencil and the Θ-range.* Given any order-cubic T_0; it is well known that there are in general three order-cubics (say T_1, T_2, T_3) of which T_0 is the common Hessian. Let Θ_n be the class-cubic apolar to T_n; and let Δ_n be the Hessian of Θ_n; then (1·3) Δ_n is the Cayleyan of T_n ($n = 1, 2, 3$).

It is known that the cubics T_n ($n = 0, 1, 2, 3$) belong to a pencil, the base-points of which are the points of inflexion of every cubic of the pencil; this pencil will be termed the *T-pencil*. Dually, the cubics Θ_n, Δ_n belong to a range, the base-lines of which are the cuspidal tangents of every cubic of the range; this range will be termed the Θ-*range*.

3. *Polar and Pole Conics, Lines and Points.* 3·1. The polar conic, q. T_n, of a point (Y) will be denoted by $S_n(Y)$; the polar line (briefly the *polar*), q. T_n, of Y will be denoted by $p_n(Y)$‡. If Z be contained by $S_n(Y)$, then Y is contained by $p_n(Z)$.

If Y be a point of a (fixed) line y, then the conics $S_n(Y)$ constitute a pencil, projective with the range [Y]. The base-points of the pencil are the four *poles*, q. T_n, of y; the polar, q. T_n, of any one of these poles is y. The lines [$p_n(Y)$] are tangents to a class-conic projective with the range [Y]. The polar conics, q. any given cubic, of all the points of its plane constitute a net.

The polar conics, q. the cubics of the T-pencil, of a fixed point P, themselves constitute a pencil projective with the T-pencil; the base-points of the pencil of conics are the points of contact of the tangents (other than that at P) to that cubic of the T-pencil which

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If $f \equiv a_x{}^3 = 0$ be the order-cubic, then $\Pi = u_\pi{}^3 = ST - T\Sigma = 0$ is the class-cubic. If $f \equiv x^3 + y^3 + z^3 + 6kxyz = 0$, then $\Pi \equiv \xi^3 + \eta^3 + \zeta^3 + 6\kappa\xi\eta\zeta = 0$, where $2k\kappa + 1 = 0$.

† *V.*, e.g. Wieleitner, *Th. d. eb. alg. Kürv. höh. Ordnung*, Leipzig, 1905, p. 233.

‡ If $T_n \equiv a_x{}^3 = 0$, and $Y \equiv (y_1, y_2, y_3)$, then $S_n(Y) \equiv a_x{}^2 a_y = 0$, and $p_n(Y) \equiv a_x a_y{}^2 = 0$.

contains P. The polars of P q. the cubics of the T-pencil also constitute a pencil projective with the T-pencil.

3·2. The *mixed polar** $p_n(Y, Z)$, q. T_n, of any two points Y, Z is the polar, q. $S_n(Y)$, of Z, and also the polar, q. $S_n(Z)$, of Y. If the mixed polar of a particular two of three given points contains the third, then (in general)† the mixed polar of every two points of the three contains the third, and the three points constitute an *apolar triad* q. the cubic in question. An indefinite number of pairs of points have the same mixed polar q. a given cubic; for if x denote any line, and Y any point, and if Z be the pole, q. $S_n(Y)$, of x; then x is $p_n(Y, Z)$.

3·3. More generally, we may consider the line which is the polar, q. T_n, of any class-conic σ; this line will be denoted by $p_n(\sigma)\ddagger$. If the same line x be the mixed polar of two pairs of opposite vertices of a quadrilateral, then x is also the mixed polar of the remaining pair of vertices§, and is the polar of every conic of the range of which the sides of the given quadrilateral are the base-lines.

3·4. The pole conic, q. Θ_n, of a line (y) will be denoted by $\Sigma_n(y)$, and the pole (-point) of y q. Θ_n by $\varpi_n(y)$; while $\varpi_n(y, z)$ will denote the mixed pole of two lines y, z, and $\varpi_n(s)$ the pole of an order-conic s, q. Θ_n. The definitions of these elements are dual to those of (3·1, 2, 3).

4. *Corresponding Points of the Hessian.* 4·1. It is well known that $S_n(P_0)$ degenerates into a pair of lines if and only if P_0 is a point of the Hessian T_0 of T_n ($n = 1, 2, 3$); and that the meet P_n of the pair of lines is also a point of T_0. P_n will be termed the *corresponding point of the nth species*, or briefly the *n-correspondent* of P_0. Every point P_0 of T_0 has thus three correspondents, P_1, P_2, P_3, which are points of T_0. P_0 is also the *n*-correspondent of P_n, and P_l of P_m ($l, m, n = 1, 2, 3$).

P_0, P_n are also termed *conjugate poles* q. T_n ($n = 1, 2, 3$).

4·2. Any point P_0 of T_0, and its three correspondents P_1, P_2, P_3 constitute a *Maclaurin tetrad*∥. The tangents to T_0 at the four points of such a tetrad meet at a fifth point P_0' of T_0; P_0' is the common *tangential* of the four points P_n. If the meet of $P_0 P_n$, $P_l P_m$ be denoted by P_n' ($l, m, n = 1, 2, 3$); then P_n' is a point of T_0, viz. the *n*-correspondent of P_0'.

4·3. $P_0 P_n$, $P_l P_m$ are tangents to Δ_n, the Cayleyan of T_n, and are *corresponding tangents* in the sense dual to that of (4·1), being

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* If, further, $Z \equiv (z_1, z_2, z_3)$, then $p_n(Y, Z) \equiv a_x a_y a_z = 0$.

† If Y, Z are both contained by T_0, then $p_n(Y, Z)$ may be indeterminate; *v.* (7·35), below.

‡ If $\sigma \equiv a\xi^2 = 0$, then $p_n(\sigma) \equiv a_x a_a^2 = 0$; *v.* Clifford, *Proc. Lond. Math. Soc.* 2 (1869), p. 116 = *Papers*, p. 115.

§ *V.* (7·36), below.

∥ Maclaurin, *De linearum geometricarum proprietatibus* (ed. De Jonquières, *Mélanges de géométrie pure*, Paris, 1856, p. 228).

conjugate polars q. Θ_n. Since P_0P_n, P_lP_m meet on T_0, we see that of the three tangents to Δ_n from any point of T_0, two are corresponding tangents; the third being the join of the given point and its n-correspondent.

4·4. It has been noted that the polar conic of a point, q. T_n, degenerates into a pair of lines if and only if the given point is a point of T_0. In particular, P_0P_n, P_lP_m constitute the polar conic, q. T_n, of P_0'. Dually, the conic pole, q. Θ_n, of a line degenerates into a pair of points if and only if the given line is a tangent to Δ_n; i.e. if the line is the join of n-correspondents P_0, P_n on T_0; and the degenerate conic consists of the points P_l, P_m ($l, m, n = 1, 2, 3$). The Hessian of a given order-cubic is thus the locus of those pairs of points which constitute pole conics of the apolar class-cubic, and the Cayleyan of a given order-cubic is the envelope of those pairs of lines which constitute polar conics of the given order-cubic.

5. *Twelve-point Configurations.* 5·1. Let any line meet T_0 at the points A_0, B_0, C_0. Then (with the notation of (4·1) the twelve points A_n, B_n, C_n ($n = 0, 1, 2, 3$) determine a configuration of symbol $(12_4, 16_3)$, here termed a *dodecad**; the points of a dodecad thus consist of three Maclaurin tetrads.

In particular, the points A_0, B_n, C_n; A_n, B_0, C_n; A_n, B_n, C_0 are in line ($n = 1, 2, 3$) and the points A_l, B_m, C_n are in line ($l, m, n = 1, 2, 3$); and if any three points in line be omitted, then the remainder determine a configuration of symbol $(9_2, 6_3)$. Since the tangents to T_0 at any three points in a line meet T_0 again at three points in another line (the *satellite* of the given line), therefore (4·2) the twelve points A_n, B_n, C_n are the points of contact (other than the given points) of the tangents to T_0 from three given collinear points of T_0.

5·2. From the definition (5·1), the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are in perspective two by two, $A_0B_0C_0$ being the common axis of perspective. Let F, G, H be respectively the centres of perspective of $A_2B_2C_2$, $A_3B_3C_3$; $A_3B_3C_3$, $A_1B_1C_1$; $A_1B_1C_1$, $A_2B_2C_2$. Since A_1, B_1, C_1 are respectively the meets of B_2C_3, B_3C_2; C_2A_3, C_3A_2; A_2B_3, A_3B_2; therefore the points F, G, H are in line.

Again, the triangles $A_2B_3C_3$, $A_3B_2C_2$ are in perspective, axis $A_0B_1C_1$, centre F; and A_1, B_0, C_0 are respectively the meets of B_3C_2, B_2C_3; C_3A_3, C_2A_2; A_2B_2, A_3B_3. Let G'', H' be respectively the centres of perspective of the triangles $A_3B_2C_2$, $A_1B_0C_0$; $A_1B_0C_0$, $A_2B_3C_3$; then F, G', H' are in line. Proceeding similarly with the pairs of triangles $A_3B_2C_3$, $A_2B_3C_2$; $A_3B_3C_2$, $A_2B_2C_3$; two other lines, each containing F, of the same type as FGH may be obtained. Thus all four lines of this type (lines of three centres of perspec-

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* Maclaurin, *loc. cit.* This, and all other configurations of the same symbol here treated, are of the type denoted by de Vries [*Acta math.* 12 (1888), p. 63] ($12_4, 16_3$) A.

tive) contain F; and one of these lines corresponds to each of the lines $A_0B_0C_0$, $A_0B_1C_1$, $A_1B_0C_1$, $A_1B_1C_0$, regarded each as the common axis of perspective of three triangles taken two by two.

Treating each of the lines of the $[ABC]$ configuration in turn as such an axis, we apparently obtain 48 centres of perspective; but each centre (e.g. F) occurs four times, and thus we have 12 distinct centres of perspective, lying three by three in 16 lines, four of which contain each centre. In other words, the twelve centres of perspective themselves determine a configuration of symbol $(12_4, 16_3)$, the *central dodecad* of the given dodecad.

Again, the line A_3A_1 contains two points G, G' of the central dodecad; and (e.g. from the quadrilateral of which the vertices are A_3, A_1, B_3, B_1, C_0, C_2) the point-pairs A_3, A_1; G, G' are harmonic. Generalizing, we have the theorem*: The centres of perspective of the pairs of triangles in perspective determined by points of a configuration of symbol $(12_4, 16_3)$ determine another configuration of the same symbol: the 24 points of the two configurations determine a configuration of symbol $(24_3, 18_4)$; each line of the 24-point configuration contains a pair of points of each of the 12-point configurations; and the two point-pairs on every such line are harmonic. The relation between the two 12-point configurations is reciprocal.

5·3. The nine points A_n, B_n, C_n ($n = 1$, 2, 3) and the corresponding centres of perspective F, G, H determine a configuration of symbol $(12_4, 16_3)$, and type A (5·1, footnote), and are therefore all contained by a cubic ($\mathfrak{T}_0$, say). Similarly any twelve points, of which nine are those which remain when any three points in a line of the $[ABC]$ configuration are omitted, and the other three are the centres of perspective of the three pairs of triangles having the omitted line as common axis of perspective, are contained by a cubic.

5·4. The tangents (other than that at A_0) from A_0 to the cubic T_0 touch that cubic at four points, $[L]$ say, of a Maclaurin tetrad (4·2). The tetrad $[L]$, and the tetrads $[M]$, $[N]$ similarly related to B_0, C_0 respectively, determine a dodecad. The central dodecad of this dodecad is of fundamental importance for the subject of this paper.

Let us denote any one of the four points $[L]$ by L_0, and any one of the four points $[M]$ by M_0. The line L_0M_0 contains one of the four points $[N]$; let that point be denoted by N_0. If we now denote the n-correspondents of L_0, M_0, N_0 by L_n, M_n, N_n respectively ($n = 1$, 2, 3), then the notation for the $[LMN]$ dodecad is complete, and is analogous to that for the $[ABC]$ dodecad. Since A_0 is the common tangential of the points $[L]$, therefore (4·2) the lines

* Proved independently by de Vries, *Acc. de Amsterdam* (3) 5 (1886), p. 105 and Caporali, *Mem. di geom.*, Napoli (1888), p. 338. *V.* also de Vries, *loc. cit.*, p. 210, *Acta math.* 12 (1888), p. 1 and *Acc. de Amst.* (3) 7 (1890), p. 177 = *Arch. néerl.* 25, p. 57.

L_0L_n, L_lL_m meet at A_n (l, m, $n = 1$, 2, 3); similarly for the points $[M]$, $[N]$ and B_n, C_n respectively.

5·5. The twelve points of the central dodecad of the dodecad $[LMN]$ will be denoted by the letters I, O, U, with suffixes according to the appended table, which specifies the 18 lines of the 24-point configuration determined by the twelve points L, M, N and the twelve points I, O, U:

$$\begin{array}{lll}
I_0I_aL_2L_3; & I_0I_\beta M_2M_3; & I_0I_\gamma N_2N_3 \\
I_\beta I_\gamma L_0L_1; & I_\gamma I_a M_0M_1; & I_a I_\beta N_0N_1 \\
O_0O_a L_3L_1; & O_0O_\beta M_3M_1; & O_0O_\gamma N_3N_1 \\
O_\beta O_\gamma L_0L_2; & O_\gamma O_a M_0M_2; & O_a O_\gamma N_0N_2 \\
U_0U_a L_1L_2; & U_0U_\beta M_1M_2; & U_0U_\gamma N_1N_2 \\
U_\beta U_\gamma L_0L_3; & U_\gamma U_a M_0M_3; & U_a U_\beta N_0N_3
\end{array}$$

It will be noted (5·2) that I_0, O_0, U_0 are in line; that I_0, O_ν, U_ν; I_ν, O_0, U_ν, and I_ν, O_ν, U_0 are in line ($\nu = \alpha$, β, γ); and that I_λ, O_μ, U_ν are in line (λ, μ, $\nu = \alpha$, β, γ).

5·6. The twelve points $[I$, O, $U]$ are contained by a cubic. From the symmetry of the relation between the points $[I$, O, $U]$ and the points $[L$, M, $N]$, the four points $[I]$ constitute a Maclaurin tetrad on this cubic; and so for the points $[O]$, $[U]$. The meet of the lines I_0I_a, $I_\beta I_\gamma$ (i.e. L_2L_3, L_0L_1) is A_1; similarly I_0I_β, $I_\gamma I_a$; I_0I_γ, $I_a I_\beta$ meet at B_1, C_1, respectively. The points A_1, B_1, C_1 are therefore (4·2) three points of a Maclaurin tetrad on this cubic. It is not difficult to prove that F is the fourth point of the tetrad; that the cubic is therefore $\mathfrak{T}_0$ (5·3); and that the points $[I$, O, $U]$ are thus the points of contact of the tangents to $\mathfrak{T}_0$ (other than those at F, G, H) from F, G, H respectively*.

6. *The Points $[F$, G, $H]$; $[I$, O, $U]$ as Poles.* 6·1. Since (4·4) the points A_0, A_1 constitute Σ_1 (A_2A_3), therefore every line containing A_0 is to be regarded as a tangent to Σ_1 (A_2A_3); thus ϖ_1 (A_0P) is contained by the line A_2A_3, whatever point may be denoted by P. In particular, ϖ_1 ($A_0B_0C_0$) is contained by A_2A_3, and similarly by B_2B_3, C_2C_3. Thus, *F is the pole, q. Θ_1, of $A_0B_0C_0$.* Similarly G, H *are the poles, q. Θ_2, Θ_3 respectively, of $A_0B_0C_0$;* and (cf. 3·1)† *The line FGH is the locus of the poles of $A_0B_0C_0$ q. the class-cubics of the Θ-range.*

6·2. Since (5·2) F is the centre of perspective of the triangles $A_2B_3C_3$, $A_3B_2C_2$; therefore (6·1) F is the pole, q. Θ_1, of $A_0B_1C_1$. Generalizing: $A_0B_0C_0$, $A_0B_1C_1$, $A_1B_0C_1$, $A_1B_1C_0$ are the four polars,

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* It follows that the cubic $\mathfrak{T}_0$, i.e. the *general* cubic in this special aspect, has the properties discussed by Sommerville, *Proc. Edin. Math. Soc.* 33, Pt 2 (1914–15), p. 85, and previously (1909) in an unpublished paper by the present writer. The non-euclidean treatment as usual obscures the threefoldness of the properties in question.

† A reference containing the abbreviation *cf.* is generally to the dual (1·3) of the theorem indicated.

q. Θ_1, of F; a theorem which might have been obtained otherwise. It is, in fact, well known* that the four polars of any given point q. any given class-cubic are the sides of a quadrilateral inscribed in the (order-cubic which is the) Cayleyan of that class-cubic; that opposite vertices of the quadrilateral are correspondents, of the appropriate species, on that Cayleyan; and that the sides of the diagonal triangle of the quadrilateral are those tangents to the Hessian of the given class-cubic which correspond (4·3) to the three tangents to that Hessian from the given point. There are thus three species of quadrilaterals inscribed in an order-cubic (e.g. T_0) regarded as the common Cayleyan of three class-cubics $(\Theta_1, \Theta_2, \Theta_3)$. which may be termed the 1-, 2-, 3-inscribed quadrilaterals.

The four sides of an *n*-inscribed quadrilateral touch the pole conic, q. Θ_n, of any line containing the point which is the common pole (q. Θ_n) of the four sides. Thus, if the four polars, q. Θ_n, of any points X, X' be denoted by $[x]$, $[x']$ respectively; then the eight lines $[x]$, $[x']$ all touch a conic, viz. $\Sigma_n (XX')$. Conversely, the two tangents from any point of T_0 to any pole conic q. Θ_n have the same pole q. Θ_n: if one of the two tangents from P_0 be P_0Q_0, where P_0. Q_0 are points of T_0; then the other is P_0Q_n, where Q_n is the *n*-correspondent of Q_0.

6·3. As in (6·2), $L_0M_0N_0$, $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ are the four polars, q. Θ_1, of I_0; with like theorems for O_0, U_0, etc. As in (6·1), the poles, q. Θ_2, of $L_0M_0N_0$, $L_0M_1N_1$, $L_1M_0N_1$, $L_1M_1N_0$ are O_0, O_a, O_β, O_γ respectively: and the poles, q. Θ_3, of the same lines are U_0, U_a, U_β, U_γ respectively.   (6·31)

It follows that there are three distinct partitions (one corresponding to each of the three class-cubics $\Theta_1, \Theta_2, \Theta_3$) of the sixteen lines of any $(12_4, 16_3)$ configuration into four groups of four lines each. In particular, for the lines $[L, M, N]$; in the partition (say the 1-partition) which corresponds to Θ_1 the four groups of lines are the polars, q. Θ_1, of the points $I_0, I_a, I_\beta, I_\gamma$. Every such group of four lines determines a quadrilateral: and every conic inscribed in such a quadrilateral is a pole conic, q. Θ_1, of some line containing that one of the four points $[I]$ which is the pole, q. Θ_1, of every side of the quadrilateral. The groups of lines of the 2-, 3-partitions are the polars, q. Θ_2, Θ_3, of the points $[O]$, $[U]$ respectively. Every group of the *m*-partition clearly consists of four lines one of which belongs to each of the groups of the *n*-partition $(m, n = 1, 2, 3)$.   (6·32)

6·4. The theorems of (6·1, 2, 31) are particularized cases of the general theorem: *If P_0, Q_0 denote any two points of T_0, and if P_n, Q_n denote the n-correspondents of P_0, Q_0 respectively; then P_lP_m, Q_lQ_m meet at $\varpi_n (P_0Q_0)$ $(l, m, n = 1, 2, 3)$.* A further important case is obtained by writing (e.g.) L_0, L_2 for P_0, Q_0. We have then (for $l = 2, m = 3, n = 1$): $\varpi_1 (L_0L_2)$ is the meet of L_2L_3, L_0L_1; i.e. is A_1.

* Cayley, *Phil. Trans.* 147 (1857), p. 415 = *Papers*, 2, p. 381.

Generalizing: A_n is the common pole, q. Θ_n, of L_0L_l, L_0L_m, L_mL_n, L_nL_l (l, m, $n = 1$, 2, 3); similarly for B_n, C_n and the points $[M]$, $[N]$ respectively.

6·5. The polar conic, q. T_1, of A_0 consists (4·4) of the line-pair L_2L_3, L_0L_1, i.e. (5·5) of the line-pair I_0I_a, I_sI_γ; similarly the polar conics, q. T_1, of B_0, C_0 respectively consist of the line-pairs I_0I_β, $I_\gamma I_a$; I_0I_γ, I_aI_β. Thus (3·1) *The points* $[I]$ *are the poles of* $A_0B_0C_0$ q. T_1; similarly *the points* $[O]$, $[U]$ *are the poles of* $A_0B_0C_0$ q. T_2, T_3 *respectively.* Further: The conics of the pencil of which the base-points are the four points $[I]$ are the polar conics, q. T_1, of the points of the line $A_0B_0C_0$; similarly for the points $[O]$, $[U]$ and the cubics T_2, T_3 respectively.

6·6. From (6·5) we have the theorem (dual to 6·2): The points $[I]$ are the vertices of a quadrangle circumscribed to Δ_1, and any one of the three pairs of opposite sides of the quadrangle are corresponding tangents to Δ_1; similarly for the points $[O]$, $[U]$ and Δ_2, Δ_3 respectively.   (6·61)

The diagonal triangle of the quadrangle determined by the four points $[I]$ is (5·6) $A_1B_1C_1$; this amounts to the following general theorem: The vertices of the diagonal triangle of the quadrangle determined by the four poles, q. T_n, of any line are the n-correspondents of the points common to T_0 and that line ($n = 1$, 2, 3).   (6·62)

6·7. The pole, q. Θ_1, of $L_0M_0N_0$ is I_0, and the polar, q. T_1, of I_0 is $A_0B_0C_0$ (6·3, 5); but $A_0B_0C_0$ is (5·4) the satellite, q. T_0, of $L_0M_0N_0$. Generalizing: *The polar (q. any order-cubic) of the pole (q. the apolar class-cubic) of any line is the satellite (q. the Hessian of the given order-cubic) of that line.*

6·8. From (6·5), the points I_β, I_γ have the same polar q. T_1; but (5·5) I_β, I_γ are contained by the join of L_0, L_1 and (5·2) are harmonic q. those points. Also (4·3) L_0L_1 is a tangent to Δ_1. Generalizing: *If P, P' be a pair of points contained by any tangent t to Δ_n, and harmonically conjugate to the pair of n-correspondent points common to t and T_0; then P, P' have the same polar q. T_n ($n = 1$, 2, 3).*   (6·81)

Of the last theorem the following is a dual: *If p, p' be a pair of tangents from any point P on T_0, harmonically conjugate to the pair of corresponding tangents from P to Δ_n; then p, p' have the same pole q. Θ_n ($n = 1$, 2, 3).*   (6·82)

6·9. Whatever point may be denoted by X_0; the polar, q. $S_1(X_0)$, of A_1 is (3·2) the polar, q. $S_1(A_1)$, of X_0. But $S_1(A_1)$ consists (4·1) of two lines meeting at A_0; the polar, q. $S_1(X_0)$, of A_1 therefore contains A_0. Similarly the polars, q. $S_1(X_0)$, of B_1, C_1 contain B_0, C_0 respectively. Therefore the triangle $A_1B_1C_1$ is in perspective with its polar triangle q. $S_1(X_0)$, $A_0B_0C_0$ being the axis of perspective. But (6·6) $A_1B_1C_1$ is the diagonal triangle of the

quadrangle determined by the four points [*I*]; and (6·5) $A_0B_0C_0$ is the polar, q. T_1, of any one of those four points. Generalizing, we have the following: *Given the polar conic, q. T_n, of a variable point, and a quadrangle of which the vertices are the four poles, q. T_n, of a fixed line; then the axis of perspective of the diagonal triangle of the given quadrangle and its polar triangle q. the given conic is the given line* ($n = 1, 2, 3$).

7. *Quadratic Transformations.* 7·1. In the newer geometry of the euclidean plane triangle, the generalization of which is the principal subject of this paper, a considerable part is played by isogonal conjugates (Winkelgegenpunkte). It will be recalled that the fundamental theorem on such points is as follows: If ABC be a triangle, and P any point of its plane; and if Q, R, S be any points of the plane such that the angles BAP, CBP, ACP are respectively congruent to QAC, RBA, SCB; then the lines AQ, BR, CS in general meet at a point P'. The relation between P and P' is reciprocal; and each of the points P. P' is termed the isogonal conjugate of the other q. the triangle ABC.

If, however, P be a point of the circumcircle of ABC, then the lines AQ, BR, CS are parallel: a theorem generally expressed by the statement that the isogonal conjugate, q. a triangle, of any point of the circumcircle of that triangle is a point at infinity: or again, as follows: The circumcircle is the isogonal transformation of the line at infinity.

More generally, if the locus of P is a line (not containing A or B or C) then the locus of P' is a conic circumscribing ABC; each locus being the isogonal transformation of the other.

The isogonal transformation is of course a quadratic transformation*, the singular points of the transformation being A, B, C, and the double points the incentre and excentres of the triangle ABC (as meets of the bisectors of the angles of ABC).

7·2. A transformation projectively identical with the isogonal transformation is determined by any plane quadrangle, as follows. Let the pairs of opposite sides a, a'; b, b'; c, c' of the quadrangle meet at A, B, C respectively, and let P be any point of the plane ABC; then there is in general one and only one point Q of the plane such that the pencils $(a, a'; AP, AQ)$, $(b, b'; BP, BQ)$, $(c, c';$ $CP, CQ)$ are harmonic; Q being the meet of the polars of P q. the conics of that pencil of which the base points are the vertices of the quadrangle. Those vertices are therefore the double points, and the points A. B. C the singular points, of the transformation, in which P, Q are conjugate points.

The transformation of any straight line not containing a singular point is a conic containing the three singular points. The trans-

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* If $P \equiv (\alpha, \beta, \gamma)$ and $P' \equiv (\alpha', \beta', \gamma')$, where ABC is the triangle of reference, and the coordinates are trilinear, then $\alpha\alpha' = \beta\beta' = \gamma\gamma'$.

formation of any pencil of straight lines (vertex P, distinct from A or B or C) is a pencil of conics (base-points A, B, C and the conjugate of P), and the two pencils are projective. The transformation of any conic containing two of the singular points, but not the third, is another conic containing those two points.

7·3. The relation to the cubic of quadratic transformations has been discussed by previous writers*; but the intimate relation between particular transformations of the quadratic type and the polar properties of the cubic, and the inter-relations of the special cases of such transformations now to be considered, do not seem to have previously been made clear. The question will therefore be discussed a little more fully than is strictly necessary for our present purpose.

We have seen (3·1, 6·6) that the polar conics, q. T_n, of the points of a line constitute a pencil, of which the base-points are the four poles, q. T_n, of that line; and that the vertices of the diagonal triangle of the quadrangle determined by the four poles are the n-correspondents of the points common to T_0 and the given line ($n = 1, 2, 3$). The polars of any given point q. the conics of a pencil all contain another point, the two points being apolar to every conic of the pencil. We may therefore give the following definition: If two points, P_0, P_n, are apolar to the polar conic, q. T_n, of every point of a given line x; then P_0, P_n are *conjugates of the nth species* or briefly *n-conjugates* (*base x*) ($n = 1, 2, 3$). (7·31)

P_0, P_n are thus conjugate points for the quadratic transformation of which the poles, q. T_n, of x are the double points, and the n-correspondents of the points common to T_0 and x the singular points. (7·32)

Again, if X denote any point of x, then the polar of P_0 q. S_n (X) contains P_n, and therefore (3·2) the polar of X q. S_n (P_0) contains P_n. Thus P_n *is the pole of x q.* S_n (P_0), and conversely. (7·33)

If P_0 be a point of T_0, then S_n (P_0) degenerates into two lines meeting at the n-correspondent (4·1) of P_0. Thus in this case the polar of *every* point q. S_n (P_0) contains the n-correspondent of P_0, and therefore *n-correspondent points of* T_0 *are n-conjugates to every base*. (7·34)

In the case of (7·34), p_n (P_0, P_n) is indeterminate. In every other case, P_0, P_n are *n-conjugates* (*base x*) *if and only if x is* p_n (P_0, P_n). (7·35)

If $A_0 B_0 C_0$ be the base, then (6·5, 6) the four points $[I]$ are the double points, and the points A_1, B_1, C_1 the singular points of the 1-transformation. If P_0, P_1 denote any pair of 1-conjugate points (base $A_0 B_0 C_0$) then the line-pairs $A_1 P_0$, $A_1 P_1$: $I_0 I_a$, $I_\beta I_\gamma$ are harmonic. The pencils of lines $A_1 [P_0]$, $A_1 [P_1]$ are therefore in involu-

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* E.g. Oguru, *Tohûku Math. J.* 4 (1913), p. 132.

tion, $I_0 I_a$, $I_\beta I_\gamma$ being the double lines of the involution, which may be termed the 1-*involution on* A_1*.* But (4·1) B_0, B_1; B_2, B_3 are pairs of 1-correspondents on T_0, and therefore (7·34) are pairs of 1-conjugates (base $A_0 B_0 C_0$). Thus $A_1 (B_1 B_2 P_0 P_0') \barwedge A_1 (B_0 B_3 P_1 P_1')$, i.e. (since A_1, B_0, C_1; A_1, B_3, C_2 are in line)

$$A_1 (B_1 B_2 P_0 P_0') \barwedge A_1 (C_1 C_2 P_1 P_1')$$

is an involution. Similarly

$$B_1 (C_1 C_2 P_0 P_0') \barwedge B_1 (A_1 A_2 P_1 P_1')$$

and

$$C_1 (A_1 A_2 P_0 P_0') \barwedge C_1 (B_1 B_2 P_1 P_1')$$

are involutions. Conversely, if two of the three involution-projectivities hold, so must the third, and the points P_0, P_1; P_0', P_1' must be 1-conjugates (base $A_0 B_0 C_0$).

The points $[O]$, $[U]$ are respectively the double points, and the points A_2, B_2, C_2; A_3, B_3, C_3 respectively the singular points, of the 2-, 3-transformations (base $A_0 B_0 C_0$); and the projectivity

$$A_n (B_n B_m P_0 P_0') \barwedge A_n (C_n C_m P_n P_n')$$

is an involution, which may be termed the *n-involution on* A_n; similarly for B_n, C_n (m, $n = 1, 2, 3$). Clearly two pairs P_0, P_n; P_0', P_n' of n-conjugate points (base $A_0 B_0 C_0$) in general determine the n-involution on A_n (or B_n or C_n); and it is a necessary and sufficient condition for the n-conjugacy (base $A_0 B_0 C_0$) of two points that their joins with two of the three points A_n, B_n, C_n should be conjugate in the appropriate involution. Thus if two pairs of opposite vertices of a quadrilateral are n-conjugates, then (by the involution property of the quadrilateral) the remaining pair of vertices are also n-conjugates. (7·36)

Since n-correspondents Q_0, Q_n on T_0 are n-conjugates to every base; therefore $A_n [Q_0] \barwedge A_n [Q_n]$. This is a particularized case of the general theorem: If P_0, Q_0 denote any points on T_0, and Q_n the n-correspondent of Q_0; then $P_0 [Q_0] \barwedge P_0 [Q_n]$.

Hence, or otherwise, the n-involution on A_n is determinate when A_n is fixed (on T_0). Therefore, if A_0 be fixed, and B_0, C_0 be the remaining points common to T_0 and a *variable* line containing A_0; and if R_n be the n-conjugate (base $A_0 B_0 C_0$) of a (fixed) point R_0; then the line $A_n R_n$ is fixed; i.e. the locus of R_n is a line containing A_n. (7·37)

Now let

$$A_1 (B_1 B_2 P_1 Q_1) \barwedge A_1 (C_1 C_2 P_0 Q_0) \quad \ldots\ldots\ldots\ldots(1),$$

$$B_1 (C_1 C_2 P_1 Q_1) \barwedge B_1 (A_1 A_2 P_0 Q_0) \quad \ldots\ldots\ldots\ldots(2),$$

$$C_1 (B_1 B_2 P_1 Q_1) \barwedge C_1 (A_1 A_2 P_0 Q_0) \quad \ldots\ldots\ldots\ldots(3),$$

* V. (7·37), below.

and let the first of these projectivities be an involution. We shall prove that (in general) the other two projectivities must also be involutions, and that therefore P_0, P_1; Q_0, Q_1 must be 1-conjugates (base $A_0B_0C_0$).

For if not, let P_0', Q_0' be respectively the 1-conjugates of P_1, Q_1; then [7·36, (1)] $A_1P_0P_0'$; $A_1Q_0Q_0'$ are in line. Also [7·36, (2)]

$$B_1 (A_1A_2P_0'Q_0') \barwedge B_1 (C_1C_2P_1Q_1)$$
$$\barwedge B_1 (A_1A_2P_0Q_0)$$
$$\barwedge B_1 (A_2A_1Q_0P_0)$$

and thus the projectivity

$$B_1 (A_1A_2P_0'Q_0') \barwedge B_1 (A_2A_1Q_0P_0) \quad \dots\dots\dots\dots(4)$$

is an involution.

But the lines P_0P_0', Q_0Q_0' meet at A_1; thus, from (4) and the involution property of the quadrilateral, the lines P_0Q_0, $P_0'Q_0'$ meet on A_2B_1. Using (3), we have similarly that P_0Q_0, $P_0'Q_0'$ meet on A_2C_1. Hence *either* P_0Q_0 contains A_2, which is not in general the case; *or* P_0, Q_0 are 1-conjugates (base $A_0B_0C_0$) of P_1, Q_1 respectively. That one of the three given projectivities should be an involution is thus (in general) a sufficient condition that all three should be involutions.   (7·38)

7·4. Since A_2, A_3 are 1-correspondents on T_0, and therefore 1-conjugates (base $A_0B_0C_0$); therefore A_1A_2, A_1A_3 are conjugate in the 1-involution on A_1. But (5·2) A_1A_2, A_1A_3 contain H, G respectively, and therefore A_1G, A_1H are conjugate in the 1-involution on A_1; similarly for B_1, C_1. Thus: *G, H are 1-conjugates (base $A_0B_0C_0$)*; similarly *H, F; F, G are respectively 2-, 3-conjugates (base $A_0B_0C_0$)**.

These are (6·1) particularized cases of the general theorem: *If x denote any line, and X any point on x; then $\varpi_m (x)$, $\varpi_n (x)$ are apolar to $S_l (X)$ (l, m, n = 1, 2, 3).*

Dually: *If T_n', T_n'' denote the two order-cubics (other than T_n) of which Δ_n is the Cayleyan; then the polars of X q. T_n', T_n'' are apolar to $\Sigma_n (x)$ (n = 1, 2, 3).*

7·5. Again, whatever 1-conjugate points (base $A_0B_0C_0$) be denoted by P_0, P_1; the lines A_1P_0, A_1P_1 are harmonic q. $I_0I,$, $I_\beta I_\gamma$. i.e. (5·5) q. L_2L_3, L_0L_1. Similarly (if P_2, P_3 denote the 2-, 3-conjugates, base $A_0B_0C_0$, of P_0) A_2P_0, A_2P_2; A_3P_0, A_3P_3 are harmonic q. L_3L_1, L_0L_2: L_1L_2, L_0L_3 respectively. Thus (7·2) A_1P_1, A_2P_2, A_3P_3 meet at a point (P_a, say), the conjugate of P_0 in the quadratic transformation determined by the quadrangle [L]. Likewise B_1P_1, B_2P_2, B_3P_3: C_1P_1, C_2P_2, C_3P_3 meet at points (P_β, P_γ) respectively. P_a, P_β, P_γ may be termed respectively the α-, β-, γ-conjugates of P_0: and A_1, A_2,

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* *V.* (14·3) for an extension of this theorem.

A_3; B_1, B_2, B_3; C_1, C_2, C_3 are respectively the singular points of the α-, β-, γ-transformations.   (7·51)

Writing B_1 for P_0, we see that B_1, C_1 are α-conjugates; more generally B_n, C_n *are α-conjugates* ($n = 1$, 2, 3); similarly C_n, A_n; A_n, B_n *are respectively β-, γ-conjugates.*   (7·52)

7·6. Now for the 1-transformation (base $A_0B_0C_0$), B_1 is a singular point and (7·36) A_2, A_3 are conjugates. Thus, if P_0 denote any point of the line $A_2B_1C_3$, then the 1-conjugate P_1 of P_0 is a point of $A_3B_1C_2$. But for the α-transformation, A_3 is a singular point and (7·5) B_1, C_1 are conjugates; thus the α-conjugate (P_0', say) of P_1 is a point of $A_3B_2C_1$. But P_0 is the 1-conjugate of P_1; and (7·5) the 1-conjugate and the α-conjugate of P_1 are in line with A_1. Therefore P_0', the α-conjugate of the 1-conjugate of P_0 (say the 1α-conjugate of P_0), is the meet of A_1P_0 and $A_3B_2C_1$. Similarly P_0' is the 1-conjugate of the α-conjugate (say the α1-conjugate) of P_0.

Again, if P_a denote the α-conjugate of P_0, then similarly A_1, P_1, P_a are in line; P_a is a point of $A_2B_3C_1$; and each of the points P_1, P_a is the 1α- and also the α1-conjugate of the other.

Further, if A_1P_0 meet $A_3B_1C_2$ at Q_0; then Q_0', the meet of A_1P_0 and $A_2B_3C_1$ is similarly the 1α- and also the α1-conjugate of Q_0; A_1P_1 meets $A_2B_1C_3$, $A_3B_2C_1$ at Q_1, Q_a the 1-, α-conjugates respectively of Q_0; and Q_1, Q_a are at once 1α- and α1-conjugates.

Now let R_0 denote any other point of A_1P_0; then R_1, the 1-conjugate of R_0, is a point of A_1P_1; and R_0', the α-conjugate of R_1, is a point of A_1P_0.

Projecting from B_1, and remembering that P_a is the 1-conjugate of P_0', we have (7·36) by 1-conjugates

$$(P_0P_0'Q_0R_0) \barwedge (P_1P_aQ_1R_1),$$

and projecting from A_2, we have by α-conjugates

$$(P_1P_aQ_1R_1) \barwedge (P_0'P_0Q_0'R_0').$$

Thus the 1α-conjugate of R_0 is the conjugate of R_0 in the involution-range in which P_0, P_0'; Q_0, Q_0' are conjugate pairs; similarly the α1-conjugate of R_0 is the conjugate of R_0 in that involution-range. Thus the same point is the 1α- and the α1-conjugate of R_0. But R_0 represents any point. Terming the operation which transforms a point into its n-conjugate the n-operation ($n = 1$, 2, 3, α, β, γ) and generalizing: *Any one of the operations* (1-, 2-, 3-) (*base $A_0B_0C_0$*) *is, for every operand which does not involve a singular point for either of the transformations concerned, commutative with any one of the operations* (α-, β-, γ-).

7·7. We have seen (4·1, 7·34) that, if P_0 denote any point of T_0, and P_1, P_2, P_3 respectively the 1-, 2-, 3-conjugates of P_0 (any base), then P_2, P_3 are themselves 1-conjugates.

Conversely, if $A_0B_0C_0$ be the base; if P_0, P_2 be any two points

whatever; and if P_1, P_3 be the 1-conjugates of P_0, P_2 respectively; then (7·36)

$$B_1 (C_1 C_2 P_0 P_3) \barwedge B_1 (A_1 A_2 P_1 P_2)$$
$$\barwedge B_1 (A_2 A_1 P_2 P_1),$$

similarly $C_1 (B_1 B_2 P_0 P_3) \barwedge C_1 (A_2 A_1 P_2 P_1).$

But, if further P_2, P_3 be respectively the 2-conjugates of P_0, P_1, then the projectivity

$$A_2 (B_1 B_2 P_0 P_3) \barwedge A_2 (C_1 C_2 P_2 P_1)$$

is an involution. Also the triangles $A_2 B_1 C_1$, $A_1 B_2 C_2$ are perspective; and thus (7·38), unless $P_1 P_2$ contains A_1, then all three projectivities

$$A_2 (B_1 B_2 P_0 P_3) \barwedge A_2 (C_1 C_2 P_2 P_1)$$
$$B_1 (C_1 C_2 P_0 P_3) \barwedge B_1 (A_2 A_1 P_2 P_1)$$
$$C_1 (B_1 B_2 P_0 P_3) \barwedge C_1 (A_2 A_1 P_2 P_1)$$

are involutions. Similarly, unless $P_1 P_2$ contains B_1, then the three projectivities

$$A_1 (C_1 C_2 P_0 P_3) \barwedge A_1 (B_2 B_1 P_2 P_1)$$
$$B_2 (C_1 C_2 P_0 P_3) \barwedge B_2 (A_1 A_2 P_2 P_1)$$
$$C_1 (A_1 A_2 P_0 P_3) \barwedge C_1 (B_2 B_1 P_2 P_1)$$

are involutions; and, unless $P_1 P_2$ contains C_1, then the three projectivities

$$A_1 (B_1 B_2 P_0 P_3) \barwedge A_1 (C_2 C_1 P_2 P_1)$$
$$B_1 (A_1 A_2 P_0 P_3) \barwedge B_1 (C_2 C_1 P_2 P_1)$$
$$C_2 (A_1 A_2 P_0 P_3) \barwedge C_2 (B_1 B_2 P_2 P_1)$$

are involutions. Thus in any case the three projectivities of one or other of the three sets are all involutions. If this be true, e.g. of the first set, then (7·36) P_2, P_3 are respectively 2-conjugates of P_0, P_1 (*base* $A_0 B_3 C_3$); whence P_0, P_2 are apolar to the polar conics (q. T_2) of (at least) three points not in line*. Remembering that T_0 is the Jacobian of any three such conics, both P_0 and P_2 are therefore points of T_0. Generalizing: *The m-, n-operations (any base) are commutative if and only if the operands are points of the cubic* T_0 (m, $n = 1, 2, 3$). (7·71)

Again, if $A_1 (B_1, C_2; B_2, C_1; P_0, P_2)$ and $B_1 (C_1, A_2; C_2, A_1; P_0, P_2)$ simultaneously specify conjugate pairs of rays in involution pencils, then, as in (7·36), P_0, P_2 are 2-conjugate points (base $A_3 B_3 C_0$); and if $A_1(B_1, C_2; B_2, C_1; P_0, P_2)$ and $C_1(A_1, B_2; A_2, B_1; P_0, P_2)$ simultaneously specify conjugate pairs of rays in involution pencils, then P_0, P_2 are 2-conjugate points (base $A_3 B_0 C_3$). Thus, as in (7·71), if all three sets of conditions hold simultaneously, then P_0, P_2 are conjugate points q. the polar conics (q. T_2) of (at least) three points not in

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* E.g. any two points of $A_0 B_0 C_0$ and any point (except A_0) of $A_0 B_3 C_3$.

line; and are therefore 2-correspondent points of T_0. Thus: A_m (B_m, C_n; B_n, C_m; P_0, P_n); B_m (C_m, A_n; C_n, A_m; P_0, P_n); C_m (A_m, B_n; A_n, B_m; P_0, P_n) *simultaneously specify conjugate pairs of rays in involution pencils, if and only if P_0, P_n are n-correspondent points of T_0 (m, n = 1, 2, 3).*   (7·72)

Further, we have seen that if X_0, Y_0, Z_0 be respectively 1-, 2-, 3-conjugates of a given point of T_0 (any base), then the pairs Y_0, Z_0; Z_0, X_0; X_0, Y_0 are respectively 1-, 2-, 3-conjugates; and that the pairs G, H; H, F; F, G are respectively 1-, 2-, 3-conjugates (base $A_0B_0C_0$). Now let Z_0, Y_0 be respectively the 2-, 3-conjugates (base $A_0B_0C_0$) of *any* given point X_0; then (7·51) A_2Z_0, A_3Y_0 meet at X_a, the α-conjugate of X_0. But A_2, A_3 are 1-conjugates; if Y_0, Z_0 are also 1-conjugates, then (7·36) A_2Y_0, A_3Z_0 meet at the 1-conjugate X_{a1} of X_a; also (7·6) X_{a1} is X_{1a}, which is a point of A_1X_0. Thus A_1X_0, A_2Y_0, A_3Z_0 are concurrent, their common point being X_{1a}; and, by the symmetry, this point must also be Y_2, and Z_3. Thus X_1, Y_2, Z_3 are identical; i.e. X_0, Y_0, Z_0 are respectively the 1-, 2-, 3-conjugates of the same point, W_0, say. Thus the 1-, 2-operations are commutative when X_0 is the operand. X_0 is therefore (7·71) a point of T_0, on which it follows that X_0, Y_0, Z_0, W_0 constitute a Maclaurin tetrad.

The argument fails if X_a is A_1, for in that case X_{a1} is indeterminate; then X_0, Y_0, Z_0 are respectively F, G, H. Thus: *Given three points, X_0, Y_0, Z_0, such that Y_0, Z_0; Z_0, X_0; X_0, Y_0 are respectively 1-, 2-, 3-conjugates (base $A_0B_0C_0$); then either X_0, Y_0, Z_0 are three points of a tetrad on T_0, or X_0, Y_0, Z_0 are respectively F, G, H.*

This theorem may also be enunciated in the form: The 2-conjugate of the 1-conjugate of a given point is in general the 3-conjugate of that point if and only if the point be contained by T_0: which with (7·71) amounts to the statement that: *The identical operation and the 1-, 2-, 3-operations form an Abelian group, if and only if the operand be a point of T_0.*   (7·73)

7·8. In the case of any point P_0, of which P_1, P_2, P_3 are respectively the 1-, 2-, 3-conjugates (base $A_0B_0C_0$), we have seen (7·5) that A_1P_1, A_2P_2, A_3P_3 meet at a point P_a. If P_0 be a point of T_0, then by the theory of Maclaurin (5·1) P_a is a point of T_0, viz. the third point common to T_0 and the line A_0P_0; similarly for P_β, P_γ. Thus the cubic T_0 is anallagmatic not only for the 1-, 2-, 3- but also for the α-, β-, γ-operations. Conversely, it may be proved that $A_0P_0P_a$ are in line only when P_0 is a point of T_0, and thus that T_0 may be regarded as the locus of a point P_0 such that $A_0P_0P_a$ are in line. The cubic $\mathfrak{T}_0$ (5·3) is also anallagmatic for each of the same six operations, the parts played in the theory of the cubic T_0 by the set of three operations 1-, 2-, 3- (base $A_0B_0C_0$) and the set of three operations α-, β-, γ-, being reversed in the case of $\mathfrak{T}_0$.

8. *Relations between Pole Conics and the Dodecad.* 8·1. Let I, I' denote any two of the four poles, q. T_1, of $A_0B_0C_0$; and let X_0, X_1 denote any two points (not on T_0) harmonically conjugate q. I, I'. Then (7·35, 6) $A_0B_0C_0$ is $p_1(X_0, X_1)$. Generalizing: If P, P' denote any two of the four poles, q. T_n, of any line p, and if Q, Q' denote any two points (not on T_0) harmonically conjugate q. P, P'; then $p_n(Q, Q')$ is p. This theorem is the converse of (6·81).

8·2. The dual of the above theorem is the following: If p, p' denote any two of the four polars, q. Θ_n, of any point P, and if q, q' denote any two lines (not touching Δ_n) harmonically conjugate q. p, p'; then $\varpi_n(q, q')$ is P ($n = 1, 2, 3$).

8·3. Now, whatever 1-conjugate points (base $A_0B_0C_0$) be denoted by Q_0, Q_1; then (7·51) A_1Q_0, A_1Q_1; L_0L_1, L_2L_3 are harmonic. Also (6·4) L_0L_1, L_2L_3 are two of the polars, q. Θ_3, of A_3. Thus (8·2) A_3 is $\varpi_3(A_1Q_0, A_1Q_1)$. In the special case where Q_0, Q_1 are points of T_0, this result may be generalized as follows: If P_0, Q_0 denote any points of T_0, and Q_1 the 1-correspondent of Q_0; then P_3 is $\varpi_3(P_0Q_0, P_0Q_1)$; or again (6·2): *If two of the polars, q. Θ_m, of a given point meet at P_0 (on T_0); then their mixed pole, q. Θ_n, is the n-correspondent of P_0 (m, $n = 1, 2, 3$).*

8·4. Quoting (8·3) in the form: P_3 is $\varpi_3(P_0Q_0, P_0Q_1)$; let us write A_0, B_0 for P_0, Q_0 respectively. Then A_2 is $\varpi_3(A_0B_0C_0, A_1B_1C_1)$, i.e. A_2 is the pole, q. $\Sigma_3(A_0B_0C_0)$, of B_1C_1; similarly B_2, C_2 are respectively the poles, q. $\Sigma_3(A_0B_0C_0)$, of C_1A_1, A_1B_1. Thus the conic q. which $A_1B_1C_1$, $A_2B_2C_2$ are polar triangles is $\Sigma_3(A_0B_0C_0)$. Similarly $A_2B_2C_2$, $A_3B_3C_3$; $A_3B_3C_3$, $A_1B_1C_1$ are pairs of polar triangles q. $\Sigma_1(A_0B_0C_0)$, $\Sigma_2(A_0B_0C_0)$ respectively. Generalizing: *If P_0, Q_0, R_0 denote three collinear points of T_0, and if P_n, Q_n, R_n denote the n-correspondents of P_0, Q_0, R_0 respectively; then the triangles $P_mQ_mR_m$, $P_nQ_nR_n$ are polar to $\Sigma_l(P_0Q_0R_0)$ (l, m, $n = 1, 2, 3$).* Thus F, G, H are the poles of $A_0B_0C_0$ q. $\Sigma_1(A_0B_0C_0)$, $\Sigma_2(A_0B_0C_0)$, $\Sigma_3(A_0B_0C_0)$ respectively; which agrees with (6·1).

8·5. Since A_0 is the meet of B_mC_m, B_nC_n, and similarly for B_0, C_0; therefore (8·4) the polars of A_0, B_0, C_0 q. $\Sigma_l(A_0B_0C_0)$ are A_mA_n, B_mB_n, C_mC_n respectively. Thus A_0, A_m; B_0, B_m; C_0, C_m are apolar to $\Sigma_l(A_0B_0C_0)$, and therefore (1·2) $\Sigma_l(A_0B_0C_0)$, considered as an *order*-conic, is apolar to every conic of the range inscribed in the quadrilateral of which A_0, A_m; B_0, B_m; C_0, C_m are opposite vertices. But (6·2) these conics are pole-conics q. Θ_m, and every pole conic (q. Θ_m) which touches $A_0B_0C_0$ is a conic of the range in question. Thus: *Whatever line may be denoted by u; $\Sigma_l(u)$, regarded as an order-conic, is apolar to every pole-conic, q. Θ_m, which touches u (l, $m = 1, 2, 3$).*

8·6. It follows from (8·4) that when A_1, B_1, C_1 and the conic $\Sigma_3(A_0B_0C_0)$ are given, then the cubic T_0 is determinate, as follows. $A_2B_2C_2$ is the polar triangle, q. $\Sigma_3(A_0B_0C_0)$, of $A_1B_1C_1$;

A_0 is the meet of the lines B_1C_1, B_2C_2; and one and only one cubic contains A_0 and the nine meets of the three lines A_1B_2, B_1C_2, C_1A_2 with the three lines A_1C_2, B_1A_2, C_1B_2. From this standpoint T_0 may be termed the *proper cubic* of the triangle $A_1B_1C_1$ and the conic Σ_3 $(A_0B_0C_0)$; similarly T_0 is the proper cubic of the triangle $A_1B_1C_1$ and the conic Σ_2 $(A_0B_0C_0)$.

9. *Theorems on Apolarity.* 9·1. We have seen (6·4) that two of the polars, q. Θ_1, of A_1 are L_0L_2 and L_3L_1. These lines meet (5) at A_2, and (4·3) are corresponding tangents to Δ_2. Also A_1 represents any point on T_0, and A_2 is the 3-correspondent of A_1; moreover (6·2) the four polars, q. Θ_n, of any point are the common tangents of the pole conics, q. Θ_n, of all lines containing that point. Thus the theorem of (6·4) may be generalized as follows: *If P_0 denote a (fixed) point on T_0, and X any variable point; then, of the common tangents to the range of conics Σ_l (P_0X), two meet at P_m, and are corresponding tangents to Δ_n $(l, m, n = 1, 2, 3)$.*   (9·11)

Writing A_0, B_0 for P_0, X respectively, and putting $l = 3$, $m = 1$, $n = 2$, then the tangents from A_1 to Σ_3 $(A_0B_0C_0)$ are corresponding tangents to Δ_2; i.e. if Q_0 be on T_0 and one of the tangents be $A_1Q_0Q_2$ then the other is $A_1Q_1Q_3$. Hence, and by the symmetry: *The tangents from A_1 to Σ_m $(A_0B_0C_0)$ are corresponding tangents to Δ_n, and conjugates in the* 1-*involution on A_1 $(m, n = 2, 3)$.* Similarly for the tangents from B_1 or C_1 to Σ_m $(A_0B_0C_0)$. But (cf. 3·1) the pole-conics of $A_0B_0C_0$ q. all the class-cubics of the Θ-range constitute a range. Thus: *If Θ_n denote any class-cubic of the Θ-range, and Σ_n $(A_0B_0C_0)$ the pole conic, q. Θ_n, of $A_0B_0C_0$; then the tangent-pairs from A_1 (or B_1 or C_1) to the conics of the range Σ_n $(A_0B_0C_0)$ are conjugate pairs in the* 1-*involution on A_1 (or B_1 or C_1).* Similarly for 2-, 3-.   (9·12)

Again, if x denote any line, and P_0, P_n n-correspondent points on T_0 $(n = 1, 2, 3)$; and if A_nP_0 touch Σ_n (x); then (6·2) A_nP_n touches Σ_n (x). But (7·34) P_n is the n-conjugate (base $A_0B_0C_0$) of P_0. Thus: *Whatever line may be denoted by x; the tangent-pairs from A_n (or B_n or C_n) to Σ_n (x) are conjugate pairs in the n-involution on A_n (or B_n or C_n) $(n = 1, 2, 3)$.*   (9·13)

9·2. Since (7·36) I_0I_a, I_3I_γ are the double lines of the 1-involution on A_1, therefore (9·12) I_0I_a, $I_\beta I_\gamma$ *are apolar to* Σ_n $(A_0B_0C_0)$ $(n = 2, 3)$; similarly for I_0I_β, $I_\gamma I_a$, I_0I_γ, I_aI_β. Therefore (1·2) every conic of the pencil determined by the four points $[I]$ is apolar to Σ_n $(A_0B_0C_0)$; i.e. (6·5) *the polar conic, q. T_1, of every point of $A_0B_0C_0$ is apolar to* Σ_n $(A_0B_0C_0)$. Generalizing: *If x denote any line, and X any point of x; then S_m (X) is apolar to Σ_n (x) $(m, n = 1, 2, 3)$.*   (9·21)

In particular, $O \, O_\gamma$, O_0O_a are apolar to Σ_3 $(A_0B_0C_0)$, and meet at A_2. Thus *the poles, q. Σ_3 $(A_0B_0C_0)$, of $O \, O_\gamma$, O_0O_a are the points (D_1, D_1', say) of $B_1 \, C_1$ contained by $O_0O_a, O_\circ O_\gamma$ respectively.* Moreover, since (7·36) $O \, O_\gamma$, O_0O_a; A_2B_2, A_2C_2 are harmonic; therefore D_1, D_1'; B_1, C_1 *are harmonic.*   (9·22)

9·3. We have seen (8·4) that A_n, B_n, C_n are the poles, q. $\Sigma_l\,(A_0B_0C_0)$, of B_mC_m, C_mA_m, A_mB_m respectively. Hence, reciprocating the pencil $A_m\,(B_mC_n,\ B_nC_m;\ P_0P_n)$ q. $\Sigma_l\,(A_0B_0C_0)$, and projecting from A_m the range so obtained, we may write the theorem (7·72) in the following form: *Given two points P_0, P_n; then the line-pairs A_mP_0, A_mP_n; B_mP_0, B_mP_n; C_mP_0, C_mP_n are simultaneously apolar q. $\Sigma_l\,(A_0B_0C_0)$ if and only if P_0, P_n are n-correspondents on $T_0\,(l,\ m,\ n=1,\ 2,\ 3)$.*

10. *Poloconics.* 10·1. In the generalization of the geometry of circles associated with the triangle a remarkable part is played by the poloconic. The discussion of conics of this type has been postponed because of the simplicity of the treatment by quadratic transformations.

It was known to Cayley* that the envelope (termed by him the lineo-polar envelope) of the polars, q. any cubic, of the points of a given line is a conic. The conic is also the locus of points of which the polar conic, q. the given cubic, touches the given line, and is now generally termed the *pure poloconic* of the line q. the cubic. In the present paper, the pure poloconic, q. T_n, of the line x will be denoted by $C_n\,(x)\,(n=1,\ 2,\ 3)$.

The *mixed poloconic* of two lines seems to have been introduced by Cremona†. It may be defined as the locus of the poles of either of the lines q. the polar conics (q. a given cubic) of all points of the other; or, more symmetrically, as the locus of a point P such that the given lines are apolar to the polar conic (q. the given cubic) of P. The mixed poloconic, q. T_n, of the lines x, y will here be denoted by $C_n\,(x,\ y)\,(n=1,\ 2,\ 3)$.

The more general notion of the *poloconic, q. a given order-cubic, of a given order-conic* appears to be much less familiar. It derives from Hilbert‡, and has been discussed by H. S. White§ and G. Manfredini∥. The poloconic, q. a given order-cubic, of a given order-conic may be defined as the locus of a point P such that the given order-conic is apolar to the polar conic (q. the given cubic) of P; the polar conics being regarded as *class*-conics. The poloconic, q. T_n, of the order-conic s will be denoted by $C_n\,(s)$¶. The pure and mixed poloconics previously defined are clearly included as special cases.

10·2. Now from the general theory of quadratic transformations or otherwise, it is evident that the 1-transformation (base $A_0B_0C_0$) of any line meeting T_0 at P_0, Q_0, R_0, and not containing A_1 or B_1 or C_1 is a (non-degenerate) conic containing A_1, B_1 and C_1

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* *Loc. cit.* (6·2).
† *Teoria geom. d. curve piane,* Bologna, 1862, p. 111.
‡ "Lettre adressée à M. Hermite," *Liouville,* (4), 4 (1888), p. 249.
§ *Trans. Amer. Math. Soc.* 1 (1900), p. 1.
∥ *Giorn. di Mat.* 39 (1901), p. 145.
¶ If $T_n \equiv a_x^3 \equiv b_x^3 = 0$, and $s \equiv a_x^2 = 0$, then $C_n\,(s) \equiv a_x b_x\,(aba)^2 = 0$.

and the 1-correspondents of P_0, Q_0, R_0. The conic is, in fact, the locus of the poles of the line $P_0Q_0R_0$ q. the conics of the pencil determined by the four points $[I]$ as base-points; and the conics of that pencil are (6·5) the polar conics, q. T_1, of the points of $A_0B_0C_0$. Generalizing: *The n-transformation (base $A_0B_0C_0$) of any line $P_0Q_0R_0$ is $C_n\,(A_0B_0C_0,\ P_0Q_0R_0)\ (n = 1,\ 2,\ 3)$.* (10·21)

The mixed poloconic, q. T_n, of two lines thus meets T_0 at the n-correspondents of the common points of T_0 and those lines. It is conversely true that if a conic (C_n, say) meets T_0 at the n-correspondents of the points common to T_0 and one line, then the remaining common points of C_n and T_0 are the n-correspondents of the points common to T_0 and another line, and that C_n is the mixed poloconic, q. T_n, of the two lines. In particular, $C_n\,(x,\,y)$ touches T_0 if and only if the lines x, y meet on T_0. (10·22)

In (10·21). regarding $A_0B_0C_0$ as fixed, and letting $P_0Q_0R_0$ tend to coincidence with $A_0B_0C_0$, we have the theorems: The n-transformation (base $A_0B_0C_0$) of $A_0B_0C_0$ is $C_n\,(A_0B_0C_0)$; and: The pure poloconic, q. T_n, of any line x touches T_0 at the n-correspondents of the common points of T_0 and x; and conversely: Given any three points on T_0 which are the n-correspondents of three points in line, then there is a conic which touches T_0 at the three given points. A conic may thus have triple contact of one or other of three distinct species (corresponding to the three values of n) with a given cubic.

(10·23)

From (7·2, 10·21): *If k denote a fixed line, and x a variable line containing a fixed point, then the conics $[C_n\,(k,x)]$ constitute a pencil projective with the pencil $[x]$.* (10·24)

From the definition, $C_n\,(A_0B_0C_0)$ is the envelope of the polars, q. T_n, of the points X_0 of $A_0B_0C_0$; also (10·23) $C_n\,(A_0B_0C_0)$ is the n-transformation (base $A_0B_0C_0$) of $A_0B_0C_0$, i.e. the locus of the n-conjugates X_n of the points X_0. But (7·33) X_n is the pole of $A_0B_0C_0$ q. $S_n\,(X_0)$; thus: The polar, q. T_n, of any point of a line k touches $C_n\,(k)$ at the n-conjugate (base k) of the given point.

(10·25)

Further, if a (variable) line x meet k at a (fixed) point K_0; then $C_n\,(k,\ x)$, being the n-transformation (base k) of x, contains the n-conjugate (K_n, say) of K_0; and (10·25) K_n is the point of contact of $p_n\,(K_0)$ with $C_n\,(k)$; similarly $C_n\,(k,\ x)$ contains the point of contact of $p_n\,(K_0)$ with $C_n\,(x)$. Thus: The mixed poloconic, q. any cubic, of any two lines contains the points of contact of the pure poloconics (q. that cubic) of the two lines with the polar (q. that cubic) of the meet of the two lines. (10·26)

The mixed poloconic, q. T_n, of two lines is in general a non-degenerate conic. Let us consider, however, the case in which one of the two lines (x) meets T_0 at P_0, Q_0, R_0 and the other (x') meets T_0 at P_n, Q_0', R_0', where P_0, P_n are n-correspondents ($n = 1,\ 2,\ 3$).

Then (10·22) $C_n(x, x')$ contains the points P_n, Q_n, R_n, P_0, Q_0', R_n', where Q_n, Q_n', R_n, R_n' are the n-correspondents of Q_0, Q_0', R_0, R_0' respectively. But since P_0, Q_0, R_0; P_n, Q_0', R_0' are triads of points in line; therefore (5·1) P_0, Q_n, R_n; P_n, Q_n', R_n' are triads of points in line. Thus it is a sufficient condition for the degeneracy of $C_n(x, x')$ that each of the lines x, x' contains the n-correspondent of a point common to T_0 and the other. It is easy to see that the condition is also necessary. Hence: The mixed poloconic, q. T_n, of two lines degenerates into two lines if and only if each of the given lines contains the n-correspondent of a point common to T_0 and the other ($n = 1, 2, 3$). (10·27)

10·3. Alternative synthetic proofs of the existence of the pure and mixed poloconics may be found in the text-books. The proofs, by the authors cited, of the existence of the poloconic of a conic depend on the theory of invariants. A synthetic proof here follows.

Let P_0, Q_0, R_0 denote any three of the common points of T_0 and any conic s. Let R_0P_0, P_0Q_0 meet T_0 again at V_0, W_0 respectively; let P_1, Q_1, R_1, V_1, W_1 be the 1-correspondents of P_0, Q_0, R_0, V_0, W_0 respectively; and let X denote a variable point of s.

Then (10·21) $C_1(P_0Q_0W_0, R_0X)$ contains the four (fixed) points P_1, Q_1, R_1, W_1; and $C_1(P_0R_0V_0, Q_0X)$ contains the four (fixed) points P_1, Q_1, R_1, V_1. Let the fourth common point of the two conics be Y. Then the line-pairs P_0Q_0, R_0X; P_0R_0, Q_0X are (10·1) apolar to $S_1(Y)$; and hence the (order-) conic s (of the pencil determined by P_0, Q_0, R_0, X as base-points) is (1·2) apolar to $S_1(Y)$.

Conversely, whatever point may be denoted by Y, then the conic $P_1Q_1R_1W_1Y$ is the mixed poloconic, q. T_1, of the line $P_0Q_0W_0$ and some line r containing R_0; and the conic $P_1Q_1R_1V_1Y$ is the mixed poloconic, q. T_1, of the line $P_0R_0V_0$ and some line q containing Q_0. If X denote the meet of q, r; then we may prove by exhaustion that, if now Y be such that s is apolar to $S_1(Y)$, then X is a point of s, unless the lines Q_0R_0, R_0P_0, P_0Q_0 contain P_1, Q_1, R_1 respectively. In that case P_0, Q_0, R_0, P_1, Q_1, R_1 are the vertices of a quadrilateral of the first species (6·2) inscribed in T_0; P_1, Q_1, R_1 are in line; and the conic s is the mixed poloconic, q. T_1, of the line $P_1Q_1R_1$ and some other line. This case has already been considered (10·21). In the non-degenerate case now under consideration, X must be a point of s.

Again, as X varies on s, then (10·24) the pencil of conics $C_1(P_0Q_0W_0, R_0X)$ is projective with the pencil of lines R_0X; and the pencil of conics $C_1(P_0R_0V_0, Q_0X)$ is projective with the pencil of lines Q_0X; but since Q_0, R_0 are points of the conic s, therefore $Q_0[X] \overline{\wedge} R_0[X]$; thus the two pencils of conics are projective. Now every conic of both pencils contains the three points P_1, Q_1, R_1; there is therefore a conic $(P_1Q_1R_1V_1W_1)$ common to the

two pencils. Moreover, this conic is self-correspondent in the projectivity; for it may be regarded either as $C_1 (P_0 Q_0 W_0, R_0 P_0)$, or as $C_1 (P_0 R_0 V_0, Q_0 P_0)$.

Now in any quadratic transformation in which P_1, Q_1, R_1 are singular points, let V_0', W_0' be respectively conjugate to V_1, W_1. Then the transformation of the projective pencils of conics $[C_1 (P_0 Q_0 W_0, R_0 X)]$ and $[C_1 (P_0 R_0 V_0, Q_0 X)]$ are projective pencils of lines $[w_0']$, $[v_0']$, say, of which the vertices are W_0', V_0' respectively; moreover, since the conic $P_1 Q_1 R_1 V_1 W_1$ is self-correspondent in the projectivity $[C_1 (P_0 Q_0 W_0, R_0 X)] \barwedge [C_1 (P_0 R_0 V_0, Q_0 X)]$, therefore the line $W_0' V_0'$ is self-correspondent in the projectivity $[w_0'] \barwedge [v_0']$. Thus $[w_0'] \overline{\barwedge} [v_0']$; the meets of homologous lines of the two pencils $[w_0']$, $[v_0']$ are therefore in a line, y say; and transforming back, the locus of the point Y is (7·2) a conic, s' say (the transformation of y), containing the points P_1, Q_1, R_1.

But if S_0 denote *any* one of the three points, other than P_0, Q_0, R_0, common to T_0 and s; then the fourth common point of $C_1 (P_0 Q_0 W_0, R_0 S_0)$ and $C_1 (P_0 R_0 V_0, Q_0 S_0)$ is (10·21) the 1-correspondent of S_0. Summing up, and generalizing: The locus of a point Y such that a given order-conic s is apolar to $S_n (Y)$ considered as a class-conic is a conic (s') which contains the n-correspondents of the six points common to s and T_0 $(n = 1, 2, 3)$.

Since s contains the n-correspondents of the six points common to s' and T_0; therefore s is $C_n (s')$. The relation between a conic and its poloconic is thus reciprocal.

10·4. If s_1, s_2 denote any two order-conics; if s_1', s_2' denote $C_n (s_1)$, $C_n (s_2)$ respectively; and if P' denote any point common to s_1', s_2'; then s_1 and s_2, and therefore (1·2) every conic of the pencil determined by s_1 and s_2, are apolar to $S_n (P')$. P' is thus a point of the poloconic, q. T_n, of every conic of that pencil. Hence: The poloconics, q. any order cubic, of the conics of a pencil, themselves constitute a pencil.   (10·41)

The relation between the conics of the two pencils is one—one and reciprocal; the pencils are therefore projective.   (10·42)

If the conic s_2 (say) degenerates into a line $(p,$ say$)$ counted twice; then the conics of the pencil determined by s_1, s_2 become the conics which have double contact, on p, with s_1. The corresponding degenerate form of (10·41) is the following: If any number of conics have double contact on a given line; then their poloconics, q. any order-cubic, constitute a pencil, one conic of which is the pure poloconic of the given line.   (10·43)

If both the conics s_1, s_2 degenerate into lines $(p_1,\ p_2,$ say$)$ counted twice; then the remaining conics of the pencil degenerate into line-pairs harmonic q. p_1, p_2. The corresponding degenerate form of (10·41) is the following: If two line-pairs p_1, p_2; q_1, q_2 be harmonic; then the mixed poloconic, q. any order-cubic, of q_1, q_2

is a conic of the pencil determined by the pure poloconics of p_1, p_2. (10·44)

10·5. From the general theory of quadratic transformations, the transformation of a conic which does not contain any one of the three singular points is a quartic. The 1-transformation (base $A_0B_0C_0$) of a conic (not containing either A_1 or B_1 or C_1) therefore meets the poloconic (q. T_1) of the given conic at eight points. Of these, six are (7·34, 10·3) the 1-correspondents of the common points of T_0 and the given conic. We proceed to determine the other two.

Let s denote a conic which meets T_0 at six points P_0, none of which is A_1 or B_1 or C_1; and let P_1 be the 1-correspondent of P_0, and therefore the 1-conjugate (base $A_0B_0C_0$) of P_0. Let $A_0B_0C_0$ be denoted by u, and let s meet u at the points Q_0, R_0. Then the points A_1, B_1, C_1, Q_0, R_0 determine a conic (s', say) which meets s in two further points (Q_0', R_0', say). Let u' be the line determined by Q_0', R_0'. Since Q_0, R_0, Q_0', R_0' are contained by a circumconic of $A_1B_1C_1$; therefore (10·21) Q_1, R_1, Q_1', R_1', the 1-conjugates of Q_0, R_0, Q_0', R_0' respectively, are in a line, v, say; and C_1 (s') consists of the lines u, v. Also C_1 (u, u') is (10·21) the conic $A_1B_1C_1Q_1'R_1'$. Thus Q_1', R_1' are common to the conics C_1 (s'), C_1 (u, u') and therefore (10·41) to the poloconic (q. T_1) of every conic of the pencil determined by s' and (u, u'). In particular, Q_1', R_1' are points of C_1 (s); and therefore Q_0', R_0' are the two points, not contained by T_0, of which the 1-conjugates (base $A_0B_0C_0$) are common to $C_1(s)$ and the 1-transformation of s. Thus: *There are (in general) two points, not contained by T_0, of a conic s, of which the n-conjugates (base $A_0B_0C_0$) are contained by the poloconic (q. T_n) of s. If s meet $A_0B_0C_0$ at Q_0, R_0, and s' be the conic containing A_n, B_n, C_n, Q_0, R_0; then the two points in question are the two remaining points common to s and s' ($n = 1, 2, 3$).*

10·6. The poloconic, q. a given class-cubic, of a given class-conic may be defined as the envelope of a line p such that the given class-conic is apolar to the pole conics (q. the given class-cubic) of p; the pole conics being regarded as *order*-conics. The poloconic, q. Θ_n, of the class-conic σ will be denoted by $\Gamma_n(\sigma)$. The mixed poloconic, q. Θ_n, of two points (U, V) and the pure poloconic, q. Θ_n, of a point (U) (which need not separately be defined) will be denoted by $\Gamma_n(U, V)$ and $\Gamma_n(U)$ respectively. The theorems for such conics, dual to those treated above, need not be separately stated.

11. *Relations between Conics associated with the T-pencil and the Θ-range.* 11·1. Let P_0 denote any one of the six points common to T_0 and any order-conic s; and let P_n denote the n-correspondent of P_0 ($n = 1, 2, 3$). Then (10·3) $C_m(s)$, $C_n(s)$ meet T_0 at P_m, P_n respectively; and (4·1) P_m, P_n are themselves l-correspondents. Thus: *The poloconics, q. T_m and T_n, of any given conic are mutually poloconics q. T_l ($l, m, n = 1, 2, 3$).*

11·2. The polar conic, q. T_1, of A_0 consists (4·4) of the two
lines $A_1L_0L_1$, $A_1L_2L_3$; the six points common to T_0 and $S_1(A_0)$
are thus A_1 counted twice, and the four points $[L]$. $C_1\{S_1(A_0)\}$
meets T_0 at the 1-correspondents of these six points; viz. at A_0
counted twice, and (4·1) at the four points $[L]$. Since $C_1\{S_1(A_0)\}$
contains the four points $[L]$, therefore (3·1) it is the polar conic
of A_0 q. some cubic of the T-pencil; and since $C_1\{S_1(A_0)\}$ touches
T_0 (at A_0) it follows that that cubic is T_0. Thus $C_1\{S_1(A_0)\}$ is
$S_0(A_0)$; similarly $C_1\{S_1(B_0)\}$, $C_1\{S_1(C_0)\}$ are respectively $S_0(B_0)$,
$S_0(C_0)$. Also, if X denote any point of $A_0B_0C_0$, then (3·1) the conics
$S_1(X)$ constitute a pencil; and (10·42) the conics $C_1\{S_1(X)\}$ there-
fore constitute a pencil projective with the first pencil. It follows
that $C_1\{S_1(X)\}$ is $S_0(X)$ whatever point of $A_0B_0C_0$ may be denoted
by X; i.e. the theorem is true if X denotes any point whatever.
Thus: The polar conics, q. T_0 and T_n, of any given point are mutually
poloconics q. T_n ($n = 1, 2, 3$).   (11·21)

Hence $C_n\{S_n(X)\}$ is $S_0(X)$, and $C_m\{S_0(X)\}$ is $S_m(X)$. But
(11·1) $C_m\{C_n(s)\}$ is $C_l(s)$, whatever conic may be denoted by s;
i.e.: The polar conics, q. T_m and T_n, of any given point are mutually
poloconics q. T_l ($l, m, n = 1, 2, 3$).   (11·22)

11·3. Again (3·1) the polar conics of any given point q. the
cubics of the T-pencil themselves constitute a pencil; and (11·2)
the poloconics, q. T_n, of that pencil constitute the same pencil.
The pencil of conics is therefore (10·42) an involution pencil, and
two conics of the pencil must be self-poloconics, or, as they have
been termed, *autopoloconics*, q. T_n ($n = 1, 2, 3$). White* and
Manfredini* have shewn analytically that these are the poloconics
of the given point q. the two cubics, other than T_n, of which Δ_n is
the Cayleyan. Our notation enables us to give an easy synthetic
proof of the dual theorem.

For (cf. 10·3) a class-conic σ is an autopoloconic q. Θ_1 if and
only if the six tangents common to σ and Δ_1 are three pairs of
corresponding tangents; and we have already (9·12) shewn that the
tangents from A_1, B_1, and C_1 to $\Sigma_3(A_0B_0C_0)$ are corresponding
tangents to Δ_2. Thus $\Sigma_3(A_0B_0C_0)$, that is, the pole conic q. Θ_3
of *any* line, is an autopoloconic q. Θ_1; similarly the pole conic
q. Θ_2 of any line is an autopoloconic q. Θ_1. Conversely, if any
(class-) conic σ be an autopoloconic q. Θ_1, then the six common
tangents of σ and Δ_1 are three pairs of corresponding tangents to
Δ_1; and (4·3) the meet of each pair is a point of T_0. Without loss
of generality we may denote two of these points by B_2, C_2, and the
pairs of corresponding tangents to Δ_1, which meet at B_2, C_2, by
b, b'; c, c' respectively. Thus σ is a conic of the range determined
by b, b', c, c' as base-lines; and of the conics of this range we know
that two, viz. $\Sigma_3(A_0B_0C_0)$ and $\Sigma_2(A_0B_1C_1)$ are autopoloconics q. Θ_1.

* *Loc. cit.* (10·1).

If σ be a conic distinct from these two, then, by the involution, the harmonic conjugate with respect to them of σ is $\Gamma_1(\sigma)$, which is thus distinct from σ. Hence no conics of the range, other than the two specified, are autopoloconics q. Θ_1. Generalizing: The pole conics of every line q. Θ_m and Θ_n, and no other conics, are autopoloconics q. Θ_l (l, m, $n = 1$, 2, 3). (11·31)

In particular, any two m-correspondent points of T_0 constitute (4·4) a pole conic q. Θ_m. Thus: *Any two m-correspondent points of T_0 constitute their own mixed poloconic q. Θ_n (m, $n = 1, 2, 3$).* (11·32)

Now let σ denote any class-conic which is not an autopoloconic q. Θ_1, and σ' the poloconic, q. Θ_1, of σ. Then σ, σ' determine a range of conics; and since (cf. 10·3) each of the conics σ, σ' is the poloconic, q. Θ_1, of the other, therefore the poloconics, q. Θ_1, of the conics of the range constitute the same range, which is thus (cf. 10·42) an involution range. Two autopoloconics q. Θ_1 are therefore included in the range. If both of these were pole conics q. Θ_2, then (cf. 3·1) every conic of the range would be a pole conic q. Θ_2, and therefore an autopoloconic q. Θ_1, contrary to assumption. Thus not more than one of the autopoloconics of the range can be a pole conic q. Θ_2, and similarly not more than one can be a pole conic q. Θ_3; i.e. of the two autopoloconics of the range one is a pole conic q. Θ_2, and the other a pole conic q. Θ_3. We may denote the two conics by $\Sigma_2(y)$ and $\Sigma_3(z)$; and without loss of generality we may represent z by $A_0B_0C_0$. Now (9·13) the tangents from A_2 to $\Sigma_2(y)$ are conjugate lines in the 2-involution on A_2 (7·36); and (9·12) the tangents from A_2 to $\Sigma_3(A_0B_0C_0)$ are conjugate lines in that involution. Thus the involution pencil, vertex A_2, of which conjugate lines are the pairs of tangents from A_2 to the conics of the range determined by $\Sigma_2(y)$ and $\Sigma_3(A_0B_0C_0)$, is the 2-involution on A_2; similarly for B_2, C_2. The point-pairs of the range of conics are therefore 2-conjugate points (base $A_0B_0C_0$); i.e. (7·35) $A_0B_0C_0$ is the mixed polar, q. T_2, of any one of the three point-pairs of the range. Hence (3·3) $A_0B_0C_0$ is the polar, q. T_2, of every conic of the range, and, in particular, of σ. Generalizing: *The range of conics determined by any given class-conic (σ) and its poloconic, q. Θ_l, is an involution range, of which the double conics are $\Sigma_m(y)$ and $\Sigma_n(z)$*[*]; *where y, z are the polars, q. T_n, T_m respectively, of σ and of every other conic of the range (l, m, $n = 1$, 2, 3).* Such a range of conics will be termed an *autopolo-range q. Θ_l*, or briefly a *l-autopolo-range*. (11·33)

If P, P' denote any point pair of the range, then $\Gamma_l(P, P')$ is a conic of the range; which may thus be regarded as determined by (P, P') and $\Gamma_l(P, P')$. Hence[†]: *$\Sigma_m\{p_n(P, P')\}$ is a conic of*

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[*] The dual of this part of the theorem is proved analytically by Manfredini, *loc. cit.* (10·1).

[†] Unless P, P' are n-correspondents on T_0, in which case $p_n(P, P')$ is indeterminate.

the range determined by (P, P') and $\Gamma_l (P, P')$; in particular $\Sigma_m (A_0 B_0 C_0)$ *is a conic of the range determined by* (P_0, P_n) *and* $\Gamma_l (P_0, P_n)$, *where* P_0, P_n *denote any pair of n-conjugate points (base* $A_0 B_0 C_0$) (11·34); and again: *Whatever line may be denoted by y; if* P_0 *be the meet of two common tangents of* $\Sigma_m (A_0 B_0 C_0)$ *and* $\Sigma_n (y)$, *then* P_n *is the meet of the other two common tangents* $(l, m, n = 1, 2, 3)$.

(11·35)

11·4. We have seen (11·3) that, if Θ_l denote any class-cubic, and Θ_m, Θ_n the other two class-cubics having the same Cayleyan as Θ_l; then a range determined by two conics, of which one is the pole conic of any line q. Θ_m, and the other the pole conic of any line q. Θ_n, is an autopolo-range q. Θ_l. Dually, if T_n denote any order-cubic, and T_n', T_n'' the other two order-cubics which have the same Cayleyan as T_n; then a pencil determined by two conics, of which one $\{S_n' (X)\}$ is the polar conic of any point q. T_n', and the other $\{S_n'' (Y)\}$ is the polar conic of any point q. T_n'', is such that the poloconic, q. T_n, of any conic of the pencil is a conic of the pencil, and that $S_n' (X)$, $S_n'' (Y)$ are the double conics of the pencil. Such a pencil of conics will be termed an *autopolo-pencil q.* T_n, or briefly an *n-autopolo-pencil* $(n = 1, 2, 3)$. Any order-conic and its polo-conic q. T_n determine such a pencil. (11·41)

The case already (11·3) referred to, where the pencil consists of the polar conics of a given point (X) q. all the cubics of the T-pencil is of special interest. Such a pencil of conics is an autopolo-pencil q. every cubic of the T-pencil. If the cubic T_1 be selected; then (11·2) $S_0 (X)$, $S_1 (X)$; $S_2 (X)$, $S_3 (X)$ are conjugate pairs in the pencil of conics $(\Pi_1$, say); and (with the notation above) $S_1' (X)$, $S_1'' (X)$ are the double conics. Thus, of the three involution pencils of conics determined by pairing in the three possible ways the four conics $S_0 (X)$, $S_1 (X)$, $S_2 (X)$, $S_3 (X)$; the three pairs of conics $S_n' (X)$, $S_n'' (X)$ $(n = 1, 2, 3)$ are the double conics. Hence (e.g.) $S_2' (X)$, $S_2'' (X)$; $S_3' (X)$, $S_3'' (X)$ are conjugate pairs in the pencil Π_1. (11·42)

11·5. We may obtain the mixed poloconic of a pair of lines as the reciprocal, q. the polar conic of their meet, of a certain pole conic.

For, whatever point may be denoted by X, then (6·2) $\Sigma_1 (FX)$ touches the four lines $A_0 B_0 C_0$, $A_0 B_1 C_1$, $A_1 B_0 C_1$, $A_1 B_1 C_0$; also, if Y denote any point of $A_0 B_0 C_0$, then (6·5) $S_1 (Y)$ contains the four points $[I]$, and $A_1 B_1 C_1$, the diagonal triangle of the four points $[I]$, is therefore apolar to $S_1 (Y)$. Thus the reciprocal, q. $S_1 (Y)$, of $\Sigma_1 (FX)$, contains the points A_1, B_1, C_1, and is therefore (10·21) the poloconic, q. T_1, of $A_0 B_0 C_0$ and some other line. But for our present purpose $A_0 B_0 C_0$ simply denotes either of the tangents from Y to a certain pole conic q. Θ_1. Thus the poloconic in question is that of $A_0 B_0 C_0$ and the other tangent from Y to that pole conic. Generaliz-

ing: *If p, p' denote the tangents from any point P to any pole conic $q.$ Θ_n; then the reciprocal, $q.$ $S_n(P)$, of that pole conic is $C_n(p, p')$ $(n = 1, 2, 3)$.*

11·6. We shall now shew that the mixed poloconic (q. a given order-cubic) of a certain line-pair is also the pure poloconic (q. the apolar class-cubic) of a certain point.

Let the four polars, $q.$ Θ_1, of A_1, B_1, C_1 be termed respectively the four lines $[l]$, the four lines $[m]$, and the four lines $[n]$. The lines $[l]$ are (6·4) L_0L_2, L_0L_3, L_1L_2, L_1L_3. If X denote a point such that $\Gamma_1(X)$ contains A_1, then (10·6; cf. 10·1) X must be a point of one or other of the lines $[l]$; similarly for B_1, C_1 and the lines $[m]$, $[n]$ respectively. Thus $\Gamma_1(X)$ circumscribes the triangle $A_1B_1C_1$ if and only if a line l, a line m and a line n are concurrent at X. But (5·5) any one of the eight points $[O]$, $[U]$ is such a point of concurrence; and it is easy to see that there are no other such points. Thus: *There are eight and only eight pure poloconics, $q.$ Θ_1, which circumscribe the triangle $A_1B_1C_1$; these eight conics consist of two sets of four conics each, viz. the poloconics, $q.$ Θ_1, of the four points $[O]$ and those of the four points $[U]$.* (11·61)

Since $\Gamma_1(O_0)$ contains A_1, B_1, C_1; therefore (10·21) $\Gamma_1(O_0)$, regarded as an order-conic, is $C_1(A_0B_0C_0, x)$, where x is some line to be determined. But (6·5) $A_0B_0C_0$ is $p_2(O_0)$; thus, from the symmetry, x must be $p_3(O_0)$. Generalizing: *Whatever point may be denoted by P; $\Gamma_l(P)$, regarded as an order-conic, is $C_l\{p_m(P), p_n(P)\}$ $(l, m, n = 1, 2, 3)$.* (11·62)

Again, whatever 2-conjugate points (base $A_0B_0C_0$) may be denoted by P_0, P_2; $\Gamma_1(P_0, P_2)$ is (11·34) a conic of the range of class-conics determined by $\Sigma_3(A_0B_0C_0)$ and the point-pair P_0, P_2. Making P_0, P_2 coincident we have (7·36): *$\Gamma_1(O_\lambda)$ has double contact with $\Sigma_3(A_0B_0C_0)$, pole O_λ ($\lambda = 0, \alpha, \beta, \gamma$)*; or more generally: *Whatever point may be denoted by P; $\Gamma_l(P)$ has double contact with $\Sigma_m\{p_n(P)\}$, pole P ($l, m, n = 1, 2, 3$).* (11·63)

12. *F- and Φ-Conics of Pole and Polar Conics.* 12·1. The further relations which we shall require between the conics associated with the T-pencil and the Θ-range involve the F- and Φ-conics of Salmon*. It will be recalled that the Φ-conic of two order-conics is that class-conic which is the envelope of a line meeting the given conics in pairs of points which separate each other harmonically; and reciprocally, that the F-conic of two class-conics is that order-conic which is the locus of a point from which the pairs of tangents to the given conics separate each other harmonically. The Φ-conic of two conics therefore touches the eight tangents which touch the given conics at their common points; and the F-conic of two conics contains the eight points of contact, with the given conics, of their

* *Conic Sections*, ed. 10 (1896), pp. 343 ff.

four common tangents. We shall denote the F-conic of two class-conics σ, σ' by $F(\sigma, \sigma')$; and the Φ-conic of two order-conics, s, s' by $\Phi(s, s')$*.   (12·11)

Much use will be made of the well-known theorems (12·12—12·16) which follow:

The Φ-conics of a given conic and all the conics of a given pencil constitute a range which is projective with the given pencil; and reciprocally for F-conics.   (12·12)

If a conic s (regarded as an order-conic) be apolar to a conic s' (regarded as a class-conic); then the reciprocal, q. s, of s' is $F(s, s')$; and the reciprocal, q. s', of s is $\Phi(s, s')$.   (12·13)

Given two harmonic pairs of conics of a pencil; then the Φ-conic of either pair is a conic of the *range* determined by the other pair; and reciprocally for F-conics.   (12·14)

The F-conic of a class-conic and a point-pair contains the point-pair; and reciprocally for the Φ-conic.   (12·15)

The Φ-conic of two line-pairs $(p, p'; q, q')$ is a conic which touches all four lines; if p, p' meet at P, then the points of contact with p, p' of the Φ-conic are contained by the polar of P q. the line-pair (q, q').   (12·16)

We shall prove two further theorems on the F-conic. Let P denote any point on the F-conic of two class-conics σ_1, σ_2; let p be the polar, q. σ_1, of P; and let σ_m, σ_n be any conjugate pair of conics of the range determined by σ_1, σ_2 as double conics. Let r_m, r_m' be the tangents from P to σ_m, and R_m, R_m' respectively the poles, q. σ_1, of r_m, r_m'. Now (12·11) the tangent-pairs from P to σ_1, σ_2 separate each other harmonically; therefore, by reciprocation q. σ_1; PR_m, PR_m' (say r_n, r_n') are conjugate in the involution pencil determined at P by the range of conics, and thus r_n, r_n' are the tangents from P to some conic of the range. Moreover, this conic must be σ_n. For let r_1, r_1'; r_2, r_2' be the tangent pairs from P to σ_1, σ_2 respectively; we need only prove that the polars of some one point Q (other than P) q. the line-pairs r_1, r_1'; r_2, r_2': r_m, r_m'; r_n, r_n' are harmonic. Taking Q on r_1, the proof is easy. But r_n, r_n' are the polars, q. σ_1, of the meets of p with r_m, r_m'. Thus: *If P be any point of the F-conic of two class-conics σ_1, σ_2; if p be the polar, q. σ_1, of P; and if σ_m, σ_n be any conjugate pair of conics of the range determined by σ_1, σ_2 as double conics: then the tangents from P to σ_m meet p on the reciprocal (q. σ_1) of σ_n.*   (12·17)

In particular, if Q, Q' denote any one of the three point-pairs of the range, and σ_m that conic of the range which is conjugate to the point-pair Q, Q'; then the tangents from any point P of the F-conic to σ_m meet p, at points, R, R', say, on the polars, q. σ_1, of Q, Q'. If now P denote either of the points common to the F-conic and the line QQ', then R, R' coincide (at the pole, q. σ_1, of QQ'), and

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* If $s \equiv a_x{}^2 = 0$, and $s' \equiv a_x'{}^2 = 0$, then $\Phi(s, s') \equiv (aa'u)^2 = 0$.

σ_m therefore touches the F-conic at P. Thus: *The F-conic of two class-conics has double contact, on the join of any one of the point-pairs of the range determined by the two class-conics, with that class-conic of the range which is harmonically conjugate (q. the two class-conics) to the given point-pair.* (12·18)

12·2. We shall now prove a theorem regarding the Φ-conic of two polar conics.

From (6·4) two of the polars, q. Θ_1, of B_1 are M_0M_2, M_1M_3; and two of the polars, q. Θ_1, of C_1 are N_0N_3, N_1N_2. Thus (6·2) $\Sigma_1(B_1C_1)$ touches the four lines M_0M_2, M_1M_3, N_0N_3, N_1N_2. (α)

Also (8·4), whatever points of T_0 may be denoted by Y_0, Z_0, the polar of Y_3 q. $\Sigma_1(Y_0Z_0)$ is Y_0Z_2; where Y_3, Z_2 denote the 3-, 2-correspondents of Y_0, Z_0 respectively. Writing B_1, C_1 for Y_0, Z_0 respectively, and remembering (4·1), the polar of B_2 q. $\Sigma_1(B_1C_1)$ is B_1C_3; similarly the polar of C_3 q. $\Sigma_1(B_1C_1)$ is C_1B_2. (β)

But (4·4) the line-pairs M_0M_2, M_1M_3; N_0N_3, N_1N_2 respectively constitute $S_2(B_0)$, $S_3(C_0)$. Therefore (12·16) the Φ-conic of $S_2(B_0)$, $S_3(C_0)$ touches all four lines M_0M_2, M_1M_3, N_0N_3, N_1N_2. Also (5·4) B_2, C_3 are the meets of M_0M_2, M_1M_3; N_0N_3, N_1N_2 respectively. Again, the polar (p, say) of B_2 q. the line-pair $S_3(C_0)$ contains C_3, the meet of that line-pair; and since (3·2) p is also the polar of C_0 q. $S_3(B_2)$, which (4·1) consists of a line-pair meeting at B_1, therefore p contains B_1. Thus the polar of B_2 q. the line-pair N_0N_3, N_1N_2 is B_1C_3; similarly the polar of C_3 q. the line-pair M_0M_2, M_1M_3 is C_1B_2. Therefore (12·16) C_1B_2, B_1C_3 contain the points of contact of the two line-pairs M_0M_2, M_1M_3; N_0N_3, N_1N_2 with their Φ-conic, which must thus (α, β) be $\Sigma_1(B_1C_1)$. Generalizing the notation: If Q_0, R_0 denote any two points of T_0, and Q_1, R_1 the 1-correspondents of Q_0, R_0 respectively; then $\Phi\{S_2(Q_0), S_3(R_0)\}$ is $\Sigma_1(Q_1R_1)$. (γ)

Further, since $S_1(Q_0)$, $S_1(R_0)$ degenerate into line-pairs meeting at Q_1, R_1 respectively; therefore Q_1R_1 is $p_1(Q_0, R_0)$. (δ)

In particular, if Q_0 denote any point of T_0, and Q_1 the 1-correspondent of Q_0; then $\Phi\{S_2(Q_0), S_3(A_0)\}$; $\Phi\{S_2(Q_0), S_3(B_0)\}$: $\Phi\{S_2(Q_0), S_3(C_0)\}$ are respectively $\Sigma_1(Q_1A_1)$, $\Sigma_1(Q_1B_1)$, $\Sigma_1(Q_1C_1)$. (ϵ)

But (3·1) if R_0 now denote a (variable) point of $A_0B_0C_0$, then the conics $S_3(R_0)$ constitute a pencil; thus (12·12) as R_0 varies on $A_0B_0C_0$, while Q_0 remains fixed, the conics $\Phi\{S_2(Q_0), S_3(R_0)\}$ constitute a range projective with the pencil $S_3(R_0)$. (ζ)

Also (ϵ) three of the conics, and thus (cf. 3·1) all the conics, of the range are pole conics, q. Θ_1, of some line containing Q_1; let R_1 denote any (variable) point such that $\Phi\{S_2(Q_0), S_3(R_0)\}$ is $\Sigma_1(Q_1R_1)$. (η)

Again (v. δ), the polar, q. $S_1(R_0)$, of Q_0 is a line containing Q_1: the polars of Q_0 q. the pencil $[S_1(R_0)]$ therefore constitute a pencil, $Q_1[R_1']$, say, projective with the pencil $[S_1(R_0)]$. (θ)

Now (cf. 3·1) $\qquad Q_1\,[R_1]\; \overline{\wedge}\; [\Sigma_1\,(Q_1R_1)],$

$(\zeta,\,\eta)$ $\qquad\qquad [\Sigma_1\,(Q_1R_1)]\; \overline{\wedge}\; [S_3\,(R_0)],$

$(3\cdot1)$ $\qquad\qquad\quad [S_3\,(R_0)]\; \overline{\wedge}\; [S_1\,(R_0)],$

(θ) $\qquad\qquad\quad [S_1\,(R_0)]\; \overline{\wedge}\; Q_1\,[R_1'],$

$\therefore$ $\qquad\qquad\qquad Q_1\,[R_1]\; \overline{\wedge}\; Q_1\,[R_1'].$ $\qquad\qquad(\iota)$

Moreover, if R_0 coincides with A_0, B_0, C_0, then (ϵ) Q_1R_1 coincides with Q_1A_1, Q_1B_1, Q_1C_1 respectively, and $(v.\ \delta)$ the like is true of Q_1R_1'. Therefore (ι) the lines Q_1R_1, Q_1R_1' are identical in all cases. Also $A_0B_0C_0$ represents any line, and therefore R_0 may be taken to denote any point whatever. Hence $(\eta,\ \theta)$: If Q_0 denote any point of T_0, and R_0 an arbitrary point; then $\Phi\,\{S_2\,(Q_0),\ S_3\,(R_0)\}$ is $\Sigma_1\,\{p_1\,(Q_0,\ R_0)\}$.

Now keeping R_0 fixed, and letting Q_0 coincide, first with A_0, B_0, C_0 and secondly with any point of $A_0B_0C_0$, the restriction on Q_0 may similarly be removed. Thus, generalizing the suffixes: *If Q_0, R_0 denote any points whatever; then $\Phi\,\{S_m\,(Q_0),\,S_n\,(R_0)\}$ is $\Sigma_l\,\{p_l\,(Q_0, R_0)\}$ $(l,\,m,\,n = 1,\,2,\,3)$.*

12·3. We shall require several theorems related to that last proved.

If Q_0, R_0 denote any points, and if $S_2\,(Q_0)$, $S_3\,(R_0)$; s, s' be harmonic pairs of conics of a pencil; then $(12\cdot14)$ *one conic of the range determined by s, s' is $\Phi\,\{S_2\,(Q_0),\ S_3\,(R_0)\}$;* i.e. $(12\cdot2)$ *is $\Sigma_1\,\{p_1\,(Q_0,\ R_0)\}$.* $\qquad\qquad(12\cdot31)$

Again $(10\cdot1)$, $S_m\,(Q_0)$, considered as a class-conic, is apolar to $S_n\,(R_0)$, if and only if Q_0 is a point of $C_m\,\{S_n\,(R_0)\}$, i.e. $(11\cdot22)$ of $S_l\,(R_0)$; and in this case $(12\cdot13)$ the Φ-conic of $S_m\,(Q_0)$, $S_n\,(R_0)$ is also the reciprocal, q. $S_m\,(Q_0)$, of $S_n\,(R_0)$. But $(3\cdot1)$ if Q_0 be a point of $S_l\,(R_0)$, then R_0 is a point of $p_l\,(Q_0)$. Thus $(12\cdot2)$: *If Q_0 denote any point, and R_0 a point of $p_l\,(Q_0)$; then the reciprocal, q. $S_m\,(Q_0)$, of $S_n\,(R_0)$ is $\Sigma_l\,\{p_l\,(Q_0, R_0)\}$ $(l,\,m,\,n = 1,\,2,\,3)$.* $\qquad(12\cdot32)$

From $(7\cdot4)$, $p_2\,(H,\ F)$, which is the polar of F q. $S_2\,(H)$, is $A_0B_0C_0$; the polar of A_0 q. $S_2\,(H)$ therefore contains F. But the polar of A_0 q. $S_2\,(H)$ is also the polar of H q. $S_2\,(A_0)$, i.e. q. a line-pair meeting at A_2; and therefore contains A_2. Thus the polar of A_0 q. $S_2\,(H)$ is A_2F, i.e. $(5\cdot2)$ A_2A_3; similarly the polars, q. $S_2\,(H)$, of B_0, C_0 are B_2B_3, C_2C_3 respectively. But by the symmetry the polars of A_0, B_0, C_0 q. $S_3\,(G)$ are also A_2A_3, B_2B_3, C_2C_3 respectively; and $(8\cdot5)$ the same lines are the polars of A_0, B_0, C_0 respectively q. $\Sigma_1\,(A_0B_0C_0)$. Thus $S_2\,(H)$, $S_3\,(G)$, $\Sigma_1\,(A_0B_0C_0)$ have double contact on $A_0B_0C_0$. Also $(6\cdot1)$ G, H are respectively $\varpi_2\,(A_0B_0C_0)$, $\varpi_3\,(A_0B_0C_0)$. Generalizing: *Whatever line may be denoted by x; $\Sigma_l\,(x)$, $S_m\,\{\varpi_n\,(x)\}$, $S_n\,\{\varpi_m\,(x)\}$ have double contact on x $(l,\,m,\,n = 1,\,2,\,3)$.* This is partly a particular case of $(12\cdot2)$.

Since (6·1) O_0, U_0 are respectively $\varpi_2 (L_0 M_0 N_0)$, $\varpi_3 (L_0 M_0 N_0)$; therefore in particular $\Sigma_1 (L_0 M_0 N_0)$, $S_2 (U_0)$, $S_3 (O_0)$ have double contact on $L_0 M_0 N_0$; with similar theorems involving I_0, etc.

(12·33)

From (12·2), $\Phi \{S_1 (Q), S_2 (R)\}$, being $\Sigma_2 \{p_3 (Q, R)\}$ is also $\Phi \{S_1 (R), S_2 (Q)\}$, similarly $\Phi \{S_1 (Q), S_3 (R)\}$ is $\Phi \{S_1 (R), S_3 (Q)\}$. Also $\Phi \{S_1 (Q), S_1 (R)\}$ is $\Phi \{S_1 (R), S_1 (Q)\}$. Moreover, if $S_n (Q)$, $S_n (R)$ denote the polar conic of Q, R respectively q. *any* cubic T_n of the T-pencil, then (3·1, 12·12)

$$[\Phi \{S_1 (Q), S_n (R)\}] \barwedge [S_n (R)] \barwedge [S_n (Q)] \barwedge [\Phi \{S_1 (R), S_n (Q)\}],$$

and therefore $\Phi \{S_1 (Q), S_n (R)\}$ is identical with $\Phi \{S_1 (R), S_n (Q)\}$ whatever cubic of the T-pencil be denoted by T_n. Generalizing: *Whatever points be denoted by Q, R and whatever cubics of the T-pencil be denoted by T_m, T_n; $\Phi \{S_m (Q), S_n (R)\}$ is identical with $\Phi \{S_m (R), S_n (Q)\}$*; and dually for F-conics of pole conics. (12·34)

12·4. In a special case the F-conic of two pole conics may be identified as a poloconic. For we have seen (6·2) that if Q_0, Q_2 be any 2-correspondent points of T_0, such that $A_1 Q_0$ touches any pole conic q. Θ_2, then $A_1 Q_2$ also touches that conic; and (9·3) that the lines $A_1 Q_0$, $A_1 Q_2$ are apolar to $\Sigma_3 (A_0 B_0 C_0)$. Thus the tangents from A_1 to any pole conic $\Sigma_2 (y)$ q. Θ_2 are apolar to $\Sigma_3 (A_0 B_0 C_0)$; i.e. A_1 is a point of the F-conic of $\Sigma_2 (y)$, $\Sigma_3 (A_0 B_0 C_0)$. Similarly B_1, C_1 are points of that F-conic. Writing z for $A_0 B_0 C_0$, we see (10·21) that the F-conic of $\Sigma_2 (y)$ and $\Sigma_3 (z)$ is the poloconic, q. T_1, of z and some other line; and by the symmetry the latter line must be y. Thus the F-conic in question is $C_1 (y, z)$. Generalizing: *Whatever lines may be denoted by y, z; $F \{\Sigma_m (y), \Sigma_n (z)\}$ is $C_l (y, z)$ $(l, m, n = 1, 2, 3)$.*

Remembering that, when P_0, P_n are n-correspondents on T_0, then P_0, P_n constitute $\Sigma_n (P_l P_m)$, and using (12·15), we see that (9·3) is partly a case of this theorem [for the conics $\Sigma_m (A_0 B_0 C_0)$, $\Sigma_n (P_l P_m)$]. (12·41)

Again, $C_1 (y, A_0 B_0 C_0)$ is $F \{\Sigma_2 (y), \Sigma_3 (A_0 B_0 C_0)\}$, and therefore (12·11) contains the eight points of contact of the common tangents of $\Sigma_2 (y)$, $\Sigma_3 (A_0 B_0 C_0)$. Its reciprocal, q. $\Sigma_3 (A_0 B_0 C_0)$, therefore touches the four common tangents, and is thus a conic of the range determined by $\Sigma_2 (y)$, $\Sigma_3 (A_0 B_0 C_0)$. But (10·21) $C_1 (y, A_0 B_0 C_0)$ contains A_1, B_1, C_1; therefore (8·4) its reciprocal, q. $\Sigma_3 (A_0 B_0 C_0)$, touches $B_2 C_2$, $C_2 A_2$, $A_2 B_2$. Generalizing: *A conic of the range determined by $\Sigma_m (A_0 B_0 C_0)$ and any pole conic q. Θ_n is inscribed in the triangle $A_n B_n C_n$*; or, what is (11·34) the same thing: *A conic of the range determined by $\Sigma_m (A_0 B_0 C_0)$ and any pair of n-conjugate points (base $A_0 B_0 C_0$) is inscribed in $A_n B_n C_n$ $(m, n = 1, 2, 3)$.*

(12·42)

A case of (12·42) of special interest for our purpose arises when $m = 3$, $n = 1$, and the pair of n-conjugate points are coincident.

In this case the point of coincidence must (7·36) be one of the four points $[I]$; if it be I_λ, then the range of conics in question degenerates into the conics which have double contact, pole I_λ, with $\Sigma_3 (A_0 B_0 C_0)$. Also four and only four distinct conics (in general) touch three given lines and have double contact with a given conic. Hence: *The poles of double contact of the four conics, inscribed in the triangle $A_1 B_1 C_1$, which have double contact with $\Sigma_3 (A_0 B_0 C_0)$, are the four points $[I]$**.   (12·43)

A more general theorem on double contact, of which (11·62) may be regarded as a special case, is derived at once from (12·18). For, whatever lines may be denoted by y, z; the class-conics $\Sigma_m (y)$, $\Sigma_n (z)$ determine (11·33) a l-autopolo-range of which they are the double conics; and if P, P' denote any one of the three point-pairs of the range, then (11·34) (P, P'), $\Gamma_l (P, P')$ are conjugate conics of the range. Moreover (12·41), $F \{\Sigma_m (y), \Sigma_n (z)\}$ is $C_l (y, z)$. Thus (12·18): *If P, P' denote any one of the three point-pairs of the range of class-conics determined by $\Sigma_m (y)$, $\Sigma_n (z)$; then $\Gamma_l (P, P')$ has double contact, on the line PP', with $C_l (y, z)$* $(l, m, n = 1, 2, 3)$.   (12·44)

Further, if x, y denote any two lines, and $[\Theta_n]$ denote the Θ-range; then (cf. 3·1) the conics $[\Sigma_n (x)]$ constitute a range, and the conics $[F \{\Sigma_n (x), \Sigma_1 (y)\}]$ therefore (12·12) constitute a pencil projective with the given range. Moreover (12·41) the conics $F \{\Sigma_2 (x), \Sigma_1 (y)\}$, $F \{\Sigma_3 (x), \Sigma_1 (y)\}$ are respectively $C_3 (x, y)$, $C_2 (x, y)$; and (11·1) each of these conics is the poloconic, q. T_1, of the other. The pencil of conics is therefore a 1-autopolo-pencil. Again, if T_1', T_1'' denote the two other order-cubics which have the same Cayleyan as T_1, and $S_1' (X)$, $S_1'' (X)$ the polar conics of any point X q. those order-cubics respectively; and if Θ_1', Θ_1'' denote the class-cubics apolar (1·3) to T_1', T_1'' respectively, as Θ_1, $\Sigma_1' (x)$, $\Sigma_1'' (x)$ the pole conics of x, and $\varpi_1' (x, y)$, $\varpi_1'' (x, y)$ the mixed poles of x, y q. those class-cubics respectively: then (cf. 12·2) the conics $F \{\Sigma_1' (x), \Sigma_1 (y)\}$; $F \{\Sigma_1'' (x), \Sigma_1 (y)\}$ are respectively $S_1'' \{\varpi_1'' (x, y)\}$; $S_1' \{\varpi_1' (x, y)\}$. The last two conics are therefore (11·4) the double conics of the 1-autopolo-pencil. Further, if $\Sigma_1''' (x)$ be such that $\Sigma_1 (x)$, $\Sigma_1''' (x)$; $\Sigma_1' (x)$, $\Sigma_1'' (x)$ are harmonic in the given range of pole conics, then

$$F \{\Sigma_1 (x), \Sigma_1 (y)\}, F \{\Sigma_1''' (x), \Sigma_1 (y)\}$$

are conjugate in the 1-autopolo-pencil. Thus: *Given the range $[\Sigma_n (x)]$ of conics which are the pole conics of a given line x q. all the class-cubics of the Θ-range; and the pole conic $\Sigma_1 (y)$ of a given line y q. Θ_1: then the conics $[F \{\Sigma_n (x), \Sigma_1 (y)\}]$ constitute a 1-autopolo-pencil projective with the given range; the double conics of the pencil are $F \{\Sigma_1' (x), \Sigma_1 (y)\}$ and $F \{\Sigma_1'' (x), \Sigma_1 (y)\}$; and two pairs of con-*

* *V.* also (19·21).

jugate conics of the pencil are $C_2(x, y)$, $C_3(x, y)$ *and* $F\{\Sigma_1(x), \Sigma_1(y)\}$, $F\{\Sigma_1'''(x), \Sigma_1(y)\}$; *where* $\Sigma_1'(x)$, $\Sigma_1''(x)$ *denote the pole conics of* x q. *the other two class-cubics of the* Θ-*range which have the same Hessian as* Θ_1, *and* $\Sigma_1(x)$, $\Sigma_1'''(x)$; $\Sigma_1'(x)$, $\Sigma_1''(x)$ *are harmonic pairs in the given range of conics.* (12·45)

In the special case of (12·45) where y is identical with x, we have the theorem: $\Sigma_1(x)$, $C_2(x)$, $C_3(x)$ *are three conics of a pencil.* It is unnecessary further to particularize in this case. (12·46)

13. *Apolar Quadrilaterals.* 13·1. From (8·5), whatever line may be denoted by x; $\Sigma_m(x)$, regarded as an *order*-conic, is apolar to every pole conic (q. Θ_n) which touches x (m, $n = 1$, 2, 3). Let p denote a tangent to $\Sigma_3(A_0B_0C_0)$; then $\Sigma_1(p)$, regarded as an order-conic, is apolar to $\Sigma_3(A_0B_0C_0)$. Hence (12·13) the reciprocal, q. $\Sigma_1(p)$, of $\Sigma_3(A_0B_0C_0)$ is $F\{\Sigma_1(p), \Sigma_3(A_0B_0C_0)\}$, i.e. (12·41) is $C_2(p, A_0B_0C_0)$. Therefore, if p' denote any other tangent to $\Sigma_3(A_0B_0C_0)$, then the pole of p' q. $\Sigma_1(p)$ is a point of $C_2(p, A_0B_0C_0)$; i.e. $\varpi_1(p, p')$ is a point (P_2, say) of $C_2(p, A_0B_0C_0)$. By the symmetry, P_2 is also a point of $C_2(p', A_0B_0C_0)$. But (10·21) $C_2(p, A_0B_0C_0)$, $C_2(p', A_0B_0C_0)$ are the 2-transformations (base $A_0B_0C_0$) of p, p' respectively; and thus P_2 is the 2-conjugate (base $A_0B_0C_0$) of the meet (P_0, say) of p, p'. Again (11·33), if two of the common tangents to $\Sigma_2(y)$ and $\Sigma_3(A_0B_0C_0)$ meet at P_0, then the other two meet at P_2. Thus the mixed pole, q. Θ_1, of any pair of common tangents to $\Sigma_2(y)$ and $\Sigma_3(A_0B_0C_0)$ is the meet of the other pair of common tangents. Also (11·31) $\Sigma_2(y)$, $\Sigma_3(A_0B_0C_0)$ are two autopolo-conics, one of each type, q. Θ_1. Generalizing: Given two autopolo-conics, one of each type, q. Θ_n; then the mixed pole, q. Θ_n, of any pair of common tangents to the given conics is the meet of the other pair* ($n = 1$, 2, 3).

A quadrilateral such that the meet of any pair of opposite sides of it is the mixed pole, q. Θ_n, of the other pair is termed an *apolar quadrilateral* q. Θ_n. It is conversely true that the sides of any apolar quadrilateral q. Θ_n are the common tangents of two autopolo-conics, one of each type, q. Θ_n. (13·11)

In particular, whatever points may be denoted by X, Y; the common tangents (t, t', t'', t''', say) of $\Sigma_1(FX)$, $\Sigma_2(GY)$ constitute a quadrilateral apolar to Θ_3. But (6·1) one of these common tangents (t, say) is $A_0B_0C_0$; and (13·11) $\varpi_3(t, t')$ is the meet of t'', t'''. Thus the meet of t'', t''' is the pole of t' q. $\Sigma_3(A_0B_0C_0)$. Hence: *Whatever points may be denoted by X, Y; the common tangents, other than $A_0B_0C_0$, of $\Sigma_1(FX)$, $\Sigma_2(GY)$ constitute a triangle apolar to $\Sigma_3(A_0B_0C_0)$.* (13·12)

14. *Miscellaneous Theorems.* 14·1. Let any two lines meet T_0 at the points P_0, Q_0, R_0; P_0', Q_0', R_0', and let P_n (etc.) denote the

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* The dual theorem, on quadrangles apolar to an order-cubic, is proved analytically by Manfredini, *loc. cit.* (10·1).

n-correspondent of P_0 (etc.). Then (10·22) the six points P_n, Q_n, R_n, $P_n{'}$, $Q_n{'}$, $R_n{'}$ are contained by a conic; these six points and the three points P_0, Q_0, R_0 are therefore contained by the degenerate cubic consisting of the conic and the line $P_0Q_0R_0$. The nine points P_0, Q_0, R_0, P_n, Q_n, R_n, $P_n{'}$, $Q_n{'}$, $R_n{'}$ are thus the base-points of a pencil of cubics; and any cubic which contains eight of the points must also contain the ninth. Thus (6·2): Any three collinear points of a cubic, the n-correspondents of those three points, and the n-correspondents of any other three collinear points of the cubic are the base-points of a pencil of cubics, one of which is the given cubic; and any cubic which contains the six vertices of an inscribed quadrilateral, of any species, of a given cubic also contains the vertices opposite to three collinear vertices of another inscribed quadrilateral of that species.   (14·11)

Again, with the same notation, a pole conic (Σ_n, say) q. Θ_n touches (6·2) the eight sides of the two quadrilaterals of which P_0, Q_0, R_0; $P_0{'}$, $Q_0{'}$, $R_0{'}$ are respectively collinear vertices. But P_0, Q_0, R_0, P_n, Q_n, R_n are the vertices of a quadrilateral inscribed in *any* cubic (V_0, say) of the pencil considered. Σ_n is therefore a pole conic q. the class-cubic Υ_n, related to V_0 as Θ_n is related to T_0. Hence $P_n{'}$, $Q_n{'}$, $R_n{'}$ are three vertices of a quadrilateral inscribed in V_0; and thus V_0 meets the sides of the triangle $P_n{'}Q_n{'}R_n{'}$ again at three points in a line, the remaining side of that quadrilateral. Moreover (6·2) since Σ_n is a pole conic q. Υ_n, therefore the line touches Σ_n. Conversely, one and only one cubic of the pencil determined by the nine points P_0, Q_0, R_0, P_n, Q_n, R_n, $P_n{'}$, $Q_n{'}$, $R_n{'}$ contains any given point ($P_0{''}$, say, distinct from $Q_n{'}$, $R_n{'}$) of $Q_n{'}R_n{'}$, and this cubic must meet the lines $R_n{'}P_n{'}$, $P_n{'}Q_n{'}$ on the tangent (other than $Q_n{'}R_n{'}$) from $P_0{''}$ to Σ_n. Thus: *Every cubic of the pencil determined by three collinear points of a given cubic, the n-correspondents of those points, and the n-correspondents of any three other points in line of the given cubic, meets the sides of the triangle determined by the last three points at three further points in line.*

(14·12)

14·2. From (3·1), the T-pencil, the pencil of conics constituted by the polar conics of a given point q. all the cubics of the T-pencil, and the pencil of lines constituted by the polars of a given point q. all the cubics of the T-pencil are projective with each other. Also, if X denote any point, $T_n{'}$, $T_n{''}$ the two other cubics which have the same Cayleyan as T_n, and $S_n{'}(X)$, $S_n{''}(X)$ the polar conics of X q. $T_n{'}$, $T_n{''}$ respectively; then (11·2, 4) $S_n{'}(X)$, $S_n{''}(X)$ are the double conics of the involution pencil of conics in which $S_0(X)$, $S_n(X)$; $S_l(X)$, $S_m(X)$ are pairs of conjugate conics. Thus, if $p_n{'}(X)$, $p_n{''}(X)$ be the polars of X q. $T_n{'}$, $T_n{''}$ respectively, then *the double lines of the pencil in involution determined by $p_0(X)$, $p_n(X)$; $p_l(X)$, $p_m(X)$ as pairs of conjugate lines are $p_n{'}(X)$, $p_n{''}(X)$ (l, m,*

$n = 1, 2, 3$). Hence *the line-pair $p_m'(X)$, $p_m''(X)$ is a conjugate pair in the involution determined by $p_n'(X)$, $p_n''(X)$ as double lines* (m, $n = 1, 2, 3$). Corresponding theorems hold of the order-cubics of the T-pencil, and of the class-cubics of the Θ-range.

14·3. Let Θ_1', Θ_1'' denote the other two class-cubics which have the same Hessian as Θ_1; let Θ_m, Θ_n denote any pair of class-cubics of the Θ-range, harmonically conjugate q. Θ_1', Θ_1''; and let $\Sigma_1'(x)$, $\Sigma_1''(x)$, $\Sigma_m(x)$, $\Sigma_n(x)$ denote the pole conics, q. Θ_1', Θ_1'', Θ_m, Θ_n respectively, of any line x. Since (4·4) $\Sigma_1(A_2A_3)$ consists of the point-pair A_0, A_1, therefore (12·15) $F\{\Sigma_1(A_2A_3), \Sigma_m(x)\}$ (say F_m) contains A_0, A_1. Let the other four common points of T_0 and F_m be denoted by P_0, Q_0, R_0, S_0. Then (12·45) the conic $F\{\Sigma_1(A_2A_3), \Sigma_n(x)\}$ (say F_n) is the poloconic, q. T_1, of F_m, and therefore (10·3) meets T_0 at $A_1, A_0, P_1, Q_1, R_1, S_1$. A_0 is thus a point common to F_m, F_n and the lines B_0C_0, B_1C_1. Let B_0C_0, B_1C_1 meet F_m, F_n again at Y, Z respectively. Then (12·11) the line-pairs B_0C_0, A_1Y; B_1C_1, A_1Z are apolar to $\Sigma_m(x)$, $\Sigma_n(x)$ respectively.   (α)

Also (since A_0, A_1, P_0, Q_0, R_0, Y are six points of a conic)

$$A_1(YP_0Q_0R_0) \barwedge A_0(YP_0Q_0R_0) \barwedge A_0(B_0P_0Q_0R_0),$$

and (7·37)

$$A_0(B_0P_0Q_0R_0) \barwedge A_0(B_1P_1Q_1R_1) \barwedge A_0(ZP_1Q_1R_1),$$

while (since A_0, A_1, P_1, Q_1, R_1, Z are six points of a conic)

$$A_0(ZP_1Q_1R_1) \barwedge A_1(ZP_1Q_1R_1),$$

and thus $$A_1(YP_0Q_0R_0) \barwedge A_1(ZP_1Q_1R_1).$$

A_1Y, A_1Z are therefore (7·36) conjugate lines in the 1-involution on A_1.   (β)

But (α) A_1Z contains the pole, q. $\Sigma_n(x)$, of B_1C_1; hence A_1Z contains the centre of perspective (K_0, say) of the triangle $A_1B_1C_1$ and its polar triangle q. $\Sigma_n(x)$. Also, if K_1 denotes the 1-conjugate (base $A_0B_0C_0$) of K_0, then (β) A_1Y contains K_1. Thus (α) A_1K_1 contains the pole, q. $\Sigma_m(x)$, of $A_0B_0C_0$. By the symmetry, B_1K_1, C_1K_1 contain the pole, q. $\Sigma_m(x)$, of $A_0B_0C_0$; and that pole must therefore be K_1. Thus: *If x denote any line, and $\Sigma_m(x)$, $\Sigma_n(x)$ the pole conics of x q. any pair of class-cubics of the Θ-range harmonic q. Θ_1', Θ_1''; then the pole of $A_0B_0C_0$ q. $\Sigma_m(x)$, and the centre of perspective of the triangle $A_1B_1C_1$ and its polar triangle q. $\Sigma_n(x)$, are 1-conjugates (base $A_0B_0C_0$).*

14·4. Since (6·1) the pole, q. $\Sigma_1(A_0B_0C_0)$, of $A_0B_0C_0$ is F; therefore some point of $A_0B_0C_0$ is the pole, q. $\Sigma_1(A_0B_0C_0)$, of FGH; i.e. is $\varpi_1(A_0B_0C_0, FGH)$. Let this point be denoted by P. If X denote any point of FGH, then the polar, q. $\Sigma_1(A_0B_0C_0)$, of X contains P. But (12·33) $\Sigma_1(A_0B_0C_0)$ has double contact, on $A_0B_0C_0$, with $S_3(G)$. The polars of X q. $\Sigma_1(A_0B_0C_0)$ and $S_3(G)$ therefore meet on $A_0B_0C_0$.

Thus the polar of X q. S_3 (G), i.e. p_3 (G, X), contains P; in particular p_3 (G) contains P. Similarly p_2 (H, X), and in particular p_2 (H), contain P. Also (12·33) S_1 (H), S_3 (F) have double contact on $A_0 B_0 C_0$. Hence, and from (10·43, 11·21, 2), by poloconics q. T_3; S_2 (H), S_1 (F), C_3 $(A_0 B_0 C_0)$ are three conics of a pencil. The polars of X q. these three conics are therefore concurrent; i.e. p_2 (H, X), p_0 (F, X) and the polar of X q. C_3 $(A_0 B_0 C_0)$ are concurrent. Similarly p_3 (G, X), p_0 (F, X) and the polar of X q. C_2 $(A_0 B_0 C_0)$ are concurrent. But (12·46) Σ_1 $(A_0 B_0 C_0)$, C_2 $(A_0 B_0 C_0)$, C_3 $(A_0 B_0 C_0)$ are conics of a pencil; the polars of X q. these three conics are therefore concurrent.

Now, as X tends to coincidence with F, then (7·4) p_2 (H, X) and p_3 (G, X) and (6·1) the polar of X q. Σ_1 $(A_0 B_0 C_0)$ tend to coincidence with $A_0 B_0 C_0$. Hence p_0 (F, X) and the polars of X q. C_2 $(A_0 B_0 C_0)$, C_3 $(A_0 B_0 C_0)$ tend to concurrence on $A_0 B_0 C_0$, at the limiting point of concurrence of p_2 (H, X), p_3 (G, X); i.e. at the point P. Remembering that (6·1) F, G, H are the poles of $A_0 B_0 C_0$ q. Θ_1, Θ_2, Θ_3 respectively, and generalizing: *If x denote any line, and x' the locus of the poles of x q. the class-cubics of the Θ-range; then the four lines x, $p_0 \{\varpi_l (x)\}$, $p_n \{\varpi_m (x)\}$, $p_m \{\varpi_n (x)\}$ are concurrent at $\varpi_l (x, x')$ $(l, m, n = 1, 2, 3)$.*

The application of this theorem to the points I_0, O_0, U_0, as poles (6·1) of $L_0 M_0 N_0$, should especially be noted.   (14·41)

We have seen (14·41) that p_2 (X, H) contains P for all positions of X on the line FGH. In particular, p_2 (G, H) contains P; therefore G, H, P constitute an apolar triad (3·2) q. T_2, and thus G, H are apolar to S_2 (P); also (7·4) G, H are apolar to S_1 (P). Thus G, H are apolar to every conic of the pencil determined by S_1 (P), S_2 (P); i.e. (3·1) G, H are apolar to the polar conic of P q. every cubic of the T-pencil. Generalizing as in (14·41), we have the theorem: *$\varpi_m (x)$, $\varpi_n (x)$ are apolar to the polar conic of $\varpi_l (x, x')$ q. every conic of the T-pencil.*   (14·42)

Remembering that Θ_1, Θ_2, Θ_3 denote three class-cubics which have the same Cayleyan, and dualizing: *If T_n, T_n', T_n'' denote any three order-cubics of the T-pencil which have the same Cayleyan; if X denote any point, and X' the meet of the polars of X q. the cubics of the T-pencil; if p_n' (X), p_n'' (X) denote the polars of X q. T_n', T_n'' respectively, and p_n (X, X') the mixed polar of X, X' q. T_n: then p_n' (X), p_n'' (X) are apolar to the pole conic of p_n (X, X') q. every class-conic of the Θ-range.*   (14·43)

Proceeding similarly with the theorem (7·31, 7·4): If Z denote any point of $A_0 B_0 C_0$, then G, H are apolar to S_1 (Z); we obtain the theorem: *Whatever point may be denoted by Y; p_n' (X), p_n'' (X) are apolar to Σ_n (XY).*   (14·44)

PART II.

THE GENERALIZATION OF THEOREMS OF
EUCLIDEAN GEOMETRY

15. *The Non-Euclidean Case.* 15·1. If $\Sigma_3 (A_0B_0C_0)$ be identified with the *Absolute* of non-euclidean geometry, then we have the following subsidiary identifications.

15·2. The triangle $A_2B_2C_2$ becomes (8·4) the absolute *polar triangle* of $A_1B_1C_1$. The point H therefore becomes (5·2) the *orthocentre**, and the line $A_0B_0C_0$ the *orthaxis*.   (15·21)

Again, we have seen (7·36) that if P_0, P_1 denote any pair of 1-conjugates (base $A_0B_0C_0$), then A_1B_1, A_1C_1; A_1B_2, A_1C_2; A_1P_0, A_1P_1 are pairs of conjugate lines in an involution. Thus the angles $B_1A_1P_0$, $P_1A_1C_1$ become equal; similarly for the angles $C_1B_1P_0$, $P_1B_1A_1$; $A_1C_1P_0$, $P_1C_1B_1$. Thus P_0, P_1 become *isogonal conjugates*, and the 1-transformation becomes the isogonal transformation.

In particular, the point G becomes (7·4) the *isogonal conjugate of the orthocentre*.   (15·22)

Similarly any pair (P_0, P_2) of 2-conjugates become isogonal conjugates q. $A_2B_2C_2$, the polar triangle (15·21) of $A_1B_1C_1$. The absolute polars of P_0, P_2 therefore meet the sides of the triangle $A_1B_1C_1$ *isotomically*; i.e. if the polars $(p, p'$, say) of P_0, P_2 meet B_1C_1, C_1A_1, A_1B_1 at X, X'; Y, Y'; Z, Z' respectively; then $B_1X = X'C_1$; $C_1Y = Y'A_1$; $A_1Z = Z'B_1$.   (15·23)

From (15·22) the four points $[I]$ become the *incentres*; and from (15·23) the four points $[O]$ become the *circumcentres*. The incentres and circumcentres of any triangle are thus (5·5) eight points† of a configuration of symbol $(12_4, 16_3)$.

The sides of the quadrangle determined by the four points $[I]$ become the *bisectors of the angles*; in particular, I_0I_a, $I_\beta I_\gamma$ become the bisectors of the angle A_1. The sides of the quadrangle determined by the four points $[O]$ become (9·22) the *perpendicular bisectors of the sides*; in particular, O_0O_1, $O_\beta O_\gamma$ become the perpendicular bisectors of the side B_1C_1.

We therefore obtain from (5·4, 5) the theorem: *The bisectors of any angle of a triangle in a non-euclidean plane meet the perpendicular bisectors of the opposite side at the points of a Maclaurin tetrad on a cubic; and the tetrads corresponding to the three vertices of the triangle are the points of a dodecad.*   (15·24)

The four conics $[\Gamma_1 (O)]$ (the poloconics, q. Θ_1, of the four points $[O]$) become (11·61, 3) the *circumcircles*.

We have separately proved (12·43) a theorem which may now

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* When no triangle is specified, the triangle $A_1B_1C_1$ is implied.
† An instance of the inadequacy of the non-euclidean treatment.

be stated as follows: The centres of the four circles inscribed in the triangle $A_1B_1C_1$ are the four points $[I]$.   (15·25)

15·3. A curve of some importance in the sequel is the absolute reciprocal (as it may now be called) of Δ_2 (v. 2).

Since Δ_2 is a class-cubic, therefore the absolute reciprocal (D_2, say) of Δ_2 is an order-cubic. Since (4·3) Δ_2 touches the join of every pair of 2-correspondent points of T_0, therefore (5·5) Δ_2 touches, in particular, the six sides of the quadrangle determined by the four points $[O]$. Thus (15·24) D_2 contains the three pairs of middle points of the sides of $A_1B_1C_1$. Similarly D_2 contains the absolute poles of the lines A_0A_2, B_0B_2, C_0C_2, i.e. (15·21) contains the meets of B_1C_1, C_1A_1, A_1B_1 with A_1A_2, B_1B_2, C_1C_2 respectively; and these three points are (in the non-euclidean case) the feet of the perpendiculars from the vertices of the triangle $A_1B_1C_1$ to the opposite sides respectively. D_2 also contains the absolute poles of A_1A_3, B_1B_3, C_1C_3, which are points of B_2C_2, C_2A_2, A_2B_2 respectively. Thus: *The reciprocal, q. Σ_3 $(A_0B_0C_0)$, of Δ_2 is an order-cubic D_2. When Σ_3 $(A_0B_0C_0)$ is identified with the absolute, D_2 contains the three pairs of middle points of the sides of the triangle $A_1B_1C_1$, and the feet of the perpendiculars from the vertices of that triangle on the opposite sides respectively.*

16. *The Euclidean Case.* 16·1. If Σ_3 $(A_0B_0C_0)$ be identified with the circular points at infinity of a euclidean plane, while A_1, B_1, C_1 remain actual points of the plane, then the results of (15) remain true with the following modifications.

16·2. The points A_2, B_2, C_2 become the *points at infinity* corresponding to the directions perpendicular to B_1C_1, C_1A_1, A_1B_1 respectively. The line $A_0B_0C_0$ becomes the *line at infinity.*   (16·21)

Since (15·24) the four points $[I]$ become the *in- and ex-centres*; therefore the polar conics, q. T_1, of the points of $A_0B_0C_0$, which (6·5) all contain the four points $[I]$, become the *rectangular hyperbolas to which the triangle $A_1B_1C_1$ is apolar.* Every point-pair apolar to all these hyperbolas is therefore a pair of isogonally conjugate points. In particular, the circular points at infinity, which are apolar to all rectangular hyperbolas, must be regarded as a pair of isogonally conjugate points.   (16·22)

Of the four points $[O]$, one (O_0, say) becomes the *circumcentre*, and the other three are confounded with the points at infinity A_2, B_2, C_2. The point G, isogonally conjugate to the orthocentre, is confounded with the *circumcentre*.   (16·23)

Any conic of the pencil determined by the four points $[O]$ degenerates (16·23) into the line at infinity and a line containing the circumcentre. Since in the general case (7·31) any point P_0 and its 2-conjugate (base $A_0B_0C_0$) P_2 are apolar to every conic of the pencil: therefore in the present case, if P_0 be an actual point, then P_2 becomes its *opposite q. the circumcentre*, i.e. the point such that

the circumcentre is the middle point of the segment P_0P_2; and if P_0 be a point at infinity, then P_2 is also a point at infinity.   (16·24)

Since A_3 is (4·1, 7·34) the 2-conjugate of A_1; therefore A_3 becomes (16·24) the point of the circumcircle diametrically opposite to A_1; similarly B_3, C_3 become the points of the circumcircle diametrically opposite to B_1, C_1 respectively. The sides of the triangle $A_3B_3C_3$ thus become parallel to the corresponding sides of $A_1B_1C_1$; and since (8·4) $A_2B_2C_2$, $A_3B_3C_3$ are polar to Σ_1 ($A_0B_0C_0$), therefore (16·21) Σ_1 ($A_0B_0C_0$) degenerates into the circular points at infinity. Hence 3-conjugates become isogonal conjugates q. $A_2B_3C_3$, and the points $[U]$ become the in- and ex-centres of $A_3B_3C_3$. The circular points at infinity must therefore (16·22) be regarded as conjugate in the euclidean development of the 3- as well as of the 1-transformation.   (16·25)

16·3. We have seen (9·3) that, given two points, P_0, P_2; then the line-pairs A_1P_0, A_1P_2; B_1P_0, B_1P_2; C_1P_0, C_1P_2 are simultaneously apolar to Σ_3 ($A_0B_0C_0$) if and only if P_0, P_2 are 2-correspondent points of T_0. Now lines apolar to Σ_3 ($A_0B_0C_0$) become in the euclidean case lines at right angles; and T_0 therefore becomes the locus of point-pairs which subtend right angles at all the three points A_1, B_1, C_1. Also (16·24) 2-correspondent points P_0, P_2 become *either* both actual points *or* both points at infinity. Hence 2-correspondent points of T_0 become either *diametrically opposite points of the circumcircle* or *points at infinity corresponding to directions at right angles*; and T_0 therefore degenerates into the *circumcircle and the line at infinity*.   (16·31)

Now of the four points P_n of a tetrad on T_0, let P_2 be the 2-correspondent of P_0; then (4·1) the other two points P_1, P_3 of the tetrad are 1-correspondents of P_0, P_2 respectively, and are themselves 2-correspondents. In the euclidean case let P_0 become an actual point of the circumcircle. Then (16·31) P_2 becomes the diametrically opposite point of the circumcircle; P_1, P_3, as isogonal conjugates of P_0, P_2 respectively, become points at infinity; and from (16·31), they must correspond to directions at right angles.   (16·32)

Of the twelve points of a dodecad (5·1) on T_0, it is now clear that (in general) six become (actual) points of the circumcircle, and six become points at infinity. In particular, six of the points of the dodecad $[LMN]$ (5·4) become points at infinity; and the other six must therefore (5·2, 5) become the middle points of the sides of the quadrangle determined by the in- and ex-centres. The well-known euclidean form of (15·24) at once follows.   (16·33)

The class-cubics Δ_1, Δ_3 become three-cusped hypocycloids (*v.* 17·031) of which the (common) cuspidal tangents meet at the circumcentre (*v.* 19·4). As regards Δ_2, the argument of (16·31) shews that each of the circular points at infinity must be regarded

as self-conjugate for the euclidean development of the 2-transformation. Every line containing either of the circular points must therefore be regarded as a tangent to the degenerate Δ_2. The three tangents from any actual point to Δ_2 thus become (16·31) the diameter of the circumcircle, and the isotropic lines which contain the point. Hence: *Δ_2 becomes a triad of points; viz. the circumcentre and the circular points at infinity.* (16·34)

The cubic D_2 of (15·3) also degenerates; for it contains six distinct points at infinity, viz. three (the 'external middle points') on the sides of the triangle $A_1B_1C_1$ and three others (the 'poles' of A_1A_3, B_1B_3, C_1C_3). D_2 also contains the (internal) middle points of the sides of $A_1B_1C_1$, and the foot of the perpendicular on each side from the opposite vertex. Thus: *In the euclidean case, the reciprocal, q. $\Sigma_3 (A_0B_0C_0)$ of Δ_2 degenerates into the nine-point circle and the line at infinity.* (16·35)

16·4. We can mention only a few of the interesting properties of the conics which represent the range $\Sigma_n (A_0B_0C_0)$. They become the family of confocal conics of which one ($v.$ 12·42) is the inscribed conic concentric with the circumcircle; the auxiliary circles of this conic touch (19·033) the nine-point circle. $\Sigma_2 (A_0B_0C_0)$ becomes (12·41) that conic of the family of which the director circle is the circumcircle.

17. *The Pedal Line.* 17·0*. The following theorems of euclidean geometry are well known.

The feet of the perpendiculars to the sides of a triangle (ABC) from any point (P) of the circumcircle of the triangle are in line; their line is the *pedal* or *Wallace line* of the given point†. Conversely, if the feet of the perpendiculars from a given point to the sides of a given triangle are in line, then the given point is contained by the circumcircle of the given triangle. (17·011)

The isogonal conjugate, q. the triangle ABC, of the point P is the point at infinity corresponding to the direction perpendicular to the pedal line of P. (17·012)

If the perpendicular from P to BC meets the circumcircle again at Q, then AQ is parallel to the pedal line of P. (17·021)

The pedal lines of diametrically opposite points of the circumcircle are at right angles‡. (17·022)

The envelope of the pedal lines, q. ABC, of the points of the

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* For convenience in identification, the numeration of each euclidean theorem to be generalized differs from that of the general theorem by the insertion of 0 after the point in the case of the former.

† Dr William Wallace, *Leybourn's Math. Repository*, O.S. No. 7, 25 March 1799; wrongly attributed to Simson by Servois, *Gergonne*, 4 (1814), p. 250; *v.* Muir, *Proc. Edin. Math. Soc.* 3 (1885), p. 104; Mackay, *ibid.* 9 (1891), p. 83; Archibald, *ibid.* 28 (1910), p. 64.

‡ Steiner, *Crelle*, 53 (1857), p. 231.

circumcircle of ABC is a curve of the third class inscribed in ABC (Steiner's hypocycloid)*.   (17·031)

The pedal lines of diametrically opposite points of the circumcircle meet on the nine-point circle*.   (17·032)

Steiner's hypocycloid has triple contact with the nine-point circle*, and double contact at the circular points with the line at infinity, the common tangents of the circle and the hypocycloid being perpendicular to the cuspidal tangents of the latter.   (17·033)

The pedal line of any point bisects the join of that point and the orthocentre*.   (17·041)

All these theorems are particular cases of properties of the general cubic.

17·1. Grassmann† discovered that, if A, B, C, A', B', C' be any six coplanar points, and P any point of their plane such that the three points $(PA, B'C')$, $(PB, C'A')$, $(PC, A'B')$ are in a line p; then the locus of P is an order-cubic, and the envelope of p a class-cubic. Grassmann's theorem was further discussed by Clebsch‡, who shewed that the general cubic may be generated by the method of Grassmann, and that A, B, C are correspondents, of the same species, of A', B', C' respectively.

The most interesting special case of Grassmann's theorem is that which arises when ABC, $A'B'C'$ are triangles in perspective. Berkhan§ and Sommerville ‖ have discussed this case analytically; the subjoined treatment is for the most part taken from a paper written by the author (in ignorance of Berkhan's work) in 1908, but never published.

We have seen (8·6) that, if any triangle $A_1B_1C_1$ and any conic be given, then a certain order-cubic T_0 is determined; that A_1, B_1, C_1 are respectively correspondents, of the same species, of three collinear points A_0, B_0, C_0 of T_0; and that the given conic is the pole conic [denoted by $\Sigma_3(A_0B_0C_0)$] of the straight line $A_0B_0C_0$ q. a class-cubic (Θ_3) of which T_0 is the Cayleyan.

Little difference arises if two triangles $A_1B_1C_1$, $A_2B_2C_2$ in perspective are given; for $\Sigma_3(A_0B_0C_0)$ is determined as the conic q. which $A_1B_1C_1$, $A_2B_2C_2$ are mutually polar, and T_0 may then be obtained as before. Or again, if A_0, B_0, C_0 be the meets of B_1C_1, B_2C_2; C_1A_1, C_2A_2; A_1B_1, A_2B_2 respectively, and A_3, B_3, C_3 the meets of B_1C_2, B_2C_1; C_1A_2, C_2A_1; A_1B_2, A_2B_1 respectively, then (v. 5·1) T_0 is the unique cubic which contains the twelve points $[ABC]$. These points determine a dodecad on the cubic: A_n, B_n, C_n being the n-correspondents (4·1) of A_0, B_0, C_0 respectively ($n = 1, 2, 3$).

* Steiner, *loc. cit.*

† *Crelle*, 52 (1856), p. 254.

‡ *Math. Annalen*, 5 (1872), p. 424; Clebsch-Lindemann, *Vorlesungen über Geometrie*, Leipzig (1876), Bd. 1, p. 540.

§ *Arch. d. Math. u. Phys.* (3), 11 (1907), p. 1.

‖ *Proc. 5 Int. Cong. Math.* (1911), 2, p. 93.

Now let P_0, P_1 denote any pair of 1-correspondents on T_0, and let $\Sigma_3 (A_0 B_0 C_0)$ be denoted by Σ_3 simply. Then (9·3) the line-pairs $A_2 P_0$, $A_2 P_1$; $B_2 P_0$, $B_2 P_1$; $C_2 P_0$, $C_2 P_1$ are apolar to Σ_3; i.e. $A_2 P_0$, $B_2 P_0$, $C_2 P_0$ respectively contain the poles, q. Σ_3, of $A_2 P_1$, $B_2 P_1$, $C_2 P_1$ respectively. But these poles are (8·4) the meets of $B_1 C_1$, $C_1 A_1$, $A_1 B_1$ respectively with the polar, q. Σ_3, of P_1.

Conversely, if $A_2 P_0$, $B_2 P_0$, $C_2 P_0$ meet $B_1 C_1$, $C_1 A_1$, $A_1 B_1$ respectively at three points X, Y, Z in line; let Q_0 denote the pole, q. Σ_3, of XYZ. Then the polars, q. Σ_3, of X, Y, Z are $A_2 Q_0$, $B_2 Q_0$, $C_2 Q_0$ respectively; $A_2 P_0$, $A_2 Q_0$; $B_2 P_0$, $B_2 Q_0$; $C_2 P_0$, $C_2 Q_0$ are therefore line-pairs apolar to Σ_3, and thus (9·3) P_0, Q_0 are 1-correspondents on T_0. Hence: If, and only if, P_0 be a point of T_0, then $A_2 P_0$, $B_2 P_0$, $C_2 P_0$ meet $B_1 C_1$, $C_1 A_1$, $A_1 B_1$ respectively at three points in line*.
(17·11)

Let the line $A_0 B_0 C_0$ (which may be termed the *base*) be denoted by u; and let the line of the three points $A_2 P_0$, $B_1 C_1$; $B_2 P_0$, $C_1 A_1$; $C_2 P_0$, $A_1 B_1$ of (17·11) be termed the u-(1, 2)-*pedal line* of P_0. Then we have shewn that: The u-(1, 2)-pedal line of any point of T_0 is the polar, q. Σ_3, of the 1-correspondent of that point*.
(17·12)

17·2. Let any line containing A_0 meet T_0 again at P_1, Q_3; and consider the dodecad A_n, P_n, Q_n $(n = 0, 1, 2, 3)$. $A_2 P_0$ contains Q_0, and $A_1 Q_0$ contains P_3 $(v.\ 5\cdot1)$; also P_3 is (4·1) the 2-correspondent of P_1. Thus: If P_0 *denote any point of* T_0, *and* Q_0 *the third point common to* T_0 *and* $A_2 P_0$, *then the third point common to* T_0 *and* $A_1 Q_0$ *is the 2-correspondent of the pole* (q. Σ_3) *of the* u-(1, 2)-*pedal line of* P_0.
(17·21)

Again, if P_0, P_2 denote any pair of 2-correspondents on T_0, then (4·1) P_1, P_3, the 1-correspondents of P_0, P_2 respectively, are themselves 2-correspondents. The joins of P_1, P_3 to A_1, B_1, C_1 are thus (9·3) pairs of lines apolar to Σ_3. Hence (17·12): *The joins of any vertex of the triangle* $A_1 B_1 C_1$ *and the poles of the* u-(1, 2)-*pedal lines of any pair of 2-correspondents on* T_0 *are apolar to* Σ_3.
(17·22)

17·3. From (17·12) the pole, q. Σ_3, of the u-(1, 2)-pedal line of any point of T_0 is another point of T_0. The envelope of the pedal lines is therefore a class-cubic, the reciprocal, q. Σ_3, of T_0. This class-cubic may be denoted by Π. Since (17·1) T_0 contains A_1, B_1, C_1, A_2, B_2, C_2; therefore Π touches $B_1 C_1$, $C_1 A_1$, $A_1 B_1$, $B_2 C_2$, $C_2 A_2$, $A_2 B_2$. Thus: The envelope of the u-(1, 2)-pedal lines of the points of T_0 is a class-cubic inscribed in $A_1 B_1 C_1$ and also in $A_2 B_2 C_2$*.
(17·31)

Again, as in (17·22), the poles, q. Σ_3, of the u-(1, 2)-pedal lines of any pair of 2-correspondents on T_0 are themselves 2-correspondents on T_0. Their join therefore (4·3) touches the class-cubic

* Berkhan, *loc. cit.*; Sommerville, *loc. cit.*

Δ_2 (2), and thus the meet of the given pedal lines is contained by an order-cubic D_2, the reciprocal, q. Σ_3, of Δ_2. Hence: *The locus of the meet of the u-*(1, 2)*-pedal lines of any pair of 2-correspondents on T_0 is an order-cubic.*   (17·32)

Now it is well known* that the Hessian and the Cayleyan of any order-cubic touch at all their common points; and that those points are the conjugate poles (4·1), q. the given cubic, of its inflexions (which are also those of the Hessian). In particular (2) T_0 and Δ_2, as Hessian and Cayleyan respectively of T_2, touch at the 2-correspondents of the common inflexions of T_0 and T_2. Also the reciprocal, q. any conic (say Σ_3), of any order-cubic (say T_0) is a class-cubic (say Π); the polars of corresponding points on T_0 are corresponding tangents (4·3) to Π; and the polars of the points of inflexion of T_0 are the cuspidal tangents of Π. Further, the polars, q. Σ_3, of 2-correspondents P_0, P_2 on T_0 are the pedal lines of P_1, P_3. Thus: *The curves* Π, D_2 *touch at all their nine common points; the cuspidal tangents of* Π *and its common tangents with D_2 being u-*(1, 2)*-pedal lines of 2-correspondents on T_0.*   (17·33)

17·4. Transposing the triangles $A_1B_1C_1$, $A_2B_2C_2$, and proceeding as in (17·1), we obtain the theorem: If, and only if, P_0 be a point of T_0, then A_1P_0, B_1P_0, C_1P_0 meet B_2C_2, C_2A_2, A_2B_2 respectively at three points in line. This line, which may be termed the *u-*(2, 1)*-pedal line* of P_0, is by the symmetry the polar, q. Σ_3, of P_2, the 2-correspondent of P_0.

Now the pole, q. Σ_3, of the line P_0P_3 is the mixed pole (3·4; cf. 3·2), q. Θ_3, of P_0P_3 and $A_0B_0C_0$, and is therefore the pole of $A_0B_0C_0$ q. $\Sigma_3(P_0P_3)$, which (4·4) consists of the point-pair P_1, P_2. Thus $A_0B_0C_0$ and the polar, q. Σ_3, of P_0 are apolar to the point-pair P_1, P_2. Reciprocating q. Σ_3, and remembering (8·4): *The u-*(1, 2)*- and u-*(2, 1)*-pedal lines of any point P_0 of T_0 are apolar to the point-pair P_0, H.*   (17·41)

17·5. The theorems of (17·1–4) may be generalized as follows:

Let x denote any line meeting T_0 at X_0, Y_0, Z_0; and let the n-correspondents of X_0, Y_0, Z_0 be denoted by X_n, Y_n, Z_n respectively ($n = 1, 2, 3$). We have the following: *If, and only if, P_0 be a point of T_0; then X_nP_0, Y_nP_0, Z_nP_0 meet Y_mZ_m, Z_mX_m, X_mY_m at three points in line* ($m, n = 1, 2, 3$).   (17·51)

Let their line be termed the *x-*(m, n)*-pedal line* of P_0; then: *The x-*(m, n)*-pedal line of P_0 is the polar, q. $\Sigma_l(x)$, of P_m.*   (17·52)

Thus: *The envelope of the x-*(m, n)*-pedal line of P_0 is the reciprocal, q. $\Sigma_l(x)$, of T_0* ($l, m, n = 1, 2, 3$).   (17·53)

Let the *x-*(m, n)-pedal line of P_0 be denoted by y. Then (17·52) P_m is $\varpi_l(x, y)$, and x is therefore the polar, q. $\Sigma_l(y)$, of P_m. The relation between x, y is thus reciprocal, i.e.: *If y be the x-*(m, n)*-pedal*

* Cremona, *loc. cit.* (10·1), p. 116.

line of P_0, then x is the y-(m, n)-*pedal line of* P_0, and thus *any* line is the pedal line of *any* point of T_0 q. some base.   (17·54)

A case of special interest arises when we take for base in succession each of the 16 lines of a configuration of symbol $(12_4, 16_3)$ of which the points are on T_0; e.g. the configuration of which the points are A_n, B_n, C_n ($n = 0, 1, 2, 3$). We thus obtain $16 \times 6 = 96$ related pedal lines of a point P_0 of T_0. Of such a set of 96 pedal lines, it may be proved that 16 pairs meet on each of the 3 lines P_0P_n ($n = 1, 2, 3$); 9 pairs meet on each of the 16 lines of the configuration; and 8 pairs meet on each of the 18 lines A_mA_n, B_mB_n, C_mC_n ($m, n = 0, 1, 2, 3$).   (17·55)

17·6. On the non-euclidean and euclidean interpretation of the theorems on the pedal line, little need be added to the statements of (15, 16). (16·32) may now be made definite as follows: The points of a tetrad on T_0 are represented in the euclidean case by two diametrically opposite points of the circumcircle, and the points at infinity corresponding to the directions of the pedal lines of those two points. As regards (17·41), it appears from (16·3) that (in the euclidean case) the u-$(2, 1)$-pedal line of any actual point on the circumcircle becomes the line at infinity.

It is interesting that the method of proof in (17·11) shews that the rectangular and pedal properties of the euclidean circle become, when generalized, polar reciprocal theorems ($v.$ 16·3); and that the non-metrical properties of the three-cusped hypocycloid, including its characteristic property of double contact, at the circular points, with the line at infinity, arise from the fact that it is the Hessian of a class-cubic of which its tritangent circle and the line at infinity constitute the Cayleyan.

18. *The Pedal Conic.* 18·0. In elementary geometry the pedal line of any point of the circumcircle may be regarded as constituting, with the line at infinity, a degenerate pedal circle. The principal theorems on the pedal circle are as follows:

The feet of the perpendiculars, on the sides of a triangle, from any two points isogonally conjugate q. that triangle, are contained by a circle, the *pedal circle* of the two points (or of either of them)*.   (18·011)

If a variable point be contained by a fixed diameter of the circumcircle, then the pedal circle of that point contains a fixed point; viz, the meet of the pedal lines of the extremities of the given diameter†.   (18·021)

Any pair of (real) isogonally conjugate points q. a triangle are

* Ascribed in *Encyk. d. math. Wiss.* to W. Schönborn, *Progr. Gymn. Krotoschin,* 1881.
† Fontené, *Nouv. Ann. de Math.* (4), 6 (1906), p. 55. The corresponding theorem for the isogonally conjugate point was known to Bobillier, *Gergonne,* 19 (1828–9), p. 356.

the (real) foci of a conic inscribed in the triangle; and the pedal circle of those points is the (major) auxiliary circle of the conic; and in particular: (18·031)

The pedal line of any point of the circumcircle is the vertical tangent of that inscribed parabola of which the given point is the focus*. (18·032)

18·1. In projective geometry, the dependence of the pedal properties on the theory of poloconics q. a class-cubic, which is somewhat obscured in the case of the pedal line, becomes clear in the more general case, now to be considered, of the pedal conic.

Let x denote any line; let $A_0 B_0 C_0$ be denoted by u; let $\Sigma_3 (u)$ be denoted by Σ_3 simply†; and let us consider the range of class-conics determined by $\Sigma_1 (x)$ and Σ_3 as double conics. From (12·41) A_2, B_2, C_2 are points on the F-conic of those conics, and (11·33) the conjugate of any conic σ of the range is $\Gamma_2 (\sigma)$. Applying (12·17) and remembering (8·4), we have the following: The pairs of tangents from A_2, B_2, C_2 to any class-conic σ of the range $[\sigma]$ determined by $\Sigma_1(x)$ and Σ_3 meet $B_1 C_1$, $C_1 A_1$, $A_1 B_1$ respectively in six points on a conic, which is the reciprocal, q. Σ_3, of $\Gamma_2 (\sigma)$.

Now (11·33) the point-pairs of the range $[\sigma]$ are 1-conjugates (base u); and conversely (11·34) any pair (P_0, P_1) of 1-conjugate points (base u) with Σ_3 determine such a range; the point-pair P_0, P_1 may therefore be substituted for σ above.

Again, the common tangents of Σ_3 and any conic of the range contain P_0, P_1; reciprocating q. Σ_3, we see that the reciprocal, q. Σ_3, of any conic of the range meets Σ_3 on the polars, q. Σ_3, of P_0, P_1. Therefore: *The meets, with $B_1 C_1$, $C_1 A_1$, $A_1 B_1$ respectively, of the joins to A_2, B_2, C_2 of any pair P_0, P_1 of 1-conjugate points (base u) are contained by a conic, the u-(1, 2)-pedal conic of P_0, P_1. This conic is the reciprocal, q. Σ_3, of $\Gamma_2 (P_0, P_1)$; and meets Σ_3 on the polars, q. Σ_3, of P_0, P_1.* (18·11)

The x-(m, n)-*pedal conic of* P_0, P_m may be similarly defined; where x denotes any line, and $(m, n = 1, 2, 3)$ (*v.* 17·5).

18·2. If the variable point P_0 is contained by a fixed tangent $Q_0 Q_2$ to Δ_2 (Q_0, Q_2 denoting 2-correspondents on T_0), then (cf. 10·22) $\Gamma_2 (P_0, P_1)$ touches a fixed line, viz. the tangent to Δ_2 corresponding (4·3) to $Q_0 Q_2$; i.e. (with the usual notation) the line $Q_1 Q_3$. The reciprocal, q. Σ_3, of $\Gamma_2 (P_0, P_1)$ therefore contains the pole, q. Σ_3, of $Q_1 Q_3$, which is (17·12) the meet of the u-(1, 2)-pedal lines of Q_0, Q_2. Thus: *If the variable point P_0 be contained by a fixed tangent to Δ_2, then the u-(1, 2)-pedal conic of P_0, P_1 contains a fixed point; viz. the meet of the u-(1, 2)-pedal lines of the 2-correspondents on T_0 which are contained by the given tangent.* (18·21)

18·3. Again, we have seen (12·42) that of the conics of the range

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* Wallace, *Leybourn's Math. Repos.* O.S. 1 (1797), p. 309; 2 (1798), p. 54.

† And so throughout the remainder of the paper.

$[\sigma]$ one, viz. the reciprocal, q. Σ_3, of $C_2(x, u)$, is inscribed in $A_1 B_1 C_1$; and (12·44) that $C_2(x, u)$ has double contact, on $P_0 P_1$, with $\Gamma_2(P_0, P_1)$. Reciprocating q. Σ_3, we have: *The u-(1, 2)-pedal conic of P_0, P_1 has double contact, on $P_0 P_1$, with that class-conic, of the range determined by $\Sigma_3(u)$ and the point-pair P_0, P_1, which is inscribed in $A_1 B_1 C_1$.*   (18·31)

The u-(1, 2)-pedal conics of the other two point-pairs of the range clearly have double contact with the same inscribed conic.

If P_0 be a point of T_0, then (7·34) P_1 is also a point of T_0, viz. the 1-correspondent of P_0. In that case (4·4) the points P_0, P_1 constitute $\Sigma_1(P_2 P_3)$, and thence (11·32) $\Gamma_2(P_0, P_1)$ is constituted by P_0, P_1. We thus revert to the case of (17); the u-(1, 2)-pedal conic of (P_0, P_1) degenerates into the u-(1, 2)-pedal lines of P_0, P_1; and from (18·31) at once follows: *The u-(1, 2)-pedal line of any point P_0 of T_0 touches, at its meet with the line $P_0 P_1$, that class-conic, of the range determined by $\Sigma_3(u)$ and the point-pair P_0, P_1, which is inscribed in $A_1 B_1 C_1$.*   (18·32)

18·4. Reverting to the case of (18·11); let $A_2 P_0$, $A_2 P_1$; $B_2 P_0$, $B_2 P_1$; $C_2 P_0$, $C_2 P_1$ meet $B_1 C_1$, $C_1 A_1$, $A_1 B_1$ at X, X'; Y, Y'; Z, Z' respectively. Then it may be proved that $A_1 X$, $B_1 Y$, $C_1 Z$ are concurrent if and only if P_0 is a point of the cubic $\overline{\mathfrak{C}}_0$ (5·3); that in that case $A_1 X'$, $B_1 Y'$, $C_1 Z'$ are also concurrent, and therefore P_1 is a point of $\overline{\mathfrak{C}}_0$; and that the line $P_0 P_1$ contains F. $\overline{\mathfrak{C}}_0$ may therefore be regarded as the locus of a point, such that the join of that point and its 1-conjugate (base $A_0 B_0 C_0$) contains F^*.

18·5. In the euclidean case (16) the isogonal conjugate of an actual point is in general an actual point. The polars of two isogonally conjugate points therefore both coincide with the line at infinity, and (18·11) the pedal conic becomes a circle. The isogonal conjugate of an actual point P of the circumcircle is, however, a point at infinity (17·012); and the inscribed conic, focus P, is therefore a parabola. The identity of the euclidean theorems (18·011– 18·032) with the projective theorems correspondingly numbered will now be clear.

It will have been noted that in the non-euclidean case (15) the pedal conic, not having (in general) double contact with the Absolute, is not a circle. In the case of the u-(1, 2)-pedal conic, exceptions occur only when the points P_0, P_1 are coincident; in which case the point of coincidence must (7·36, 15·24) be one of the four points $[I]$.

19. *Feuerbach's Theorem.* 19·0. Feuerbach's famous theorem is as follows:

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* $V.$ (5·6), footnote. For the euclidean case, $v.$ F. G. Taylor, *Proc. Edin. Math. Soc.* 33, Pt 2 (1914–5), p. 70; for the non-euclidean case, Sommerville, *ibid.* p. 85. $V.$ also *Encyk. d. math. Wiss.* III. AB 10, p. 1260, footnote 329 a.

The nine-point circle of any triangle touches the inscribed and escribed circles of that triangle*.   (19·031)

Let the points of contact be termed the *Feuerbach points* of the inscribed and escribed circles; then:

The Feuerbach point of the inscribed circle is the meet of the pedal lines of the extremities of that diameter of the circumcircle which contains the incentre; and similarly for the escribed circles.
(19·032)

More generally:

If the join of two isogonally conjugate points be a diameter of the circumcircle, then the pedal circle of the two points touches the nine-point circle, at the meet of the pedal lines of the extremities of the given diameter†.   (19·033)

The form assumed by Feuerbach's theorem in the non-euclidean case, whether for the four incircles or for the four circumcircles, is also well known. The course of our work will shew that the latter is the more fundamental form of the theorem. It may be enunciated as follows:

The four circumcircles of any triangle in a non-euclidean plane are all touched by four other circles‡.   (19·091)

19·1. In projective geometry, Feuerbach's theorem has analogues of two different types, in one of which the curve touched by the four conics corresponding to the in- and ex-circles is an order-cubic, and that touched by the four conics corresponding to the circumcircles is a class-cubic; while in the other the curve touched is a conic. The theorems most closely related to that of the pedal line are of the first type; while the theorems of the second type are capable of interesting extension§.

19·2. We have already seen (11·61, 3) that the conics which circumscribe $A_1B_1C_1$ and have double contact with Σ_3 are the four conics $[\Gamma_1 (O_\lambda)]$; and that the pole of double contact of Σ_3 and $\Gamma_1 (O_\lambda)$ is O_λ ($\lambda = 0, \alpha, \beta, \gamma$).

Similarly the conics which circumscribe $A_2B_2C_2$ and have double contact with Σ_3 are the four conics $[\Gamma_2 (I_\lambda)]$; and the pole of double contact of Σ_3 and $\Gamma_2 (I_\lambda)$ is I_λ. Reciprocating q. Σ_3, we have the theorem: *The conics which are inscribed in $A_1B_1C_1$ and have double contact with Σ_3 are the reciprocals, q. Σ_3, of the four conics $[\Gamma_2 (I_\lambda)]$; and the poles of double contact are the four points $[I_\lambda]$ respectively* ($\lambda = 0, \alpha, \beta, \gamma$).   (19·21)

The reciprocal, q. Σ_3, of $\Gamma_2 (I_\lambda)$ will be denoted by $R (I_\lambda)$.

19·3. Now let the three tangents to Δ_2 which contain I_λ be P_0P_2, Q_0Q_2, R_0R_2, where the named points are pairs of 2-corre-

* Feuerbach, *Eigenschaften einiger merkwürdigen Punkte*, usw., Nurnberg, 1822.
† Ascribed in *Encyk. d. math. Wiss.* to J. Griffiths, *Educ. Times*, July, 1889; but apparently due to Weill, *Nouv. Annales* (2) 19 (1880), p. 259.
‡ V. Hart, *Quart. Journ. Math.* 4 (1861), p. 260; Salmon, *ibid.* 6 (1864), p. 67.
§ The author hopes shortly to publish a further paper on this subject.

spondents on T_0. Then (cf. 10·23) $\Gamma_2(I_\lambda)$ touches Δ_2 on P_1P_3, Q_1Q_3, R_1R_3. Reciprocating q. Σ_3, and remembering (17·12), we have: *The locus (D_2) of the meets of the u-(1, 2)-pedal lines of pairs of 2-correspondents on T_0 has triple contact with each of the four conics inscribed in $A_1B_1C_1$ and having double contact with Σ_3.*   (19·31)

The conics $R(I_\lambda)$ are, in fact, 'pure' poloconics (10·1) q. an order-cubic (the reciprocal, q. Σ_3, of Θ_2) of which D_2 is the Hessian.

Further: *Each point of contact of D_2 with $R(I_\lambda)$ is the meet of the u-(1, 2)-pedal lines of a pair of 2-correspondent points on T_0 which are such that their join contains I_λ ($\lambda = 0, \alpha, \beta, \gamma$).*   (19·32)

Again, if V_0, V_2 denote any pair of 2-correspondents on T_0, and if W_0, W_1 denote the pair of 1-conjugates (base u) contained by the line V_0V_2*; then (4·4) V_0V_2 touches Δ_2, i.e. the join of W_0, W_1 touches Δ_2, and therefore (cf. 10·22) $\Gamma_2(W_0, W_1)$ touches Δ_2, the common tangent being V_1V_3. Reciprocating q. Σ_3, we have from (17·12): *If the join of two 1-conjugate points (base u) be a tangent to Δ_2, then the u-(1, 2)-pedal conic of the two given points touches D_2, at the meet of the u-(1, 2)-pedal lines of the 2-correspondent points of T_0 on the given tangent to Δ_2.*   (19·33)

19·4. The principal result of the above discussion is that Feuerbach's theorem may be regarded as a case of the fundamental theorem (10·23) that a pure poloconic q. any order-cubic has triple contact with the Hessian of that cubic. In the euclidean case the Hessian is represented (16·35) by the nine-point circle and the line at infinity; and its three points of contact with (e.g.) the inscribed circle are represented by the Feuerbach point of that circle and the circular points at infinity (v. 16·34, on Δ_2). The corresponding theorem for the (euclidean) circumcircle becomes: *The circumcircle is the common tritangent circle of the three-cusped hypocycloids which, in the euclidean case, represent the class-cubics Δ_1, Δ_3.*   (19·41)

It is easy to shew that the four conics $[\Gamma_1(U_\lambda)]$ (11·61, 3) become, in the euclidean case, the four circumconics of which the centres are the points $[I]$. Hence, and from (16·4) we have the following euclidean theorem, which may be deemed a companion of Feuerbach's theorem: *Let Σ_2 denote that conic which is confocal with an inconic and has the circumcircle as director circle. Then the four circumconics, of which the centres are the in- and ex-centres, all have double contact with Σ_2, and triple contact with a certain three-cusped hypocycloid to which the circumcircle is also tritangent.*   (19·42)

19·5. We now turn to the second type of projective analogue of Feuerbach's theorem, in which a conic plays the part of the nine-point circle of euclidean geometry. When the non-euclidean implications are removed from Hart's extension of Feuerbach's theorem, it takes the following (or an equivalent) form: The four conics $[\Gamma_1(O_\lambda)]$ all have simple contact with four other conics,

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* W_0, W_1 are the meets of V_0V_2 and its 1-transformation (base u).

which themselves all have double contact with $\Sigma_3 (A_0 B_0 C_0)$. We shall proceed to prove this theorem by methods based on the geometry of the T-pencil and the Θ-range; and to determine the relation to those families of cubics of the second group of four conics, of their points of simple contact with the conics $[\Gamma_1 (O_\lambda)]$, and of their chords of double contact with the conic $\Sigma_3 (A_0 B_0 C_0)$ (hereafter to be denoted, as above, by Σ_3 simply).

19·6. We have seen (14·41) that $L_0 M_0 N_0$, $p_2 (I_0)$, $p_1 (O_0)$ are concurrent (at P_0, say). Now since (6·5) $A_0 B_0 C_0$ (say u) is the polar, q. T_1, of I_0, and also the polar, q. T_2, of O_0, therefore (11·62) $C_3 \{u, p_2 (I_0)\}$ is $\Gamma_3 (I_0)$, and $C_3 \{u, p_1 (O_0)\}$ is $\Gamma_3 (O_0)$. Three of the common points of $\Gamma_3 (I_0)$, $\Gamma_3 (O_0)$ are therefore (10·22) A_3, B_3, C_3, and the fourth is the 3-conjugate (base u) of P_0 (say P_3). Since by definition $\Gamma_3 (I_0)$, $\Gamma_3 (O_0)$ both contain $\varpi_3 (I_0 O_0)$; therefore P_3 must be $\varpi_3 (I_0 O_0)$. Also (12·33) $S_2 (I_0)$ and $\Sigma_3 (L_0 M_0 N_0)$ have double contact on $L_0 M_0 N_0$: thus the pole, q. $S_2 (I_0)$, of $L_0 M_0 N_0$ is U_0. Hence the polar, q. $S_2 (I_0)$, of P_0 is $I_0 U_0$, i.e. (5·5) $I_0 O_0$; similarly for O_a, O_3, O_γ. Thus: *The 3-conjugate (base u) of $\varpi_3 (I_0 O_\lambda)$ is the pole, q. $S_2 (I_0)$, of $I_0 O_\lambda$ ($\lambda = 0, a, \beta, \gamma$).*　　　　　　　　(19·61)

But $\Gamma_3 (I_0)$, being $C_3 \{u, p_2 (I_0)\}$, is (10·21) the 3-transformation (base u) of $p_2 (I_0)$; and (7·36) the transformation is projective. Also if Y denote the pole, q. Θ_3, of $I_0 X$, where X is a variable point; then (cf. 3·1) the range of points $[Y]$ on the conic $\Gamma_3 (I_0)$ is also projective with the pencil $I_0 [X]$. We have thus established a projective correspondence between the points of $p_2 (I_0)$ and the lines of the pencil $I_0 [X]$; also the points of $p_2 (I_0)$ which correspond to the four lines $I_0 O_\lambda$ of the pencil are respectively (19·61) the poles of those lines q. $S_2 (I_0)$. The like is therefore true for every point of $p_2 (I_0)$; i.e. *Whatever point may be denoted by X; the 3-conjugate (base u) of $\varpi_3 (I_0 X)$ is the pole, q. $S_2 (I_0)$, of $I_0 X$*; or again, remembering that (6·5) u is $p_1 (I_0)$; and that (7·33) the 3-conjugate (base u) of any point Z is the pole of u q. $S_3 (Z)$: *The pole of $p_1 (I_0)$ q. $S_3 \{\varpi_3 (I_0 X)\}$ is the pole of $I_0 X$ q. $S_2 (I_0)$.*　　　　　　　　(19·62)

In particular, supplying for X the point I_2 (the 2-conjugate, base u, of I_0), and remembering that (7·35) $p_2 (I_0, I_2)$ is u; we see that the 3-conjugate (base u) of $\varpi_3 (I_0 I_2)$ is the meet (J_0, say) of u and $p_2 (I_0)$. J_0 is thus the meet of $p_1 (I_0)$ and $p_2 (I_0)$, and therefore (3·1) *J_0 is the common meet of the polars of I_0 q. every cubic of the T-pencil.*　　　　　　　　(19·63)

Again (12·32) the reciprocal, q. $S_1 (J_0)$, of Σ_3 is $S_2 (R_0)$, where R_0 is the pole, q. $S_3 (J_0)$, of u, and is thus (7·33) the 3-conjugate (base u) of J_0, that is (19·63) $\varpi_3 (I_0 I_2)$. Thus: *The reciprocal, q. $S_1 (J_0)$, of Σ_3 is $S_2 \{\varpi_3 (I_0 I_2)\}$.*　　　　　　　　(19·64)

Further, since (19·63) $p_2 (I_0)$ contains J_0, therefore (3·1) $S_2 (J_0)$ contains I_0; and thus $p_2 (I_0, J_0)$, being the polar of I_0 q. $S_2 (J_0)$, contains I_0. But since J_0 is a point of u, therefore (7·33) $p_2 (I_0, J_0)$,

being the polar of J_0 q. $S_2 (I_0)$, contains I_2. Hence $p_2 (I_0, J_0)$ *is*
$I_0 I_2$. (19·65)

19·7. With the notation of (14·2), let us now consider the pencil
of conics (say the pencil $[C_0]$) of which the base-points are the com-
mon points of the conics $C_1 \{p_1' (I_0)\}$ and $C_1 \{p_1'' (I_0)\}$. From (19·63),
$p_1' (I_0)$, $p_1'' (I_0)$ meet at J_0; thus (10·44) every conic of $[C_0]$ is the
mixed poloconic, q. T_1, of a pair of lines which meet at J_0 and are
harmonically conjugate q. $p_1' (I_0)$ and $p_1'' (I_0)$. Each conic of $[C_0]$
is therefore (11·5) the reciprocal, q. $S_1 (J_0)$, of that pole conic, q.
Θ_1, which touches such a pair of lines; in particular (14·2)
$C_1 \{p_3' (I_0), p_3'' (I_0)\}$ is a conic of $[C_0]$ and the reciprocal of such a pole
conic.

But (14·44) $p_1' (I_0)$, $p_1'' (I_0)$ are apolar to every conic of type
$\Sigma_1 (I_0 X)$, where X denotes any point. Thus the tangents from J_0 to
every such conic are harmonic q. $p_1' (I_0)$, $p_1'' (I_0)$; and since the
whole of the pole conics q. Θ_1 constitute a web, therefore in general
one and only one pole conic q. Θ_1 touches a given pair of lines.
Thus: *The pencil $[C_0]$ is the reciprocal, q. $S_1 (J_0)$, of the range of conics*
$[\Sigma_1 (I_0 X)]$. (19·71)

Also (14·43) $p_2' (I_0)$, $p_2'' (I_0)$ are apolar to the pole conic (q. any
class-cubic of the Θ-range) of $p_2 (I_0, J_0)$, i.e. (19·65) of $I_0 I_2$; and in
particular are apolar to $\Sigma_1 (I_0 I_2)$. Thus the two line-pairs $p_n' (I_0)$,
$p_n'' (I_0)$ $(n = 1, 2)$ are both apolar to $\Sigma_1 (I_0 I_2)$; and (14·2) that conic
therefore touches $p_3' (I_0)$, $p_3'' (I_0)$. Thus: *The reciprocal, q. $S_1 (J_0)$,
of $\Sigma_1 (I_0 I_2)$ is $C_1 \{p_3' (I_0), p_3'' (I_0)\}$*. (19·72)

We have now shewn (19·64, 72) that the reciprocals, q. $S_1 (J_0)$,
of Σ_3, $C_1 \{p_3' (I_0), p_3'' (I_0)\}$ are respectively $S_2 \{\varpi_3 (I_0 I_2)\}$, $\Sigma_1 (I_0 I_2)$;
but (12·33) the latter pair of conics have double contact, $I_0 I_2$ being
their chord of contact and $\varpi_1 (I_0 I_2)$ its pole. Thus

$$\Sigma_3, \ C_1 \{p_3' (I_0), p_3'' (I_0)\}$$

have double contact, their pole of contact being the pole, q. $S_1 (J_0)$,
of $I_0 I_2$, and their chord of contact $p_1 \{J_0, \varpi_1 (I_0 I_2)\}$. Further,
transposing suffixes and putting I_2 for X in the second form of
(19·62); the pole of $p_3 (I_0)$ q. $S_1 \{\varpi_1 (I_0 I_2)\}$ is the pole of $I_0 I_2$ q. $S_2 (I_0)$,
i.e. is the meet of $p_2 (I_0)$ and $p_2 (I_0, I_2)$, which (19·63) is J_0. Thus
$p_1 \{J_0, \varpi_1 (I_0 I_2)\}$ is $p_3 (I_0)$; and therefore: *$C_1 \{p_3' (I_0), p_3'' (I_0)\}$ has
double contact with Σ_3, the chord of double contact being $p_3 (I_0)$, and
its pole being the pole, q. $S_1 (J_0)$, of $I_0 I_2$*. (19·73)

The base points of the pencil $[C_0]$ may be identified without much
difficulty. For one pair of lines harmonically conjugate q. $p_1' (I_0)$,
$p_1'' (I_0)$ are (14·2) $p_2 (I_0)$, $p_3 (I_0)$, and therefore $C_1 \{p_2 (I_0), p_3 (I_0)\}$,
which (11·62) is $\Gamma_1 (I_0)$, is a conic of $[C_0]$; and another such pair of
lines are (14·2) $p_0 (I_0)$, $p_1 (I_0)$, whence $C_1 \{p_0 (I_0), p_1 (I_0)\}$ is a conic of
$[C_0]$. Now (6·5) $p_1 (I_0)$ is u, and therefore (10·21) $C_1 \{p_0 (I_0), p_1 (I_0)\}$
is the 1-transformation (base u) of $p_0 (I_0)$. But (14·41) $p_0 (I_0), p_3 (O_0)$,

$p_2 (U_0)$ are concurrent (at K_0, say) and thus (10·21) their respective 1-transformations $C_1 \{u, p_0 (I_0)\}$, $C_1 \{u, p_3 (O_0)\}$, $C_1 \{u, p_2 (U_0)\}$ have four common points, three of which are A_1, B_1, C_1 and the fourth K_1, the 1-conjugate (base u) of K_0. Further $C_1 \{u, p_3 (O_0)\}$ is (6·5) $C_1 \{p_2 (O_0), p_3 (O_0)\}$, i.e. (11·62) $\Gamma_1 (O_0)$; similarly, $C_1 \{u, p_2 (U_0)\}$ is $\Gamma_1 (U_0)$; and by definition $\varpi_1 (O_0 U_0)$ is a point common to $\Gamma_1 (O_0)$, $\Gamma_1 (U_0)$. Thus K_1 is $\varpi_1 (O_0 U_0)$, i.e. (5·5) $\varpi_1 (I_0 O_0)$; but $\varpi_1 (I_0 O_0)$ is a point of $\Gamma_1 (I_0)$, i.e. of $C_1 \{p_2 (I_0), p_3 (I_0)\}$. Thus $\varpi_1 (I_0 O_0)$ is a common point of $C_1 \{p_0 (I_0), p_1 (I_0)\}$ and $C_1 \{p_2 (I_0), p_3 (I_0)\}$, and therefore of all the conics of $[C_0]$; similarly the three points $\varpi_1 (I_0 O_\lambda)$ ($\lambda = \alpha, \beta, \gamma$) are the other three base-points of the pencil $[C_0]$.

Now (19·71) the conics of $[C_0]$ are the reciprocals, q. $S_1 (J_0)$, of the conics $\Sigma_1 (I_0 X)$ (X variable); and the latter conics constitute a range, the base lines of which are (6·2) the four polars, q. Θ_1, of I_0. Thus the four points $\varpi_1 (I_0 O_\lambda)$ are the poles, q. $S_1 (J_0)$, of the four base-lines. It is necessary to identify $\varpi_1 (I_0 O_0)$, in particular, as the pole, q. $S_1 (J_0)$, of that one ($L_0 M_0 N_0$) of the four base-lines of which (v. 6·31) O_0 is the pole q. Θ_2. Now (7·35) $p_1 (K_0, K_1)$ is u, which (19·63) contains J_0. Thus J_0, K_0, K_1 constitute (3·2) an apolar triad q. T_1; i.e. K_1 is the pole, q. $S_1 (J_0)$, of some line containing K_0. Also (14·41) of the four base-lines, $L_0 M_0 N_0$ is that which contains K_0; similarly for the other three base-lines. Thus: *If the polars, q. Θ_1, of I_0 be termed the four lines $[l]$; then $\varpi_1 (I_0 O_\lambda)$ is the pole, q. $S_1 (J_0)$, of that one of the lines $[l]$ of which the pole, q. Θ_2, is O_λ ($\lambda = 0, \alpha, \beta, \gamma$).*   (19·74)

19·8. We have shewn that (11·63) $\Gamma_1 (O_0)$ and (19·73)

$$C_1 \{p_3' (I_0), p_3'' (I_0)\},$$

which (19·74) contain $\varpi_1 (I_0 O_0)$, both have double contact with Σ_3, the poles of double contact being respectively O_0 and the pole, q. $S_1 (J_0)$, of $I_0 I_2$. If the join of these poles contains the point $\varpi_1 (I_0 O_0)$, then the conics $\Gamma_1 (O_0)$, $C_1 \{p_3' (I_0), p_3'' (I_0)\}$ must touch at that point. We proceed to shew that the condition is fulfilled.

For since (12·33) $S_2 (I_0)$, $S_1 (O_0)$ have double contact on $L_0 M_0 N_0$, therefore the polars of any point, and in particular of J_0, q. those two conics meet on $L_0 M_0 N_0$. Thus $p_2 (I_0, J_0)$, $p_1 (O_0, J_0)$, $L_0 M_0 N_0$ are concurrent. But (19·65) $p_2 (I_0, J_0)$ is $I_0 I_2$; also $p_1 (O_0, J_0)$ is the polar of O_0, and $L_0 M_0 N_0$ (19·74) the polar of $\varpi_1 (I_0 O_0)$, q. $S_1 (J_0)$. Reciprocating q. $S_1 (J_0)$; O_0, $\varpi_1 (I_0 O_0)$, and the pole, q. $S_1 (J_0)$, of $I_0 I_2$ are collinear. Thus: *The conics $\Gamma_1 (O_0)$, $C_1 \{p_3' (I_0), p_3'' (I_0)\}$ have simple contact at $\varpi_1 (I_0 O_0)$.*   (19·81)

19·9. Summing up, and observing that I_0 may be understood throughout as denoting *any* one of the four poles, q. T_1, of $A_0 B_0 C_0$, we have Hart's extension (for circumconics) of Feuerbach's theorem in the following form: *The four conics $\Gamma_1 (O_\mu)$ ($\mu = 0, \alpha, \beta, \gamma$) circumscribe the triangle $A_1 B_1 C_1$, and have double contact with*

$\Sigma_3 (A_0 B_0 C_0)$, *the poles of double contact being respectively the four points* O_μ; *the four conics* $C_1 \{p_3' (I_\lambda), p_3'' (I_\lambda)\}$ $(\lambda = 0, \alpha, \beta, \gamma)$ *have double contact with* $\Sigma_3 (A_0 B_0 C_0)$, *the chords of double contact being respectively the four lines* $p_3 (I_\lambda)$; *the conic* $C_1 \{p_3' (I_\lambda), p_3'' (I_\lambda)\}$ *touches the conic* $\Gamma_1 (O_\mu)$ *at* $\varpi_1 (I_\lambda O_\mu)$.   (19·91)

There is, of course, an analogous theorem (overlooked when the standpoint is non-euclidean) for the four conics $\Gamma_1 (U_\mu)$; the theorem is obtained by substituting U, 2 for O, 3 throughout (19·91). It should be noted that, since (5·5) the join of any one of the four points $[I]$ with any one of the four points $[O]$ contains some one of the four points $[U]$, therefore the sixteen points of simple contact remain unaltered, being separated, moreover into the same four groups of four points each.   (19·92)

To obtain Hart's theorem for inconics, it is only necessary to interchange I, O and substitute 2 for 1 throughout (19·91), and then reciprocate q. $\Sigma_3 (A_0 B_0 C_0)$.   (19·93)

20. *Circles associated with the Quadrilateral.* 20·0. We shall conclude this study by examining the theory of the circles associated with the quadrilateral*. The four triangles of which the sides are sides of a quadrilateral will throughout be termed the *triangles of the quadrilateral.* The theorems to be considered are the following:

The circumcircles of the four triangles of a quadrilateral have a point common to all four†.   (20·021)

Let this point be termed the *Wallace point* of the quadrilateral.

The feet of the perpendiculars to the four sides of a quadrilateral from its Wallace point, and from that point only, are in line‡.
                    (20·022)

Let this line be termed the *Wallace line* of the quadrilateral.

The focus of the parabola inscribed in the quadrilateral is the Wallace point, and its vertical tangent is the Wallace line§.
                    (20·023)

Let this parabola be termed the *Wallace parabola* of the quadrilateral.

The orthocentres of the four triangles of the quadrilateral are contained by a line parallel to the Wallace line‖.   (20·024)

Let the line of the orthocentres be termed the *orthocentric line* of the quadrilateral.

The circumcentres of the four triangles of the quadrilateral are

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* For an interesting historical note on the whole subject *v.* Clawson, *Math. Gazette,* 9 (1917–19), p. 85.

† "Scoticus," *Leybourn's Math. Repos.* N.S. 1 (1806), p. 170. For the identification of "Scoticus" with Wallace, *v.* Mackay, *Proc. Edin. Math. Soc.* 9 (1890–1), p. 87.

‡ Steiner, *Gergonne,* 18 (1828), p. 302; but essentially due to Wallace.

§ *V.* Wallace, *loc. cit.* (18·0).

‖ Steiner, *loc. cit.*

contained by a circle, the *circumcentric circle* of the quadrilateral*. The joins of each vertex of any triangle of the quadrilateral with the circumcentre of the triangle of which one side is the fourth side of the quadrilateral and the other two sides are the sides of the quadrilateral containing the given vertex, are concurrent on the circumcentric circle†; the point of concurrence being the isogonal conjugate, q. the given triangle of the quadrilateral, of the point at infinity corresponding to the direction perpendicular to the fourth side of the quadrilateral. (20·042)

The circumcentric circle contains the Wallace point*.

(20·045)

Let A, X; B, Y; C, Z denote the pairs of opposite vertices of the quadrilateral, Y, Z being points of the *segments* CA. AB respectively: let each pair of angles BAC, BXY; ABC, CYZ; ACB, BZY be termed *opposite angles* of the quadrilateral; and, of the twelve bisectors of these six angles. let any two internal (or any two external) be termed *bisectors of the same type*, and an internal and an external bisector *bisectors of opposite types*; then:

The six meets of bisectors of the same type of opposite angles of a quadrilateral are the vertices of a second quadrilateral; and the six meets of bisectors of opposite types of opposite angles of the given quadrilateral are the vertices of a third quadrilateral‡.

(20·071)

Let the two new quadrilaterals be termed the *Mention quadrilaterals* of the given quadrilateral.

The diagonal triangles of the given quadrilateral and of its Mention quadrilaterals are in perspective two by two, the orthocentre of the diagonal triangle of the given quadrilateral being the common centre of perspective§. (20·072)

Let the diagonal triangles of the Mention quadrilaterals be termed the *Sancery triangles* of the given quadrilateral.

The director circles of the conics inscribed in the Mention quadrilaterals constitute two systems of coaxal circles; and four circles of either coaxal system each contain four of the sixteen in- and ex-centres of the triangles of the given quadrilateral*.

(20·073)

Let the two groups of four circles be termed the *Steiner circles* of the given quadrilateral.

The circumcircles of the Sancery triangles belong one to each of the two coaxal systems‡. (20·074)

The two coaxal systems are mutually orthogonal; the radical axis of either is therefore the line of centres of the other*.

(20·081)

* Steiner, *loc. cit.*
† Hermes, *Nouv. Annales* (1), 18 (1859), p. 171, Q. 476.
‡ Mention, *ibid.* (2), 1 (1862), pp. 16, 65.
§ Sancery, *ibid.* (2), 14 (1875), p. 145.

Let the two radical axes be termed the *Steiner lines* of the given quadrilateral.

The Steiner lines meet at the Wallace point*.   (20·084)

20·1. We have seen (8·6) that, given any triangle and any conic in the plane of the triangle, then a certain order-cubic, which has been called the proper cubic of the triangle and the conic, is determined; and that the given conic is a pole conic q. one of those class-cubics of which the proper cubic is the Cayleyan.

Now let any line meet the sides B_1C_1, C_1A_1, A_1B_1 of the triangle $A_1B_1C_1$ at X, Y, Z respectively; and let any conic, which, as in (8·6), may be identified with Σ_3 [i.e. $\Sigma_3 (A_0B_0C_0)$], be given. We proceed to discuss some relations between the proper cubics T_0, $T_0{}'$, $T_0{}''$, $T_0{}'''$ of the conic and the four triangles $A_1B_1C_1$, A_1YZ, B_1ZX, C_1XY respectively. The quadrilateral of which the sides are B_1C_1, C_1A_1, A_1B_1, XYZ will be termed the *quadrilateral Q*; and the four triangles, the *triangles of the quadrilateral*. The pole, q. Σ_3, of XYZ will be denoted by W_0.

We have seen that T_0 is the locus of a point P_0 which possesses an $A_0B_0C_0$-(1, 2)-pedal line in the sense of (17·12). For brevity, this line will now be termed the $(A_1B_1C_1, \Sigma_3)$-pedal line of P_0, and the line similarly related to any point $P_0{}'$ of $T_0{}'$ the (A_1YZ, Σ_3)-pedal line of $P_0{}'$, etc.

20·2. Of the nine common points of T_0, $T_0{}'$, three are A_1, B_2, C_2. If P_0 be any one of the remaining six, then the $(A_1B_1C_1, \Sigma_3)$- and (A_1YZ, Σ_3)-pedal lines of P_0 have two distinct points (on A_1B_1, A_1C_1) in common, and are therefore coincident. Let the line of coincidence be p. Then three distinct points of p are (W_0P_0, XZ); (A_2P_0, B_1X); (C_2P_0, B_1Z); and therefore p is a (B_1ZX, Σ_3)-pedal line, and P_0 is a point of $T_0{}''$; similarly P_0 is a point of $T_0{}'''$. (The argument fails for A_1, since the meets with A_1B_1, A_1C_1 of the $(A_1B_1C_1, \Sigma_3)$-pedal line of A_1 are not distinct; and for B_2 since the line B_2P_0 is not determined, and similarly for C_2.)

Now let P_0 denote a particular one of the points common to T_0, $T_0{}'$, $T_0{}''$, $T_0{}'''$, and p the (common) pedal line of P_0. Then (17·12) the pole (P_1), q. Σ_3, of p is also common to the four cubics, and is the correspondent, of a certain species, of P_0 on any one of those cubics regarded as Hessian. If $P_0{}'$ be a third common point of the four cubics, then (with similar notation) so is $P_1{}'$; the remaining two common points are thus (6·2) the remaining vertices of the quadrilateral of which P_0, P_1; $P_0{}'$, $P_1{}'$ are two pairs of opposite vertices. Hence: *Given any quadrilateral, then the proper cubics (q. any given conic, the same for all four triangles) of the four triangles of the quadrilateral have six points common to all four; and the six points are the vertices of a quadrilateral inscribed in each of the proper cubics.*   (20·21)

* Steiner, *loc. cit.*

Let the six points be termed the *Wallace points*, and the quadrilateral of which they are the vertices the *Wallace quadrilateral*, of the quadrilateral Q (q. the given conic). Remembering that (17·12) the (common) pedal line of each vertex is the polar, q. Σ_3, of the opposite vertex, we have: *The joins of a given point and the poles, q. a given conic, of the sides of a given quadrilateral meet those sides respectively at four points in line, when and only when the given point is a Wallace point, q. the given conic, of the given quadrilateral; and the six lines of collinearity are the six sides of a quadrangle.*   (20·22)

This quadrangle may be termed the *Wallace quadrangle* of the quadrilateral Q q. the given conic. It is the reciprocal, q. the given conic, of the Wallace quadrilateral.

(It will frequently be necessary below to refer to the centre and axis of perspective of a triangle and its polar triangle q. a conic. Following Salmon, this point and line will be termed respectively the *pole* and the *axis* of the given triangle q. the given conic.)

Since the Wallace quadrilateral is a quadrilateral of the first species (6·2) inscribed in T_0; therefore every conic inscribed in the Wallace quadrilateral is the pole conic, q. Θ_1, of some line. If such a conic touches one of the four polars, q. Θ_1, of any point, then it touches the other three. In particular, the conic inscribed in the Wallace quadrilateral and touching $A_0B_0C_0$ also touches B_1C_1, C_1A_1, A_1B_1. Similarly there is a conic, inscribed in the Wallace quadrilateral, which touches the three sides of the triangle A_1YZ, and also touches the axis of A_1YZ q. Σ_3. But these two conics have six common tangents, viz. the sides of the Wallace quadrilateral and the lines A_1B_1, A_1C_1; they are therefore identical. Similarly for the remaining triangles of the quadrilateral Q. Thus: *There is a conic which touches the four sides of the given quadrilateral, the axis of each of the triangles of the quadrilateral q. the given conic, and the four sides of the Wallace quadrilateral.*   (20·23)

This conic may be termed the *Wallace conic*, q. the given conic, of the given quadrilateral. It will be denoted by Ξ_1.

Now H, the pole of $A_1B_1C_1$ q. Σ_3, is (8·4) the pole, q. Σ_3, of the axis, q. Σ_3, of that triangle. The three points H', H'', H'''*, similarly related to the triangles A_1YZ, B_1ZX, C_1XY respectively, are therefore the poles, q. Σ_3, of the corresponding axes. Reciprocating the theorem (20·23) q. Σ_3, we have thus: *There is a conic which contains twelve notable points, viz. the poles, q. the given conic, of the sides of the given quadrilateral; the poles of each of the triangles of the quadrilateral q. the given conic; and the vertices of the Wallace quadrangle.*

(20·24)

This conic may be termed the *H-conic*, q. the given conic, of the quadrilateral Q, and will be denoted by K_2.

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* These are not, of course, the points similarly designated in (5·2)

Since the conic Ξ_1 is inscribed in the quadrilateral of which the sides are B_1C_1, C_1A_1, A_1B_1, $A_0B_0C_0$, therefore Ξ_1 is the pole conic, q. Θ_1, of some line (z, say) which (6·2) contains the point F; also, for the same reason, Ξ_1, regarded as a class-conic, is (8·5) apolar to Σ_3. Thus K_2, the reciprocal, q. Σ_3, of Ξ_1, is also (12·13) $F(\Sigma_3, \Xi_1)$; and is therefore (12·41) $C_2(A_0B_0C_0, z)$. But (20·24) K_2 contains the pole W_0, q. Σ_3, of XYZ; therefore (10·21) z contains W_2, the 2-conjugate (base $A_0B_0C_0$) of W_0. Thus z is FW_2, and Ξ_1 is $\Sigma_1(FW_2)$, i.e.: *If B_1C_1, C_1A_1, A_1B_1 be three sides of the given quadrilateral and $\Sigma_3(A_0B_0C_0)$ the given conic; then the Wallace conic is the pole conic, q. Θ_1, of the join of the point F and the 2-conjugate (base $A_0B_0C_0$) of the pole, q. $\Sigma_3(A_0B_0C_0)$, of the fourth side of the given quadrilateral.*   (20·25)

It should be noted that, since H' is the pole of the triangle A_1YZ q. Σ_3, therefore A_1, H', W_0 are in line; similarly B_1, H'', W_0; C_1, H''', W_0 are in line. The triangles $A_1B_1C_1$, $H'H''H'''$ are thus in perspective, centre W_0. Generalizing: *Each of the triangles of the quadrilateral Q is in perspective with one of the triangles of the quadrangle $HH'H''H'''$, the centre of perspective being in each case the pole, q. Σ_3, of the omitted side of the quadrilateral Q.*   (20·26)

The relation between the quadrangles $W_0A_2B_2C_2$, $HH'H''H'''$ is thus reciprocal, and $HH'H''H'''$ is the polar quadrangle of Q q. a certain conic.

20·3. It follows from (14·1) that the vertices of the Wallace quadrilateral and the points A_1, B_1, C_1 are the nine base-points of a pencil of order-cubics; and that one cubic of the pencil contains the three points X, Y, Z. Let this cubic, which will be termed the *Wallace cubic*, q. Σ_3, of the quadrilateral Q, be denoted by V_0. Clearly V_0 is similarly related to each of the four cubics T_0, T_0', T_0'', T_0'''.

The quadrilateral Q and the Wallace quadrilateral are quadrilaterals of the same species inscribed in V_0, and every pair of opposite vertices of either of these two quadrilaterals is therefore (6·2) a pair of conjugate poles q. an order-cubic (V_1, say) of which V_0 is the Hessian. The conic Ξ_1 is thus the pole conic, $\Xi_1(x)$, say, of some line x q. the class-cubic (Υ_1, say) apolar (1·3) to V_1, and the four sides B_1C_1, C_1A_1, A_1B_1, XYZ of the quadrilateral Q are the four polars, q. Υ_1, of a certain point.   (20·31)

Again, since opposite vertices of the Wallace quadrilateral are conjugate poles q. T_1, therefore (7·36) the involution pencil, vertex A_1, determined by the vertices of that quadrilateral is the 1-involution on A_1. But the vertices of the quadrilateral are also conjugate poles q. V_1, and A_1 is a point of V_0, the Hessian of V_1; similarly for B_1, C_1. It follows from (6·5, 7·36) that the four points $[I]$ are the poles, q. V_1, of the line XYZ, and that every conic of the pencil determined by the four points $[I]$ as base-points is the polar conic, q. V_1, of some point of XYZ. Thus if P_0 denote any point (not contained by V_0) and P_1 the 1-conjugate (base $A_0B_0C_0$) of P_0; then (7·35)

the mixed polar, q. V_1, of P_0, P_1 is XYZ. P_0, P_1 may thus be termed 1-conjugates (base XYZ), where the cubic V_0 now replaces T_0; or, for brevity, P_0, P_1 *are 1-conjugates* (*base XYZ*: V_0). (20·32)

Further (9·2) every conic of the pencil determined by the four points $[I]$ is apolar to Σ_3 (regarded as a class-conic); thus (20·32) the polar conic, q. V_1, of every point of XYZ is apolar to Σ_3. But every side of the quadrilateral Q is similarly related to Σ_3 and to V_1; and the polar conic, q. V_1, of every point of each side of the quadrilateral Q is therefore apolar to Σ_3. Since the whole of the polar conics q. V_1 constitute (3·1) a net, it follows that every polar conic q. V_1 is apolar to Σ_3. Σ_3 *is therefore* (1·3) *the pole conic*, q. Υ_1, *of some line, y;* say Σ_3 is $\Xi_1(y)$. (20·33)

We are thus led to a method, independent of the theorem of (20·21), of obtaining the Wallace cubic from the given conic Σ_3 and quadrilateral Q.

For (9·12) the 1-involution on A_1 is determined by two pairs of lines, of which one pair are A_1B_1, A_1C_1 and the other is the pair of tangents from A_1 to Σ_3; similarly for the 1-involutions on B_1, C_1. Thus, when the triangle $A_1B_1C_1$ and the conic Σ_3 are given, the four points $[I]$ are determined; and (20·32) the points $[I]$ are the poles, q. V_1, of XYZ. The poles, q. V_1, of B_1C_1, C_1A_1, A_1B_1 are similarly determinate.

But (20·31) B_1C_1, C_1A_1, A_1B_1, XYZ are the four polars, q. Υ_1, of a certain point. Their 16 poles, q. V_1, are therefore (cf. 6·32), the points of a configuration of symbol $(16_3, 12_4)$, of which the 12 lines touch the Hessian (Υ_0, say) of Υ_1. Υ_0 is therefore determinate. Also Υ_1 is the class-cubic, associated with pairs of corresponding tangents (e.g. I_0I_a, $I_\beta I_\gamma$) of known species to Υ_0, such that Υ_0 is the Hessian of Υ_1; and finally (1·3) V_0 is the Cayleyan of Υ_1.

If now, retaining the original quadrilateral Q, we substitute for Σ_3 any other pole conic q. Υ_1; then (9·13) the same involution pencils as before are determined at A_1, B_1, C_1, X, Y, Z, and therefore V_0 is again obtained as Wallace cubic. Thus, in (20·21, 31): *If there be associated with the given quadrilateral, instead of the given conic, any pole conic q. the class-cubic Υ_1, then the same Wallace cubic is obtained.* (20·34)

The particular case in which the pole conic associated with the given quadrilateral is a point-pair is of importance. Let the point-pair be E_0, E_1. Then (7·34, 20·32) E_0, E_1 are conjugates in the transformation determined by the points $[I]$ as double points; i.e. the 1-transformation (base $A_0B_0C_0$). From (20·34), the cubic V_0 remains unaltered. Let D denote the proper cubic (8·6) of $A_1B_1C_1$ and the point-pair E_0, E_1. Then the poles, q. that point-pair, of B_1C_1, C_1A_1, A_1B_1, are points of the line E_0E_1, which is therefore the axis of perspective of $A_1B_1C_1$ and its degenerate polar triangle. The meets of E_0E_1 and B_1C_1, C_1A_1, A_1B_1, are also points of D; thus D has six

points in common with E_0E_1, which is therefore part of D. Again (10·23) the 1-transformation of E_0E_1 has triple contact with D, the points of contact being A_1, B_1, C_1. Thus D consists of E_0E_1 and its 1-transformation, the conic $A_1B_1C_1E_0E_1$. If E_1' denote the third point common to V_0 and E_0E_1, then E_0', the correspondent of E_1', is a point of the conic. Thus the common points (other than A_1, B_1, C_1) of V_0 and the degenerate cubic D are E_0, E_1 each counted twice, E_0' and E_1'. Also (4·2) E_0' is the common tangential (q. V_0) of E_0, E_1. Hence, in theorem (20·21): *If the given conic be a pair of points, then the points are correspondents, of a certain species, on the appropriate Wallace cubic; and the six Wallace points are the two given points each counted twice, their common tangential q. the Wallace cubic, and its correspondent of the same species.* (20·35)

20·4. Returning to the general case; we have seen (7·4) that G, the centre of perspective of the triangles $A_1B_1C_1$, $A_3B_3C_3$, is the 1-conjugate (base $A_0B_0C_0$) of H. Thus (7·35, 20·32) the mixed polar, q. V_1, of G, H is XYZ; or, what is the same thing, G, H are 1-conjugates (base $XYZ : V_0$). Let G', G'', $G'''*$ be such that the mixed polars, q. V_1, of G', H'; G'', H''; G''', H''' are B_1C_1, C_1A_1, A_1B_1 respectively. Then A_1G', A_1H' are conjugates in the involution pencil, vertex A_1, of which one pair of conjugates is (7·36) A_1Z, A_1Y (i.e. A_1B_1, A_1C_1) and another pair (9·1, 20·34) consists of the tangents from A_1 to Σ_3; and this involution pencil is the 1-involution on A_1. Similarly B_1G'', B_1H''; C_1G''', C_1H''' are conjugates in the 1-involutions on B_1, C_1 respectively. Also (20·26) A_1H', B_1H'', C_1H''' meet at W_0; therefore (7·2) A_1G', B_1G'', C_1G''' meet at W_1, the 1-conjugate (base $A_0B_0C_0 : T_0$ *or* base $XYZ : V_0$) of W_0. Thus the triangles $A_1B_1C_1$, $G'G''G'''$ are in perspective, the centre of perspective being the point W_1 such that the mixed polar, q. V_1, of W_0, W_1 is XYZ. Similarly the triangles A_1YZ, B_1ZX, C_1XY are in perspective with $GG''''G'''$, $GG'G'''$, $GG''G'$ respectively, the centres of perspective being A_2', B_2', C_2' respectively; where A_2', B_2', C_2' are such that the mixed polars, q. V_1, of A_2, A_2'; B_2, B_2'; C_2, C_2' are B_1C_1, C_1A_1, A_1B_1 respectively. Thus: *The quadrangles $GG'G''G'''$, $W_1A_2'B_2'C_2'$ are reciprocally related, in the same manner as the quadrangles $W_0A_2B_2C_2$, $HH'H''H'''$, to the quadrilateral Q.* (20·41)

Now let Υ_1', Υ_1'' denote the other two class-cubics which have the same Hessian as Υ_1 (20·31); let Υ_1''' be the class-cubic of the Υ-range such that Υ_1, Υ_1'''; Υ_1', Υ_1'' are harmonic; let y denote the line such that (20·33) Σ_3 is the pole conic, $\Xi_1(y)$, of y q. Υ_1; and let $\Xi_1'(y)$, $\Xi_1''(y)$, $\Xi_1'''(y)$ denote the pole conics of y q. Υ_1', Υ_1'', Υ_1''' respectively.

Then (14·3) the pole, q. $\Xi_1'''(y)$, of XYZ, and the pole (H) of the triangle $A_1B_1C_1$ q. $\Xi_1(y)$, are 1-conjugates (base $XYZ : V_0$). Thus (20·41) the pole, q. $\Xi_1'''(y)$, of XYZ is G. Similarly the poles,

q. $\Xi_1'''(y)$, of B_1C_1, C_1A_1, A_1B_1 are G', G'', G''' respectively. Hence *the H-conic* $(K_2'$, say), *q. the conic* $\Xi_1'''(y)$, *of the quadrilateral* Q *contains the eight points* G, G', G'', G''', W_1, A_2', B_2', C_2'. The conic K_2' will be termed the *G-conic*, q. Σ_3, of the quadrilateral Q. (20·42)

Now consider the conic $(K_1$, say) which contains the points A_1, B_1, C_1, W_0, H. It may be shewn, as in (20·25), that K_1 is the reciprocal, q. Σ_3, of Σ_2 (GW_1). Also (13·12) one of the common tangents of Σ_1 (FW_2), Σ_2 (GW_1) is $A_0B_0C_0$, and the other three constitute a triangle apolar to Σ_3. Reciprocating q. Σ_3: the common points, other than H, of the conics K_1, K_2 determine a triangle apolar to Σ_3. But one of these common points is W_0: the other two are therefore contained by XYZ, the polar, q. Σ_3, of W_0. Thus (10·5) H, W_0, the pair of common points, not contained by XYZ, of K_1, K_2, are two points of which the 1-conjugates (base $XYZ : V_0$) are contained by the poloconic, q. V_1, of K_2. But the 1-conjugates (base $XYZ : V_0$) of H, W_0 are G, W_1 respectively. G, W_1 are therefore points of the poloconic, q. V_1, of K_2. By the symmetry, G', G'', G''', A_2', B_2', C_2' are also points of that poloconic, which is therefore K_2'. Thus: *The G- and H-conics, q. any given conic, of any given quadrilateral, are mutually poloconics q. an order-cubic of which the appropriate Wallace cubic is the Hessian.* (20·43)

But (20·25, 31, 33) K_2 is $F\{\Xi_1(x), \Xi_1(y)\}$; therefore (12·45) K_2', the poloconic, q. V_1, of K_2, is $F\{\Xi_1(x), \Xi_1'''(y)\}$. Also (cf. 12·31) one conic $(S$, say) of the *pencil* determined by $\Xi_1(y)$, $\Xi_1'''(y)$ is a polar conic q. V_1. S is therefore (1·3) apolar to $\Xi_1(x)$: moreover (20·25) $\Xi_1(y)$, i.e. Σ_3, regarded as an order-conic, is apolar to $\Xi_1(x)$. Therefore (1·2) every conic of the pencil, and in particular $\Xi_1'''(y)$ regarded as an order-conic, is apolar to $\Xi_1(x)$. Therefore (12·13)

$$F\{\Xi_1(x), \Xi_1'''(y)\}$$

is the reciprocal, q. $\Xi_1'''(y)$, of $\Xi_1(x)$, i.e. of the Wallace conic. Thus: *The G-conic, which is the H-conic of the quadrilateral Q q. a certain conic, is also the reciprocal, q. that conic, of the Wallace conic.* (20·44)

It thus appears that the Wallace conic of the quadrilateral Q q. Σ_3 is also the Wallace conic of that quadrilateral q. $\Xi_1'''(y)$, and therefore that the *H*-conic of (20·24) contains four additional identifiable points, viz. the poles, q. Σ_3, of the axes of the triangles of the quadrilateral Q q. $\Xi_1'''(y)$. Similarly: *The G-conic contains sixteen identifiable points, viz. the eight points G, G', G'', G''', A_2', B_2', C_2', W_1 and the poles, q. $\Xi_1'''(y)$, of the sides of the Wallace quadrilateral and of the four axes, q. Σ_3, of the triangles of the quadrilateral Q.* (20·45)

Finally, since (cf. 12·34) $F\{\Xi_1(x), \Xi_1'''(y)\}$ is also $F\{\Xi_1(y), \Xi_1'''(x)\}$, and since (20·33) $\Xi_1(y)$ is Σ_3; therefore: *The G-conic is the F-conic of Σ_3 and $\Xi_1'''(x)$.* (20·46)

20·5. The properties of Steiner's circles and Mention's quadrilaterals depend on those of the $(16_3, 12_4)$ configuration of (20·34). We shall first determine the corresponding properties related to the dual $(12_4, 16_3)$ configuration, which may without loss of generality be identified with that (5·4) of which the twelve points are L_n, M_n, N_n $(n = 0, 1, 2, 3)$. We shall then (20·7, 8) obtain the generalizations of Steiner's and Mention's theorems by dualizing back.

We have seen (12·2) that, whatever points may be denoted by X_0, P; $\Phi \{S_1 (X_0), S_3 (P)\}$ is $\Sigma_2 \{p_2 (X_0, P)\}$. If, in particular, P be a point of $A_0 B_0 C_0$, then $p_2 (X_0, P)$, being (3·2) the polar, q. $S_2 (X_0)$, of P, contains the pole, q. $S_2 (X_0)$, of $A_0 B_0 C_0$; i.e. (7·33) contains X_2. Thus $\Phi \{S_1 (X_0), S_3 (P)\}$ is the pole conic, q. Θ_2, of some line containing X_2; and is therefore (6·2) a conic of the range determined by the four polars, q. Θ_2, of X_2. Also (6·5), P being a point of $A_0 B_0 C_0$; $S_3 (P)$ is a conic of the pencil determined by the four points $[U]$. Thus: *The Φ-conic of $S_1 (X_0)$ and a variable conic of the pencil determined by the four points $[U]$ is a conic of the range $[\Sigma_2 (X_2 Y)]$ where Y is a variable point; similarly the Φ-conic of $S_1 (X_0)$ and a variable conic of the pencil determined by the four points $[O]$ is a conic of the range $[\Sigma_3 (X_3 Z)]$ where Z is a variable point; X_2, X_3 denoting respectively the 2-, 3-conjugates (base $A_0 B_0 C_0$) of X_0.* (20·51)

We shall denote the above ranges of conics by $[\sigma_2]$, $[\sigma_3]$ respectively.

It will appear that six conics of each of the ranges $[\sigma_2]$, $[\sigma_3]$ are of special interest for our purpose. Of the range $[\sigma_2]$, four conics are $\Sigma_2 (O_\mu X_2)$ $(\mu = 0, \alpha, \beta, \gamma)$. Each of these conics touches (6·32) four of the sixteen lines $L_l M_m N_n$ of (5·4); and each of these sixteen lines is touched by one or other of the four conics; similarly one or other of the four conics $\Sigma_3 (U_\nu X_3)$ $(\nu = 0, \alpha, \beta, \gamma)$ of the range $[\sigma_3]$ touches each of the sixteen lines $L_l M_m N_n$. Thus: *The sixteen lines $L_l M_m N_n$ may be separated in two ways into four groups of four lines each, such that one conic of the range $[\sigma_2]$ touches the four lines of each group of one partition, and one conic of the range $[\sigma_3]$ touches the four lines of each group of the other partition.* (20·52)

Again, the conic $\Sigma_2 (GX_2)$ touches (6·31) the four polars, q. Θ_2, of G; viz. $B_2 C_2$, $C_2 A_2$, $A_2 B_2$, $A_0 B_0 C_0$. The conic is therefore inscribed in the triangle $A_2 B_2 C_2$ which (6·6) is the diagonal triangle of the quadrangle determined by the four points $[O]$; similarly $\Sigma_3 (HX_3)$ is inscribed in the diagonal triangle of the quadrangle determined by the four points $[U]$. Comparing with (20·51), we see that the conic which is inscribed in the diagonal triangle of one quadrangle is the Φ-conic of $S_1 (X_0)$ and a conic circumscribed to the other quadrangle. Thus: *The Φ-conic of $S_1 (X_0)$ and one conic circumscribed to either of the quadrangles $[O]$, $[U]$ is inscribed in the diagonal triangle of the other quadrangle.* (20·53)

Now let us consider the class-conic $(\sigma$, say) which touches the

polars, q. $S_1(X_0)$, of the four points $[U]$ and also touches the axis, q. $S_1(X_0)$, of any one of the triangles of the quadrangle $[U]$; σ is therefore the dual of a H-conic (20·24). Remembering that the points $[U]$ are the poles, q. T_3, of a line $(A_0B_0C_0)$, we see by analogy with (20·25) that σ is the Φ-conic of $S_1(X_0)$ and that polar conic $[S_3(P)$, say] q. T_3 which contains the four points $[U]$ and is apolar to $S_1(X_0)$ considered as a class-conic. Since σ is the Φ-conic of $S_1(X_0)$ and a conic containing the four points $[U]$, therefore σ is a conic of the range $[\sigma_2]$; say σ is $\Sigma_2(X_2Y)$, where Y is a point to be determined. Also, since $S_3(P)$ is apolar to $S_1(X_0)$ considered as a class-conic, therefore (10·1) X_0 is a point of $C_1\{S_3(P)\}$, i.e. (11·22) of $S_2(P)$; and P is thus (3·1) a point of $p_2(X_0)$. Further, X_2Y is always (20·51) the polar, q. $S_2(X_0)$, of P; and must therefore in the present case contain X_0. Thus σ is $\Sigma_2(X_0X_2)$. Again, since $S_3(P)$ is apolar to $S_1(X_0)$ considered as a class-conic, therefore (12·13) $\Sigma_2(X_0X_2)$ is the reciprocal, q. $S_1(X_0)$, of $S_3(P)$. Similarly $\Sigma_3(X_0X_3)$, a conic of the range $[\sigma_3]$, is at once the reciprocal and the Φ-conic, q. $S_1(X_0)$, of some conic of the pencil determined by the four points $[O]$. Thus: *One conic, viz. $\Sigma_2(X_0X_2)$, of the range $[\sigma_2]$, is at once the reciprocal and the Φ-conic, q. $S_1(X_0)$, of some conic circumscribed to the quadrangle $[U]$; and one conic, viz. $\Sigma_3(X_0X_3)$, of the range $[\sigma_3]$, is at once the reciprocal and the Φ-conic, q. $S_1(X_0)$, of some conic circumscribed to the quadrangle $[O]$.*   (20·54)

If, instead of using the conic $S_1(X_0)$, we had begun with either of the conics $S_2(X_0)$, $S_3(X_0)$, we should have arrived at theorems analogous to (20·51–4), and involving also the four conics $\Sigma_1(I_\lambda X_1)$ and the conics $\Sigma_1(FX_1)$, $\Sigma_1(X_0X_1)$. There is thus as usual a three-fold reciprocity between the theorems obtained.

20·6. The case in which X_0 is a point of T_0 is of special interest. To bring the notation into accord with that of (4·2), we shall in this case denote X_0 by $P_0{}'$, and the points of the Maclaurin tetrad on T_0, of which $P_0{}'$ is the common tangential, by P_0, P_1, P_2, P_3, where, as usual, P_n is the n-correspondent of P_0, and P_l of P_m $(l, m, n = 1, 2, 3)$; and similarly for $P_1{}'$, $P_2{}'$, $P_3{}'$, the meets of $P_0P_1, P_2P_3; P_0P_2, P_3P_1; P_0P_3, P_1P_2$ respectively. The ranges corresponding to $[\sigma_2], [\sigma_3]$ (20·5), will be denoted by $[\sigma_2{}'], [\sigma_3{}']$ respectively.

Any two conics, one of each of the ranges $[\sigma_2{}'], [\sigma_3{}']$ may be denoted by $\Sigma_2(P_2{}'Y)$, $\Sigma_3(P_3{}'Z)$. Their F-conic is (12·41) $C_1(P_2{}'Y, P_3{}'Z)$; and since $P_2{}'$, $P_3{}'$ are 1-correspondents on T_0, therefore (10·27): $F\{\Sigma_2(P_2{}'Y), \Sigma_3(P_3{}'Z)\}$ *degenerates into two lines, respectively containing $P_2{}'$, $P_3{}'$.*   (20·61)

We shall denote these lines by $P_2{}'Y'$, $P_3{}'Z'$.

Again, since (4·4) $S_1(P_0{}')$ consists of the two lines P_0P_1, P_2P_3; therefore (12·15) the Φ-conic of $S_1(P_0{}')$ and any order-conic touches P_0P_1 and P_2P_3. Thus: *The tangents P_0P_1, P_2P_3 are common to all the conics of both ranges $[\sigma_2{}'], [\sigma_3{}']$.*   (20·62)

Let the remaining common tangents of $\Sigma_2 (P_2'Y)$, $\Sigma_3 (P_3'Z)$ be denoted by RR', SS', where the points R, S'; R', S are contained by P_0P_1, P_2P_3 respectively. The range determined by $\Sigma_2 (P_2'Y)$, $\Sigma_3(P_3'Z)$ is (11·33) a 1-autopolo-range, i.e. the poloconic, q. Θ_1, of any conic of the range is also a conic of the range. Also the point-pair R, S is a degenerate conic of the range; therefore $\Gamma_1 (R, S)$ is a conic of the range. But since R, S are respectively contained by P_0P_1, P_2P_3, two corresponding tangents (4·3) to Δ_1; therefore (cf. 10·27) $\Gamma_1 (R, S)$ consists of two points, respectively contained by P_0P_1, P_2P_3. Thus R', S' constitute $\Gamma_1 (R, S)$. Further (11·33), any conic of the range and its poloconic q. Θ_1 are harmonic q. $\Sigma_2 (P_2'Y)$, $\Sigma_3 (P'Z)$. It follows that R, R' are harmonic q. the points of contact with RR' of $\Sigma_2 (P_2'Y)$, $\Sigma_3 (P_3'Z)$; similarly for S, S'. Hence (12·11) RR', SS' touch the Φ-conic (Φ_1, say) of $F \{\Sigma_2 (P_2'Y), \Sigma_3 (P_3'Z)\}$ and the line-pair RS', $R'S$. But (20·61) $F \{\Sigma_2 (P_2'Y), \Sigma_3 (P_3'Z)\}$ is the line-pair $P_2'Y'$, $P_3'Z'$; and RS', $R'S$ are P_0P_1, P_2P_3 respectively. Therefore (12·16) four tangents to Φ_1 are P_0P_1, P_2P_3, $P_2'Y'$, $P_3'Z'$; thus Φ_1 touches RR', SS', P_0P_1, P_2P_3, the base lines of the range determined by $\Sigma_2 (P_2'Y)$, $\Sigma_3 (P_3'Z)$. Hence: *The Φ-conic of the line-pairs which constitute $S_1 (P_0')$, $F \{\Sigma_2 (P_2'Y), \Sigma_3 (P_3'Z)\}$ is a conic of the range determined by the two conics $\Sigma_2 (P_2'Y)$, $\Sigma_3 (P_3'Z)$.*   (20·63)

Since $\Sigma_2 (P_0'P_2')$, $\Sigma_3 (P_0'P_3')$ degenerate (4·4) into the point-pairs P_1', P_3'; P_1', P_2' respectively, we have from (20·54): *One point-pair of the range $[\sigma_2']$ is P_1', P_3'; and one point-pair of the range $[\sigma_3']$ is P_1', P_2'.*   (20·64)

The theorems of the present paragraph (20·6) have been proved only for the degenerate case in which P_0' *is a point of* T_0. It appears that when P_0' is not a point of T_0 the theorem corresponding to (20·63) cannot hold. For if it held, then the three conics $S_1 (P_0')$, $\Sigma_2 (P_2'Y)$, $\Sigma_3 (P_3'Z)$ would have a common apolar triangle; whence the common apolar triangles of the conics of the ranges $[\sigma_2']$, $[\sigma_3']$ would be identical, and this is not in general the case.   (20·65)

20·7. Just as (5·2) any configuration of symbol $(12_4, 16_3)$ and type A determines another configuration of the same symbol and type, which we have termed the central dodecad of the first; so the configuration (20·34) of symbol $(16_3, 12_4)$ and corresponding type determines another configuration of the same symbol and type, which may be termed the *axial configuration* of the first. In order to exhibit clearly the duals of the theorems of (20·5, 6) we must develop notations for the 12-line configuration of (20·34) and its axial configuration.

We have seen (6·1, 3) that the sixteen lines of the twelve-point configuration $[LMN]$ (5·4) may be obtained as follows: the four points I_0, I_α, I_β, I_γ are the poles, q. T_1, of a certain line $(A_0B_0C_0)$; the sixteen polars, q. Θ_1, of the points I_0, I_α, I_β, I_γ are the sixteen lines of the $[LMN]$ configuration. Dually, the four lines XYZ, B_1C_1,

$C_1 A_1$, $A_1 B_1$ are (20·31) the polars, q. Υ_1, of a certain point; it is therefore convenient to denote the lines XYZ, $B_1 C_1$, $C_1 A_1$, $A_1 B_1$, by i_0, i_a, i_β, i_γ respectively. The points A_1, B_1, C_1, X, Y, Z will then be denoted by $i_\beta i_\gamma$, $i_\gamma i_a$, $i_a i_\beta$, $i_0 i_a$, $i_0 i_\beta$, $i_0 i_\gamma$ respectively.

Now of the sixteen poles, q. V_1, of the lines i_0, i_a, i_β, i_γ (which throughout this paragraph will be termed simply the *poles*), it is clear, by analogy with (5) or otherwise, that four are contained by each of two lines meeting at each of the six vertices of the quadrilateral Q. These twelve lines will be termed the twelve *connectors*. Now, by analogy with (5·5), the point $i_\beta i_\gamma$ (i.e. A_1), as the meet of two connectors, may be denoted by $l_0 l_1$; similarly B_1, C_1, X, Y, Z may be denoted by $m_0 m_1$, $n_0 n_1$, $l_2 l_3$, $m_2 m_3$, $n_2 n_3$ respectively. Let *either* of the connectors containing A_1 be denoted by l_0; let *either* of the connectors containing B_1 be denoted by m_0, and let *either* of the connectors containing X be denoted by l_2. The notation for the connectors is then completely determinate: for (cf. 5·5) l_0, m_0, n_0 must be concurrent, and therefore n_0 must be that connector which contains C_1 and the meet of l_0, m_0; and similarly m_2 must be that connector which contains Y and the meet of l_2, n_0; etc.

We have now two definite quadrilaterals. of one of which the vertices are $l_0 l_2$, $m_0 m_2$, $n_0 n_2$, $l_3 l_1$, $m_3 m_1$, $n_3 n_1$. The joins of $l_3 l_1$, $m_3 m_1$; $l_3 l_1$, $m_0 m_2$; $l_0 l_2$, $m_3 m_1$; $l_0 l_2$, $m_0 m_2$ will be denoted by ι_0, ι_a, ι_β, ι_γ respectively; and the sides u_0, u_a, u_β, u_γ of the other quadrilateral may similarly be specified according to the scheme of (5·5). We have thus the following: *The pair of connectors containing any vertex of the given quadrilateral meet at four points the pair of connectors containing the opposite vertex of that quadrilateral; the twelve points thus determined by the three pairs of opposite vertices of the given quadrilateral are the vertices of two other quadrilaterals.*   (20·71)

These two quadrilaterals will be termed the *Mention quadrilaterals* of the given quadrilateral.

Again, the diagonal triangles of the quadrangles $[I]$, $[O]$, $[U]$ are respectively (6·6) $A_1 B_1 C_1$, $A_2 B_2 C_2$, $A_3 B_3 C_3$; and these three triangles are (5·2) in perspective two by two, $A_0 B_0 C_0$ being the common axis of perspective. Moreover (6·9) $A_0 B_0 C_0$ is the axis of the triangle $A_1 B_1 C_1$ and the conic $S_1 (X_0)$, whatever point may be denoted by X_0. Let us dualize this theorem, using Υ_0 and its related class- and order-cubics and conics instead of T_0 and its related order- and class-cubics and conics; writing $\Xi_1 (y)$ for $S_1 (X_0)$, and remembering that (20·33) $\Xi_1 (y)$ is Σ_3. Then: *The diagonal triangles of the given quadrilateral and of its Mention quadrilaterals are in perspective two by two, the pole, q. Σ_3, of the diagonal triangle of the given quadrilateral being the common centre of perspective.*

(20·72)

To obtain the theorems dual to those of (20·51–4) we note that (6·3) the four polars, q. Θ_2, of O_0 are the lines $L_0 M_0 N_0$, $L_0 M_2 N_2$,

$L_2M_0N_2$, $L_2M_2N_0$. For these we must substitute the points $l_0m_0n_0$, $l_0m_2n_2$, $l_2m_0n_2$, $l_2m_2n_0$; each of these points is the meet of three connectors, and is therefore one of the sixteen poles. Remembering that the Mention quadrilaterals are those determined by the four lines [ ], [u] and proceeding as in (20·72), we have from (20·51, 2): *The F-conics of Σ_3 and the conics of the ranges respectively inscribed in the Mention quadrilaterals constitute two pencils; and four conics of either pencil each contain four of the sixteen poles, q. the cubic V_1, of the sides of the given quadrilateral.* (20·73)

From (20·53), we have similarly: *The F-conic of Σ_3 and one conic inscribed in either of the Mention quadrilaterals is circumscribed to the diagonal triangle of the other.* (20·74)

Again, from (20·54): *The F-conic of Σ_3 and one conic inscribed in either of the Mention quadrilaterals is also the reciprocal, q. Σ_3, of that conic.* (20·75)

20·8. The duals of the theorems of (20·6) are but the projective forms of known theorems, any pair of points being substituted for the circular points at infinity. The only novelty in the present treatment consists in relating the theorems to the properties of the cubic. They will therefore be dealt with as briefly as possible.

Since the given conic of (20·1), there identified with Σ_3, is always (20·33) a pole conic q. a class-cubic of which the Wallace cubic is the Cayleyan, therefore the conic may in the degenerate case be represented by the point-pair E_0, E_1 of (20·35). This pair of points, as conjugate poles q. the order-cubic V_1, will now play the part of the lines P_0P_1, P_2P_3 (which are conjugate polars q. the class-cubic Θ_1) of (20·6). The line E_0E_1 is therefore to be substituted for P_1'; the point E_1', as meet of E_0E_1 and the corresponding tangent to Υ_0, for the line $P_1'P_0'$; and therefore the point E_0' for the line $P_2'P_3'$. Again, since the point-pairs P_1', P_3'; P_1', P_2' are degenerate conics of the ranges $[\sigma_2']$, $[\sigma_3']$ respectively; therefore, if E_0E_1, e; E_0E_1, e' be respectively line-pairs of the pencils of conics ($[s]$, $[s']$, say) which in the degenerate case represent those of (20·73), then the lines e, e' are those to be substituted for the points P_3', P_2' respectively. We thus obtain without trouble the following theorems, *valid only in the case where Σ_3 is replaced by any point-pair E_0, E_1.*

From (20·61): *The Φ-conic of any two conics (s, s'), one of each of the pencils $[s]$, $[s']$, degenerates into a pair of points (J, J'), one of which is contained by each of the lines e, e'; and any line containing either of these points therefore meets the conics s, s' in pairs of points which separate each other harmonically.* (20·81)

From (20·62): *All the conics of both the pencils $[s]$, $[s']$ contain the pair of points E_0, E_1.* (20·82)

From (20·63): *The F-conic of the point-pairs E_0, E_1; J, J' is a conic of the pencil determined by s, s'.* (20·83)

From (20·64) e, e' meet at E_0'. (20·84)

It appears from (20·35) that E_0' is, in the present case, a Wallace point; viz. that not contained by E_0E_1.

The conic which replaces $\Xi_1'''(y)$ (20·42) when the point-pair E_0, E_1 replaces Σ_3 (i.e. $\Xi_1(y)$) may most easily be identified by the method adopted in (20·5–7) and above. To the class-conics $\Xi_1(y)$, $\Xi_1'(y)$, $\Xi_1''(y)$ (which are the pole-conics of a given line q. three class-cubics having the same Hessian) there correspond in the case of (20·5) the order-conics $S_1(X_0)$, $S_2(X_0)$, $S_3(X_0)$ (the polar conics of a given point q. three order-cubics having the same Hessian); while to $\Xi_1'''(y)$ (the class-conic such that $\Xi_1(y)$, $\Xi_1'''(y)$; $\Xi_1'(y)$, $\Xi_1''(y)$ are harmonic in the range of pole conics of y), there corresponds the order-conic $[S_4(X_0)$, say], such that $S_1(X_0)$, $S_4(X_0)$; $S_2(X_0)$, $S_3(X_0)$ are harmonic in the pencil of polar conics of X_0. If, as in (20·6), we substitute P_0', a point of T_0, for X_0; then, with the notation of that paragraph, the conics $S_1(P_0')$, $S_2(P_0')$, $S_3(P_0')$ become the line-pairs P_0P_1, P_2P_3; P_0P_2, P_3P_1; P_0P_3, P_1P_2 respectively. The pencil $[S_n(P_0')]$, constituted of the polar conics of P_0' q. all the order-cubics of the T-pencil, is therefore that determined by P_0, P_1, P_2, P_3 as base-points; the triangle $P_1'P_2'P_3'$ being thus apolar to every conic of the pencil. Since $S_1(P_0')$, $S_4(P_0')$; $S_2(P_0')$, $S_3(P_0')$ are to be harmonic, it is easily seen that $S_4(P_0')$ is that conic (of the pencil) to which the triangle determined by P_0P_1, P_2P_3, $P_2'P_3'$ is also apolar. Moreover, the two vertices of the latter triangle which are contained by $P_2'P_3'$ separate P_2', P_3' harmonically.

Dualizing as before, the class-conic $\Xi_1'''(y)$ becomes in the degenerate case a conic to which the triangle $E_0E_1E_0'$ and also the triangle determined by e, e', E_0E_1 are apolar; moreover, e, e'; $E_0'E_0$, $E_0'E_1$ are harmonic. (20·85)

20·9. Summing up, it appears that none of the circles associated with the quadrilateral in the euclidean theorems of (20·0) are represented by circles in the non-euclidean case (15).

The circumcircles of the triangles of the quadrilateral (together with the line at infinity in each case) are represented by order-cubics, and each of the remaining circles appears as the F-conic of the Absolute and some other conic.

In particular, the circumcentric circle (the "Eight-Point Circle" of Hermes[*]) becomes a conic [the G-conic of (20·4)] which still (20·42) contains eight points representing those of Hermes, but does not contain any Wallace point.

The orthocentric line (20·024) of the given quadrilateral Q, together with the line at infinity, is represented by the H-conic, which in the non-euclidean case contains (20·24) the absolute poles of the sides of Q; and similarly for the orthocentric lines of Mention's quadrilaterals, which (together with the line at infinity in each case) are represented (cf. 20·64) by the conics of (20·75). It is curious

[*] *Loc. cit.* (20·0).

that no representative of the (euclidean) mid-diagonal line of Q appears to play any considerable part in the generalization.

The sides of Q are the polars, q. a certain class-cubic (Υ_1) of a point which is in the non-euclidean case the orthocentre (20·72) of the diagonal triangle of Q; and the sides of Mention's quadrilaterals are the polars of the same point q. the other two class-cubics which have the same Hessian as Υ_1. The sides of the three quadrilaterals thus determine a configuration of symbol $(16_3, 12_4)$, of which the sixteen points are (either in the non-euclidean or in the euclidean case) the in- and ex-centres of the triangles of Q ($v.$ 12·43, 20·32). The sides of the diagonal triangles of Q and of the Mention quadrilaterals therefore determine (cf. 5·1) a configuration of symbol $(6_3, 9_2)$, which is converted into another of symbol $(16_3, 12_4)$ by the addition of three lines meeting at the orthocentre of the diagonal triangle of Q. The further properties of the quadrilaterals and triangles discussed in Sancery's paper* are therefore those of a configuration of symbol $(18_4, 24_3)$, the dual of that first systematically discussed by de Vries and Caporali†. The Cayleyan of Υ_1 (the order-cubic V_0 which the writer has ventured to call the Wallace cubic, in honour of a mathematician whose pioneer work on the geometry of the triangle and quadrilateral is still generally overlooked) contains (20·21, 20·3) the six vertices of Q and the six Wallace points.

Although the Wallace cubic and the related class-cubics thus play a leading part in the general theory, the further (euclidean) theorems (20·023, 45, 84) involving the Wallace point seem (cf. 20·65) to belong to the degenerate case of (20·8) only. It need hardly be noted that if E_0, E_1 be identified with the circular points, then the two ranges of points $[J, J']$ become the centres of the circles of Steiner's coaxal systems (20·073), the lines e, e', which (20·85) are at right angles, become at the same time the lines of centres of those circles and the orthocentric lines of Mention's quadrilaterals, and the point E_0' becomes the euclidean Wallace point‡; while the theorem that the two systems of circles are orthogonal is derived from (20·83). The Wallace conic of (20·23) (which does not, in the general case, touch any one of the Wallace lines) clearly becomes the Wallace parabola of (20·023); while from (20·85) the conic $\Xi_1'''(y)$ becomes a rectangular hyperbola of which the centre is the Wallace point and the axes are the Steiner lines. The theorem (20·44) therefore becomes: *The circumcentric circle is the reciprocal of the Wallace parabola q. a rectangular hyperbola of which the centre is the Wallace point and the axes are the Steiner lines.*

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* *Loc. cit.* (20·0). † *Loc. cit.* (5·2).

‡ The remaining five Wallace points of (20·21) become (20·35) the circular points at infinity each counted twice, and the isogonal conjugate (a point at infinity) of the euclidean Wallace point.

Meteorology and the non-flapping flight of tropical birds. By GILBERT T. WALKER, C.S.I., Sc.D., Ph.D., F.R.S.

1. An extremely interesting paper on 'The problem of soaring flight' was published in Part II of Vol. xx of these *Proceedings* by Dr Hankin of Agra to whom all who are interested in flying must be under a deep obligation for the large amount of material that he has collected and published. While summarising 'existing evidence as to the nature of soaring flight' Dr Hankin appears, however, to have attached but little importance to what is known from aviators and meteorologists as to the actual movement of the air; and since in the absence of this knowledge the source of the energy involved is asserted to be a 'complete mystery' it appears desirable to discuss briefly what is known of the air movements in tropical countries, to examine how these may be used by birds, and then to consider how far this information affects Dr Hankin's conclusions.

I. THE FACTS ABOUT TROPICAL AIR-MOVEMENTS.

2. It is well established that in the tropics the sun heats the ground and thereby the air close to it to an extent that makes the rate of fall of temperature with height considerably greater than the adiabatic rate at which temperature would fall in a rapidly rising mass. Thus at Agra the following figures indicate approximately the normal centigrade temperatures in May in the early morning at the time of minimum temperature, and in the afternoon at the time of maximum temperature.

May	Early morning	Afternoon
Ground surface (sand)	—	64°
At 1·2 m. above ground	27°	42°
1 km.	24°	25°
2 km.	16°	17°
3 km.	8°	9°
4 km.	2°	2°
5 km.	-4°	-4°

The adiabatic rate being 1° in 100 m. it will be seen that in the early morning up to a height of 1 km. the conditions are extremely stable; and between 1 km. and 2 km. they are moderately stable. But in the afternoon the temperature difference of 22° between the surface and 1·2 m. up means phenomenal instability; the fall of 17° thence up to 1 km. involves great instability, and air rising from 1·2 m. to 2 km. would arrive there at 22°, a higher temperature than the surrounding air; and its ascent would not be checked

before 4 km. at a temperature of 2°. Under such conditions local ascending currents will be set up wherever the equilibrium is disturbed and in India aeroplanes will actually experience severe bumpiness. Similar conditions obtain in Egypt* where in the summer months bumpiness sets in about 9 hrs. and persists till about 18 hrs. The height to which bumps are felt increases as the temperature of the ground rises and measurements on 7 days at about 14 hrs. gave maximum heights to which bumps extended ranging between 4,000 and 10,000 ft. The effect of solar radiation is also shown by anemometers on towers, such as that at Allahabad at 150 ft. which indicated strong gustiness after the sun had been shining for some hours while there was much less movement at the ground level.

3. These ascending and descending currents are sufficiently widespread and powerful in April and May to reverse the direction of the ground winds over the whole region represented by Bareilly, Lucknow and Benares. In these months the pressure gradients would produce an east or south-east wind in the afternoon; but the vertical convection currents cause mingling of the lower with the upper air, thus bringing the momentum of the upper W.N.W. wind down to the ground level. The velocity then averages about 10 miles an hour at 5000 ft. above Agra and is 20 miles an hour at 10,000 ft., the direction being W.N.W. at both heights: and as the ground gets heated by the sun a surface north-westerly wind is set up which averages 6 or 7 miles an hour in the afternoon.

The growth in successively added shells of the large hailstones that are frequently associated with thunderstorms may be taken as evidence of very violent updraught, and though thunderstorms do not form every day in all places it is observed by those who send up kites and balloons that strong up-currents are usually to be met with under cumulus clouds.

4. It is obvious that air cannot be ascending over the whole of a large region unless there is a horizontal inflow of air round the boundary of the region: and over the boundaries of some regions, such as the north-western desert and the Gangetic plain of India in the afternoon of the hot weather, charts of normal winds† clearly show an inflow; but even an inflow averaging 10 miles an hour up to a height of 4 miles round a circular area 800 miles in diameter will only produce an average upward current of 0·2 mile an hour. As this is inappreciable there must be roughly as much of descending as of ascending currents. Further, we know that the air over bare

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* See *Professional Notes*, No. 20 of the London Meteorological Office, 1921, pp. 118, 119.

† See the isobar of 29″·55 in the pressure chart of 4 p.m. in May, plate 15 of the Climatological Atlas of India.

rocky or sandy soil or over a town is more heated than that over land covered with crops or trees or open water; and the result as reported by aviators is that an upward bump is felt up to a height of several thousand feet on reaching such a place and a downward bump on passing over green vegetation or water. It seems worth while to form some idea of the vertical velocities involved. If an aeroplane, possessing an angle of descent of 7°, is travelling, say at 90 m.p.h. or 132 ft. a second, what change in vertical velocity is necessary to produce a marked effect? Now it would take a downward velocity of 132 tan 7° suddenly to remove all support from the air, i.e. 16 ft. a second; and it seems reasonable to suppose that while this would be a very violent bump—such as is met with only under special conditions—probably 4 to 8 ft. a second would correspond with the descriptions given of the effects ordinarily produced. In hilly regions we shall have unequal heating caused not only by differences of the nature of the ground surface, but also by differences of its inclination to the sun's rays; slopes on which the sun shines almost at right angles will set up strong ascending currents, while those on which the incidence is oblique or which are in shadow, will tend to produce descending currents.

5. A dweller in a hill station has ample opportunity of verifying these currents, either by watching smoke or dust or feathers in the air; and at Simla within 1½ hrs. of sunrise currents of 6 to 10 ft. a second are common at heights of only 20 ft. above tree-clad slopes facing the sun; at greater heights above the trees, where the resistance to motion is less, the motion will clearly be greater. In the shade descending currents of nearly the same strength may be observed. In the plains data are harder to obtain. The ordinary equipment of an observatory does not contain any instrument for measuring vertical currents, nor are the ordinary ascents of pilot balloons adapted to afford accurate information of this kind. At Agra a rather crude recording instrument was erected by Mr Harwood on a tower at 45 ft. above ground: its readings were not calibrated, but he says that on sunny days it indicated ascending currents beginning shortly before the upward gliding of birds and ending shortly after this had ceased. In ordinary undisturbed weather these ascending currents were not recorded at night.

II. The Mechanical Properties of Birds and Aeroplanes.

6. The angle of descent of an aeroplane may be as low as 6°, but that of one with an 'alula' wing, imitating that of a bird, is said to be 5°. I would here give an extract from *The Aeroplane*, Vol. xx, No. 3, p. 62 (January, 1921): 'Assuming that Nature has succeeded in building wings as efficient as the "Alula," we may take values for the lift coefficient of 0·7 and for the L/D ratio of

21. For the complete bird the L/D and the gliding angle will not be worse than 16 and 1 in 16 respectively. For a loading of 1 lb. per sq. ft. this gives 17 m.p.h., or 24 ft. per sec., as the supporting speed, and 1·5 ft. per sec. as the rate of fall when gliding. An up-current of 1·5 ft. per sec. = about 1 m.p.h. would account for soaring in this case.' After daily opportunities for many years of watching the gliding flight of birds of many kinds from above and below and at the same level I find it hard to believe that a kite's angle of descent when travelling at an ordinary rate is as much as $3°$, though that of the more heavily loaded vulture when travelling slowly may perhaps be $5°$.

III. THE USUALLY ACCEPTED EXPLANATIONS OF THE GLIDING FLIGHT OF BIRDS.

7. In standard books the explanation usually given of the way in which birds appear capable of gliding for an indefinite period without expenditure of energy is twofold:

(*a*) they make use of differences in horizontal currents, and

(*b*) they make use of ascending currents.

Regarding both (*a*) and (*b*) there can be no doubt as to the nature of the results produced; the only question is whether these are on a large enough scale to produce the energy required.

8. Concerning (*a*) it is strangely* necessary to insist that it is as impossible to derive energy continuously from a wind that is constant in time and space as it is from a perfect calm. Inland winds are very far from constant and a wind, say of 20 miles an hour, will usually be continually changing in velocity and direction, the extreme limits of the oscillations in velocity being something like 15 miles apart.

When a bird is trying to rise as fast as it can its velocity is fairly constant: we shall call it v and corresponding to it there is a definite small angle of descent δ, i.e. if the bird glides downhill at an angle δ the velocity will remain equal to v; if the path of the bird is uphill at a small angle α the acceleration backwards will, as a first approximation, be $g(\alpha + \delta)$, and if the acceleration of the air, whose movement is assumed entirely horizontal, is f, inclined $\pi - \theta$ to the velocity v, the velocity of the bird relative to the air would decrease at a rate $g(\alpha + \delta) - f\cos\theta$. Thus if v is to be maintained constant the angle of climb must be $f\cos\theta/g - \delta$, and the bird will be using the air motion to the best advantage if its

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* Theories based on a denial of the principle have been published by M. P. Noguès, *Comptes Rendus*, Tome 170, pp. 65–68 (1920) and by Colonel de Villamil according to a summary in *Aeronautics*, Vol. xx, p. 134 (1921). The truth of the principle is obvious on superposing on the whole system a velocity equal and opposite to that of the wind.

relative velocity v is always opposite to that of the acceleration f. In that case the rate of gain of potential energy is $mvag$ or $mv(f - g\delta)$, where m is the mass of the bird, which is availing itself of the acceleration of the air to maintain its own velocity relative to the air.

9. If now we plot the air velocity as a vector OP, measured from a fixed point O, the acceleration f is the velocity of the point P and the time integral of f during any period t is the length l of arc traced by P during that period: the potential height gained will be $v(l - gt\delta)/g$. Now from an ordinary record of velocity by a Dines' anemometer, such as that in Fig. 1, p. 41, of Shaw's *Manual of Meteorology*, Part IV, it might be estimated that an hour's record contains about 25 gusts averaging about 12 miles in amplitude, so that in 1 minute we should have an arc (i.e. a sum of changes in velocity irrespective of sign) of 10 miles an hour or 15 ft. a second: but a quick run shows that the true arc is much more than this, and from that in the same diagram I estimate the velocity arc in 1 minute as about 85 ft. a second. Now there are changes of direction as well as of magnitude (see Shaw's *Manual*, Part IV, p. 45) and if the point P describes a circle of diameter d its arc is πd by comparison with $2d$, the arc described in the velocity diagram. The curve traced by P as observed is extremely irregular and the length of its arc must be roughly comparable with $85\pi/2$ or 133 in 1 minute: we shall estimate it as 2 in a second. If the bird were gliding in this wind therefore he would gain height at a rate $v(2/g - \delta)$. Now for a kite we may take v as 16* and my estimate for δ is $2°$; so the rate of climb is about ·44 ft. a second or 26 ft. a minute. Climbing will only cease when $f = g\delta$ or 1·1, corresponding to a gust-amplitude of about 6 m.p.h.

10. During a period of wind without sunshine a kite when not searching for food will often be seen gliding about in all directions, apparently aimlessly; its behaviour seems entirely consistent with this interpretation though it suggests rather more detail than that shown by a quick run anemograph of the present pattern. In a moderate breeze of force 4, averaging 16 instead of 22 miles an hour, assuming proportional eddy motion, we should expect height to be gained at an approximate rate $v(\frac{1}{22} - ·035)$ or ·17 ft. a second, or 10 ft. a minute. In this rough approximation we have neglected the small amount of energy lost in the slight banking, or heeling over, requisite for the changes of direction; the average radius of curvature being something like 40 ft. the inward acceleration will be about 6 f.s.s. so that the banking will be $\tan^{-1}\frac{6}{32}$ and the effective angle of descent will not be increased by 5 per cent.

* See *Animal Flight*, pp. 30, 31, where a cheel or kite describes a circle 12 m. in diameter in about 8 seconds.

which is inappreciable. For a vulture, if we take the angle of descent as roughly 5° and the velocity as 30*, the smallest average gust amplitude that will keep the bird in the air without flapping is easily seen to be 16, corresponding to a wind velocity in England of about 30 miles an hour, or force 6. That there is a very marked difference between the two birds in this respect is easily observed.

11. I regard the previous sections 8 to 10, in spite of the roughness of their approximations, as affording a basis for regarding Langley's 'internal work of the wind' as adequate for explaining what Hankin calls 'wind-soarability': and a first glance at the charts of a Dines' anemograph would suggest that the same explanation applies to the conditions produced in the tropics by sunshine. At Allahabad there was from 1909 to 1911 one of Dines' instruments on a tower about 150 ft. high in addition to one on the observatory tower 55 ft. high. On each some time after sunrise in clear weather there developed a marked turbulence or gustiness which died away before sunset: thus at the upper station on the mean of the month of May 1910, the amplitude of the oscillations of wind velocity was at its minimum of 1·0 m.p.h. between 22 hrs. and 23 hrs., i.e. the 23rd hour. It increased to 2·5 during the 7th hour, 4·0 during the 8th, 6·2 the 11th, and thence to the 17th varied between 7·3 and 8·2; for the 18th it was 5·3, the 19th 3·3, the 20th 1·5 and thence diminished. The mean velocity varied only from 11·5 at the 10th and 18th hours to 15·0 for the 4th. At the 55 ft. station the amplitudes of the oscillations were slightly less than half those of the 150 ft. station, ranging from a minimum of 0·3 between 19 hrs. and 23 hrs. to a maximum of 3·1 between 11 hrs. and 15 hrs.; the mean velocity had its maximum of 4·1 between 13 hrs. and 16 hrs. and its minimum of 0·8 between 19 hrs. and 22 hrs. This marked increase of eddy motion with height is what we should expect as the upward currents will be gaining in velocity up to a height of nearly 4 km. or 13,000 ft. The gust-amplitude of 6 necessary for the maintenance of a kite without flapping occurs at 150 ft. from 10 hrs. to 17 hrs.; and by extrapolation it may be inferred that at about 430 ft. the gust-amplitude will exceed 16 for the same period of 7 hrs. so that Langley's 'internal work' will enable a vulture to maintain horizontal flight without flapping.

12. Another suggestion, due to the late Lord Rayleigh†, is that a bird may use an increase of wind with height as a source of energy; and we may now examine how far this method is adequate under tropical conditions. We suppose that a bird is describing circles, of radius a, inclined to the horizontal at an angle β. If the increase of wind for each foot of height be p f.s. the wind velocity‡ will be

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* *Animal Flight*, pp. 30, 31.   † *Scientific Papers*, Vol. IV, pp. 464–467.
‡ Assumed horizontal, perpendicular to a level line in the plane of the circles.

a minimum at the lowest point of the circle and will exceed this by $ap (1 - \cos \theta) \sin \beta$ ft. a second at a point on the circle whose distance measured along the arc is $a\theta$ from the lowest point. As θ increases at a slowly varying rate v/a the rate at which the velocity of the air at the point θ increases will be $vp \sin \theta \sin \beta$, and the component of this along the tangent to the circle is $vp \sin^2 \theta \sin \beta \cos \beta$. Thus the gain in potential height due to the change in air velocity will in a complete circuit be the time-integral of $v (vp \sin^2 \theta \sin \beta \cos \beta/g - \delta)$.

Now the value round the circuit of $\int v dt$ is $2\pi a$: and, if the bird could be regarded as moving in a circle under no forces but gravity, we should have $\int v^2 \sin^2 \theta dt = \frac{1}{2} w^2 \int dt$, where w is the value of v at a level with the centre of the circle; for corresponding to each value of $\sin^2 \theta$ there will be four values of v^2 of which the sum would be accurately $4w^2$. If we take into account the slight retardations due to 'drift' and the slight accelerations due to the air movement we shall still have without serious error

and so
$$\int v^2 \sin^2 \theta dt = \tfrac{1}{2} w (2\pi a),$$

$$\int v (vp \sin^2 \theta \sin \beta \cos \beta/g - \delta) \, dt = \pi a (wp \sin \beta \cos \beta/g - 2\delta).$$

Taking $a = 30$, $\beta = 15°$, $w = 28$, we obtain as the approximation to the integral, $20 (p - \cdot 3)$ ft. Now in May 1921 the average velocity at Agra between 7 and 8 a.m. was 7·8 ft. a second at 4 ft. above ground, and was 15·7 at 45 ft.: so that on the mean of the month $p = 7\cdot9/41$, or $\cdot19$, and the average conditions below 45 ft. would not suffice for climbing. But winds vary during an hour, from day to day at the same hour and from place to place at the same time. It may, I think, be concluded that while this method will not in general be adequate to enable height to be gained by gliding it will occasionally suffice; and it will at times be a useful auxiliary when other methods require supplementing. It is noteworthy that Hankin (*Animal Flight*, p. 29) remarks that in ordinary circling flight the gain in height, both for vultures and cheels (kites), is mainly on the upwind and windward sides of the circle; for if we suppose the inclined circular path to be slowly rising we obtain the most rapid rise on the upwind portion of the circle, the windward portion being the highest.

13. For a sea gull describing circles near the stern of a steamship we may impose on the whole system a velocity equal and opposite to that of the ship, say 14 knots or 24 feet a second. We shall thus have a velocity of 24 compounded with the wind velocity in the open and a small positive*, or a negative value if an eddy forms, near the surface of the sea in the shelter of the stern. If there be a breeze from one side, the place of greatest shelter will,

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* The direction of the reversed velocity of the ship.

of course, be to one side of the stern. There will thus be a change of velocity of something like 24 in a height of about 24 ft., so that $p = 1$; and if $\beta = 30°$ and $w = 28$, and $\delta = 3°$ for a sea-gull the roughly approximate formula gives for each circle of 48 ft. in diameter a rise of 20 ft. There is thus an ample source of energy for keeping up with the ship.

14. Turning now to (*b*), the use of ascending currents, there can in my view be no question that among hills this in general provides all the energy that is needed. On the south side of the Simla ridge there is a bay with ridges projecting southwards at its east and west ends. About 2 hrs. after sunrise the western slope of the bay, having by this time been strongly heated by the sun, attracts the large birds which are flying within the bay: they may flap until they reach the slope but having reached it they glide to-and-fro over the trees for indefinite periods. From time to time some of those which are higher up will start gliding in spirals up a column, presumably of strongly heated air, and others noticing this will join them until 20 or 25 are in the column which is usually about four times as high as it is broad. After perhaps 5 minutes of this climbing, which they seem to do purely for diversion, having reached as high as they can, they break off and some of them will dive at a considerable pace back to the tree-clad slopes. An hour or two later, when a higher sun has thoroughly heated the main ridge, they glide in a spiral up to a considerable height and then float away with very little restriction as to their locality provided that it is approximately over the main ridge. By midday they may be seen 'flex-gliding' (gliding with flexed wings) in all directions, sometimes rising slowly and sometimes falling slowly; and towards sunset there is a marked disposition to congregate at the eastern end of the bay, where the slopes of Jakko and the houses of the bazaar are still strongly heated by the western sun. During the past 7 years I have not seen a bird gliding upwards in a region where, from physical causes, descending currents could be expected; and in most cases ascending air has been strongly indicated*. Further information has been derived by watching kites, vultures and eagles through a 3″ telescope freely mounted so that it is easy to follow birds for fairly long periods flying at all heights from 500 ft. below to 1000 ft. above. These observations have thrown light on several problems. It is easy to see the violent 'burbling' produced by eddy motion on the upper surface of the wings and on the back of a vulture descending rapidly at a large angle of incidence; and this burbling does not occur when the bird is slowly climbing as steeply uphill as it can, although the angle of incidence is large. The primary quills are then protruding separately from

* For similar observations in Peru see G. M. Dyott in *The Aeronautical Journal*, October 1919, Vol. xxiii, p. 526.

each other for a distance of about 7 ins., and Nature obviously attaches much importance to this feature for she employs a special device to increase the separation beyond that due to the mere fan-like spreading of the feathers: she deliberately leaves the primary quills about $1\frac{1}{2}$ ins. wide for about ·7 of their length and suddenly steps down the width to half this amount for the remaining terminal portion. The wind pressure bends up the quills, especially those at the leading edge of the wing, so that the successive shafts lie on a forward slope of something like 30°, the leading quills being higher than those behind them; and the individual quills are rotated round their shafts, the posterior margin being raised by the pressure from below. This Handley-Page-like device seems phenomenally successful, for I have watched vultures climbing spirally on a strong upward current, e.g. when a wind is blowing against a cliff, with an angle of incidence which I believe to be as great as 28°: the upward inclination of the plane of the wings as well as the angle of climb appeared to be 20° while the velocity of the birds was about 25 ft. a second. This means a vertical rate of climb of about 8·5 f.s., and estimating the vertical current at 12 f.s. makes the bird's path relative to the air descend at an angle of about 8·5° so that the angle of incidence is 20° + 8·5°. At first sight it looks as if the bird with its wings plane tilted backwards (i.e. with the leading edge of the wings higher than the following edge) cannot possibly maintain its forward motion against the resistance of the air. But if we consider the forces acting—the 'lift' (at right angles to the relative motion and so inclined forwards at 8·5° to the vertical) and the 'drift'—it is clear that the condition for maintenance of the forward motion is that, since cot 8·5° is 7·7, the lift shall exceed 7·7 times the drift; and this condition is easy to satisfy, e.g. with a Handley-Page wing (*Aeronautics*, Vol. xx, p. 129, 1921).

15. In the plains the daily programme appears to resemble very closely that in the hills. Early in the day when the stability is breaking down, and late when it is nearly established, the up-currents are only local, while at midday they are inevitably more widespread and extend higher. On some days there is much apparently aimless wandering, probably explained by unusual gustiness used as suggested in paragraph 11 above: there are usually circles interpolated from time to time; and of these some are obviously started in the teeth of a puff of wind, but most are, I believe, described round a stream of rising air. On many days circling round ascending columns seems to predominate within 100 ft. of the ground level. Thus on a recent visit to Delhi railway station I noticed near sunset a long horizontal trail of smoke with a few upward projections due to very slowly rising air. Kites were gliding about horizontally with occasional circles and one kite gliding somewhat transversely across the line of smoke described

circles round two of the upward projections. On a damp hot day, with a feeling that thunder is about, when on meteorological grounds we should expect strong updraughts to be set up locally, it is not unusual to see a score of vultures and kites gliding spirally in a column about four times as high as it is broad; and I have watched other birds, who have noticed the column and wish to enjoy it, flap along near the ground until they reach the column and glide upwards in it. I fully share Dr Hankin's view, expressed to me in conversation, that birds derive a large amount of pleasure from gliding; I have several times at Simla seen the air suddenly filled with scavenger vultures and kites when the only obvious cause was a sudden change in the weather that made the air very turbulent and gave exceptional facilities for gliding. I notice that in a paper on dust-raising winds* Dr Hankin speaks of his 'experience that if, in the hot weather, a vulture is seen at a height of 1200 metres or more, then a large dust-devil is always within two or three miles distance.' It may be that the vulture has not used the dust-devil; but it has used the ascending currents produced by the vertical instability to which the dust-devil owes its existence.

16. The reason given in paragraph 2 for believing that the larger ascending currents, as distinct from smaller ones of, say 10 ft. in diameter, must become stronger up to a height of several thousand feet are confirmed in two ways:

(*a*) Kites and vultures when ascending from near the ground have to manœuvre in order to get up, often circling round an ascending current; but when by 10 hrs. they have reached a height of 500 or 1000 ft. they seem to have no further difficulties, and can flex-glide apparently in all directions. For example, on p. 20 of his book Hankin gives 80 m. as the ordinary height of flex-gliding of a kite, while 400 m. is that of a common vulture which has higher loading and needs stronger forces to maintain it.

(*b*) In the official description of conditions experienced in the air in Egypt*, there is an account of three types of bumps which fit well with impressions derived from Indian conditions. First, there is 'disturbed air,' thoroughly churned up, which makes an aeroplane roll and pitch continuously, and does not extend more than 3000 or 4000 ft. from the ground. Secondly, there are 'small vertical currents' which cause a sudden bump of 1 to 3 seconds; of these the bottoms can in many cases be detected 'in the vicinity of a town by kites or hawks soaring': 'over hilly country the vertical speed is probably as much as 1000 ft. a minute, but considerably less over the cultivated land or flat desert.' Thirdly, there are 'large vertical currents' of ascending or descending air, 'usually noticed over the Nile valley or near the edge of the cultivated land,'

* See *Professional Notes*, No. 20 already referred to, p. 121.

and to be detected 'when gliding by carefully watching the aneroid': for the rate of ascent or descent '300 ft./min.–400 ft./min. is probably not excessive': 'these vertical currents appear to cover an area of at least $\frac{1}{2}$ to $\frac{3}{4}$ of a mile in diameter.'

17. A study of conditions in Senegal with the help of special kites and balloons and registrations of the vertical velocity of the wind has led M. Idrac to the conclusion that 'chaque fois que les oiseaux volaient à voile, et sans exception ils se trouvaient dans un zone où le vent avait une composante ascendante*.'

IV. Consideration of Dr Hankin's Arguments.

18. When reading Dr Hankin's publications it is necessary to verify that the sense in which he uses phrases is not misunderstood. When looking in his paper on 'The problem of soaring flight' for his views regarding Langley's internal work of the wind it is natural to turn to his section 5 where he considers 'the effect of lateral gusts of wind.' It may, in fact, frequently be observed that a bird flying with its wings flat heels over when it wishes to take advantage of a gust and for the gusts which are actually recorded by an anemograph there is ample time for the heeling over to be effected. The argument of Dr Hankin's section is, as is seen from its last sentence, against flight by means not of such gusts as occur in reality, but 'lateral pulsations' of which I know nothing. Similarly in his section 6 under the title 'Soaring flight is not due to the effect of ascending currents' it is clear from the references to pp. 20 and 283 of his book that he limits 'ascending currents' to currents, previously horizontal, reflected upwards from roofs or walls. It is quite natural for a bird to avoid reflected currents when it can rise several hundred feet by using ascending columns of air in the open. In his section 7 on 'Convection currents and soarability' we should come to the gist of the matter and all Dr Hankin says there is that 'ample evidence exists that convection currents in the air caused by the heat of the sun, whether at ground level (IV, p. 263) or at a height (IV, p. 23) have nothing to do with soarability.' The former reference to his book is to the beginning of Chapter XV on 'Ascending currents caused by the heat of the sun's rays': but the discussion there is limited entirely to the question whether the heat eddies that cause shimmering are the cause of soarability. It is true that these eddies and the massive upward currents are both due to the sun's heating; but it does not follow that the ascending currents stop when shimmering stops and *vice versa*. In the latter reference (p. 23) he merely demolishes the idea that reliance can be placed on the argument that because opacity of the air and soarability cease at the same time, therefore soarability depends on the movements of small masses of air.

19. It would be surprising if Dr Hankin's disbelief in the use of ascending currents were based on such slight evidence and it is desirable therefore to deal briefly with the arguments in his book: I think the chief of these are to be found in his Chapter XI. Unhappily Dr Hankin's conceptions of mechanical law are hazy in the extreme, and as the matter under consideration is mechanical his arguments are often hard to follow. Thus on p. 208 it is said that a bird gliding horizontally in unsoarable air is acted on by four chief forces, 'lift,' 'weight,' 'pull' and 'drag': of these 'the "pull" consists of the momentum of the bird' and 'acts in a horizontal direction at the centre of gravity'! Again, if the bird were to glide into soarable air 'the pull would no longer act at the centre of gravity. It would no longer consist of the momentum. It would

* *Comptes Rendus*, Tome 172, No. 19, p. 1161 (1921). See also Tome 170, No. 5, pp. 269–272 (1920).

consist of the tractive effect of soarable air on the cambered wings' (p. 209). It is puzzling enough for a momentum to be a force, but why should it cease to be one when the air becomes soarable? To draw attention to such errors would be ungenerous were it not that important and detailed conclusions are based on them. Further, the 'pull' and the 'drag' are recognized as acting at different heights, so that a couple would be formed; yet the 'lift' and the 'weight' in unsoarable air are supposed to act in the same vertical line (pp. 194, 195), which is impossible without producing rotation of the bird about a transverse axis.

20. Dr Hankin's estimates of the forces acting on the wings of birds also seem to me lacking in precision; and he appears to ignore the large amount of information derived from experiments in wind-tunnels. Obviously if there are ascending or descending currents the angle of incidence of the wings in any position will not be the same as if the air were stationary; and when discussing this purely geometrical effect Dr Hankin says on p. 198 of a vulture 'gliding with speed ahead in an ascending current' that 'the angle of incidence is about 90°. In other words, the "total pull" acts in a direction at right angles to the surface of the wing, or nearly so.' Now from p. 42 I estimate that gliding with speed ahead means a rate of at least 60 ft. (18 m.) a second and as the vertical velocity in an extensive ascending current is presumably less than 15 ft. a second the angle of incidence, the plane of the wings being horizontal (see Fig. 65), will be less than 15°, not 'about 90°.' Yet it is on this false conclusion that Dr Hankin bases his idea that the 'unknown force of soarability,' which lies at the base of his analysis of flight, acts at right angles to the wing surface. Again, I am at a loss over the diagram (Fig. 66, p. 198) showing the wings of a vulture fast flex gliding inclined at 30° *downwards*. Dr Hankin rightly says that if they were 'imitated by a power-driven aeroplane' he would expect it 'to rapidly bring the machine to the earth.' I have for years looked for this position of the wings, and have never seen it. That it is very rare in Simla I am confident; my instinct that it is rare in the plains is corroborated by Dr Hankin's own drawings (Figs. 61, 62, 63, p. 197) of vultures flex-gliding at slow, medium and fast speeds. In the first and last of these the wing is supposed to have inclinations of 0° and 30°, and the reduction of the latter by 13 per cent. in width as seen from below would be easily recognised by one accustomed to watching birds. The same argument applies to the cheel diagrams (Figs. 8, 9 on p. 39). The reason for the belief is presumably that the pressure due to 'erg-air*' beneath the wings is wrongly supposed to act at right angles to them and so the wings must slope downwards in order to get a strong normal thrust forward. But this theory is itself contradicted by Fig. 65 in which the wings of a vulture slowly flex-gliding have no slope though the bird needs a forward thrust to maintain its motion; and it is also contradicted by the diagram on p. 210, in which a vulture is gliding in soarable air with its wings sloping upwards at an angle of about 12°; if in the latter case the thrust on the wing has a backwards component, how can forward motion be maintained?

One obvious question is, how is it that erg-air can only exert pressure on the lower surface of the bird's wing whether, as in the case just quoted, it is on the forward side of the wing, or, as in the fast gliding bird, it is on the following side?

21. Another feature on which Dr Hankin relies for a refutation of the use of ascending currents is the bending up of the digital quills (pp. 202–204) 'by the unknown force of soarability': he gives (p. 203) the forces necessary to straighten the four digital quills of an adjutant as weights totalling 170 gms.; and if in order to bend these beyond the flat to the position observed in flight

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* I use Dr Hankin's convenient phrase for the air which during sun-soarability does not, according to his view, obey the ordinary mechanical laws.

we double the weights we obtain a force of 680 gms. weight while the weight of the bird (p. 140) is 7344 gms. It is not hard to believe that an eleventh of the weight should be borne by feathers of which the area is about a seventh of that of the wing (see diagram on p. 131). A vulture's digital quills are said (p. 203) to need weights of 150 gms. under flex-gliding conditions, when I estimate (see Fig. 62) that the area of the digital quills is about a ninth of the wing area. As the weight to be supported is 5·5 kilos, the estimated pressure of 1·2 kilos on the eight digitals is about twice the average pressure on the wing: it is, however, known that the shape of the wing tips on an aeroplane has a very great influence on efficiency and I therefore see no difficulty in regarding the bending of quill feathers as due to ordinary mechanical causes. That this must be so follows from the frequently observed fact that a vulture flex-gliding on a cold cloudy day (with the possibility of 'erg-air' excluded) has his wing-tips as much bent up as on a hot clear day under otherwise similar conditions. Yet Dr Hankin infers from the bending of the quills, without quoting any figures in support (p. 204), that 'if sun energy subserves soaring flight by means of ascending currents'...'it (the air) acts as if, not only in one place but all over the sky, it is ascending *en masse* at a rate of thirty or more miles an hour.'

22. Considerations of space prevent me from discussing the other arguments used against the mechanical interpretation of flight: I will merely say that the double-dip which I have often seen from the level of the bird as well as from below appears to me the exact analogue of pushing the controlling lever of an aeroplane forward when stalling is imminent—it produces a downhill glide: and that the apparent stillness of feathers close to climbing vultures is due to the very great difficulty of observing an upward velocity of 2 f.s. in such an object when considerably above the observer through field glasses held in the hand; a similar remark applies to the difficulty in the plains of finding out whether a bird is gliding slightly uphill or slightly downhill. I have not touched on the problems of dragon-flies and flying fishes because I have not observed them systematically. Regarding the 'puttung,' the ordinary boys' kite of India (paragraph 3 of 'The problem of soaring flight'), this flies at an extremely high angle in light winds with the string at an angle of about 45° near the ground and 75° near the kite. I have never seen the string hanging vertically from the kite except temporarily when there has presumably been a lull so that the kite was descending and the freshening wind carried the string with it more rapidly than it did the kite. Obviously, under existing mechanical laws, if there is a steady horizontal wind the kite cannot fly with its string vertical, and if it does fly with its string steadily vertical there must be an ascending current with no horizontal current.

23. I cannot conclude these remarks without an expression of sincere regret over my inability to appreciate the physical side of Dr Hankin's work, much though I admire the zeal, industry and skill that he has shewn as an observer. We have often talked frankly over our differences, and he has always been willing to shew me his methods and results: it is only because he has given wide publicity to his views that I have felt obliged to attempt to controvert them.

The Algebra of Symmetric Functions. By Major P. A. MacMahon.

[*Read* 30 October 1922.]

1. The necessity for an algebra of symmetric functions arose in
the first place in 1884 from the circumstance that, in the Theory of
Invariants, the seminvariants of binary quantics were shewn to be
transformations of non-unitary symmetric functions of the roots
of an equation of infinite order[*], where a non-unitary symmetric
function is defined to be

$$\Sigma \alpha_1^{p_1} \alpha_2^{p_2} \alpha_3^{p_3} \dots \alpha_s^{p_s},$$

where the number of quantities α is infinite, s may be any integer
and exponents p may be any integers $\not< 2$.

Cayley[†] was thus enabled to attack the problem of determining
the seminvariants of the quantic of infinite order which, *qua* degree,
are irreducible. In the paper (*loc. cit.*) he introduced an algorithm
for the multiplication of two non-unitary symmetric functions and
then for the first time definitely broached an algebra of these
functions.

Cayley employs the partition notation of symmetric functions
and the algorithm is the production of an abbreviated method of
multiplication on combinatory principles. In fact, one part of the
process consists, if the two functions to be multiplied be

$$(p_1 p_2 p_3 \dots p_s), \quad (q_1 q_2 q_3 \dots q_t),$$

in placing *all or any* of the numbers q_1, q_2, $\dots q_t$ underneath the
numbers p_1, p_2, $\dots p_s$ in all the really distinct ways and in assigning
a number to denote the frequency of each way. The rest of the
algorithm is automatically completed and does not involve a
possibility of errors in counting.

He was able to give a general formula for the multiplication

$$(3^a 2^b)(3^c 2^d),$$

a, b, c, d denoting repetitions of parts, and was thence able to
advance the theory of the perpetuant seminvariants.

2. The next step in the algebra was taken by the present
writer[‡] who introduced the Hammond operators D_1, D_2, D_3, $\dots$ to
the subject. The result of multiplication, being a linear function of
monomial symmetric functions, we write

$$(p_1 p_2 p_3 \dots)(q_1 q_2 q_3 \dots) = \dots + C_{r_1 r_2 r_3} \dots (r_1 r_2 r_3 \dots) + \dots.$$

* MacMahon, 'Seminvariants and Symmetric Functions,' *A.J.M.* Vol. vi, p. **131**.
† 'A Memoir on Seminvariants,' *A.J.M.* Vol. vii, 1885.
‡ MacMahon, 'The Multiplication of Symmetric Functions,' *The Messenger of
Mathematics,* New Series, No. 167, March 1885; *Combinatory Analysis,* Vol. i, p. **43**.

Then we have by the laws appertaining to the operator

$$D_{r_1} D_{r_2} D_{r_3} \dots (p_1 p_2 p_3 \dots)(q_1 q_2 q_3 \dots)$$
$$= C_{r_1 r_2 r_3} \dots D_{r_1} D_{r_2} D_{r_3} \dots (r_1 r_2 r_3 \dots)$$
$$= C_{r_1 r_2 r_3} \dots;$$

that is to say that the number $C_{r_1 r_2 r_3} \dots$ is found as the direct result of the performance of certain differential operations.

Moreover, this method is directly applicable to the multiplication of *three or more* symmetric functions. The method depends upon the law by which a Hammond operator is performed upon a product of two or more functions. The way in which D_r is performed upon a product of s functions depends upon the compositions of the number r into s parts (zero parts being taken into account). These compositions are enumerated by

$$\binom{r+s-1}{r}$$

because this number is the coefficient of x^r in $(1-x)^{-s}$.

The operation of D_r divides up, therefore, into

$$\binom{r+s-1}{r}$$

separate operations, one of which is

$$(D_{c_1} F_1)(D_{c_2} F_2) \dots (D_{c_s} F_s),$$

$c_1 c_2 \dots c_s$ being a composition of r into s parts

$$c_1 + c_2 + \dots + c_s = r.$$

Ex. gr. $\qquad\qquad D_9 F_1 F_2 F_3$
$$= (D_2 F_1)(D_0 F_2)(D_0 F_3) + (D_0 F_1)(D_2 F_2)(D_0 F_3)$$
$$+ (D_0 F_1)(D_0 F_2)(D_2 F_3) + (D_0 F_1)(D_1 F_2)(D_1 F_3)$$
$$+ (D_1 F_1)(D_0 F_2)(D_1 F_3) + (D_1 F_1)(D_1 F_2)(D_0 F_3)$$

because 2 has, into 3 parts, the six compositions

$$200, \quad 020, \quad 002$$
$$011, \quad 101, \quad 110.$$

Not all of these separate operations may be effective because $D_{c_k} F_k$ is zero unless F_k involves a part c_k.

The complete operation

$$D_{r_1} D_{r_2} D_{r_3} \dots F_1 F_2 F_3 \dots F_s$$

is performed by associating single compositions of the numbers r_1, r_2, r_3, ..., each into s parts, in all possible ways. If r_k has R_k compositions into s parts we may have to deal with

$$R_1 R_2 R_3 \dots \text{ separate operations.}$$

Each of these may be denoted by a tableau.

For if a composition of r_k be

$$c_{1k}, \; c_{2k}, \; c_{3k}, \; \ldots \; c_{sk}$$

a tableau such as

c_{11}	c_{21}	c_{31}	.	.	c_{s1}
c_{12}	c_{22}	c_{32}	.	.	c_{s2}
c_{13}	c_{23}	c_{33}	.	.	c_{s3}
.	.	.	.	.	.
.	.	.	.	.	.

denotes one of the $R_1 R_2 R_3 \ldots$ separate operations. But some of these may vanish as mentioned above. Those which do not vanish form a set of tableaux. These possess a common property, viz. the sums of the numbers in the successive rows are r_1, r_2, r_3, ... respectively and the collections of numbers in the successive columns are those which define the symmetric functions F_1, F_2, F_3, ... F_s respectively. If one is asked: What is the number of tableaux each of which possesses the property above-defined, the mathematician's answer is

$$D_{r_1} D_{r_2} D_{r_3} \ldots F_1 F_2 \ldots F_s,$$

which denotes a number which is readily evaluated. We have, in fact, a good example of the use of the Hammond operators in solving questions of enumeration of the Magic Square Class. The multiplication of symmetric functions invariably supplies an example of this kind of enumeration.

3. Let U be any linear function of symmetric functions of the same weight w.

Consider the multiplication

$$(1^{r_1}) \, U,$$

and therein the term involving the function

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots) \text{ of weight } w + r.$$

We have

$$D_{p_1} (1^{r_1}) \, U = \{(1^{r_1}) \, D_{p_1} + (1^{r_1 - 1}) \, D_{p_1 - 1}\} \, U,$$

$$D_{p_1}^2 (1^{r_1}) \, U = \{(1^{r_1}) \, D_{p_1}^2 + 2 \, (1^{r_1 - 1}) \, D_{p_1} D_{p_1 - 1} + (1^{r_1 - 2}) \, D_{p_1 - 1}^2\} \, U.$$

Now put $(1^{r_1}) = a^{r_1}$ symbolically, so that these results may be written

$$D_{p_1} (1^{r_1}) \, U = a^{r_1} (D_{p_1} + a^{-1} D_{p_1 - 1}) \, U,$$

$$D_{p_1}^2 (1^{r_1}) \, U = a^{r_1} (D_{p_1} + a^{-1} D_{p_1 - 1})^2 \, U,$$

and generally

$$D_{p_1}^{\pi_1} (1^{r_1}) \, U = a^{r_1} (D_{p_1} + a^{-1} D_{p_1 - 1})^{\pi_1} \, U,$$

and more generally

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots (1^{r_1})\ U$$

$$= a^{r_1} (D_{p_1} + a^{-1} D_{p_1-1})^{\pi_1} (D_{p_2} + a^{-1} D_{p_2-1})^{\pi_2} (D_{p_3} + a^{-1} D_{p_3-1})^{\pi_3} \dots U.$$

When this operator is developed and applied to U it is obvious that every term which involves an operator product $D_{s_1}^{\sigma_1} D_{s_2}^{\sigma_2} \dots$ of a weight $> w$ causes U to vanish. This happens whenever the associated power of a is positive.

When the power of a is zero the attached D product is of weight w and either does or does not cause U to vanish. If U involves a term

$$C_{s_1 \sigma_1\ s_2 \sigma_2\ s_3 \sigma_3 \dots}\ (s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \dots)$$

the operation of $D_{s_1}^{\sigma_1} D_{s_2}^{\sigma_2} D_{s_3}^{\sigma_3} \dots$ produces the number $C'_{s_1 \sigma_1\ s_2 \sigma_2\ s_3 \sigma_3 \dots}$. In the contrary case the operator produces zero.

The negative powers of a in the development have no real existence symbolically so that all terms involving such must be put equal to zero.

It thus appears that the effective operator upon U consists entirely of the terms in the developments which are *free from the symbol* a.

If the process of picking out this portion of the whole operator be denoted by T_a, we may write our result

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots (1^{r_1})\ U$$
$$= T_a \{a^{r_1} (D_{p_1} + a^{-1} D_{p_1-1})^{\pi_1} (D_{p_2} + a^{-1} D_{p_2-1})^{\pi_2}$$
$$(D_{p_3} + a^{-1} D_{p_3-1})^{\pi_3} \dots\}\ U.$$

This theorem enables us, from the known value of U as a sum of monomial functions, to proceed to the similar expression of

$$(1^{r_1})\ U.$$

Ex. gr. Suppose

$$U = (1^2)^2 = (2^2) + 2\ (21^2) + 6\ (1^4),$$

and that we require the term of $(1^2)^3$ which involves

$$(321),$$

$$D_3 D_2 D_1\ (1^2) \cdot (1^2)^2$$
$$= T_a \{a^2 (D_3 + a^{-1} D_2)(D_2 + a^{-1} D_1)(D_1 + a^{-1})\} (1^2)^2$$
$$= T_a \{a D_2 (D_2 + a^{-1} D_1)(D_1 + a^{-1})\} (1^2)^2$$

(because, since U is here of degree 2, we may put $D_3 = 0$)

$$= (D_2^2 + D_2 D_1^2) \{(2^2) + 2\ (21^2) + 6\ (1^4)\} = 1 + 2 = 3.$$

Thence $\qquad (1^2)^3 = \dots + 3\ (321) + \dots.$

4. The expression of

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots (1^{r_1})\; U$$

may be presented in a more convenient and suggestive form by the employment of a further symbolism.

If we write

$$\pi_k (\pi_k - 1)(\pi_k - 2) \dots (\pi_k - s + 1) = \pi_k^s \text{ symbolically},$$

we have

$$(D_{p_k} + a^{-1} D_{p_{k-1}})^{\pi_1} = D_{p_k}^{\pi_1} \exp \pi_k a^{-1} \frac{D_{p_k-1}}{D_{p_k}} \text{ effectively},$$

and thence the expression

$$T_a \left\{ a^{r_1} D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots \exp\left(\pi_1 \frac{D_{p_1-1}}{D_{p_1}} + \pi_2 \frac{D_{p_2-1}}{D_{p_2}} + \pi_3 \frac{D_{p_3-1}}{D_{p_3}} + \dots \right) a^{-1} \right\} U$$

$$= \frac{1}{r_1!} D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots \left(\pi_1 \frac{D_{p_1-1}}{D_{p_1}} + \pi_2 \frac{D_{p_2-1}}{D_{p_2}} + \pi_3 \frac{D_{p_3-1}}{D_{p_3}} + \dots \right)^{r_1} U,$$

a result which, as will be seen presently, is generalizable.

Applied to the particular case we find

$$\frac{1}{2!} D_3 D_2 D_1 \left(\frac{D_2}{D_3} + \frac{D_1}{D_2} + \frac{1}{D_1} \right)^2 (1^2)^2$$

$$= (D_2^2 + D_2 D_1^2 + D_3 D_1)\{(2)^2 + 2(21^2) + 6(1^4)\}$$

$$= 1 + 2 = 3 \text{ as above.}$$

5. Next consider the product

$$(2^{r_2} 1^{r_1})\; U.$$

We have

$$D_{p_1}(2^{r_2} 1^{r_1})\; U = \{(2^{r_2} 1^{r_1}) D_{p_1} + (2^{r_2} 1^{r_1-1}) D_{p_1-1} + (2^{r_2-1} 1^{r_1}) D_{p_1-2}\}\; U,$$

$$D_{p_1}^2 (2^{r_2} 1^{r_1})\; U$$

$$= \left\{ \begin{array}{l} (2^{r_2} 1^{r_1}) D_{p_1}^2 \qquad\qquad + (2^{r_2} 1^{r_1-1}) D_{p_1} D_{p_1-1} \\ \qquad\qquad\qquad\qquad\qquad + (2^{r_2-1} 1^{r_1}) D_{p_1} D_{p_1-2} \\ + (2^{r_2} 1^{r_1-1}) D_{p_1} D_{p_1-1} + (2^{r_2} 1^{r_1-2}) D_{p_1-1}^2 \\ \qquad\qquad\qquad\qquad\qquad + (2^{r_2-1} 1^{r_1-1}) D_{p_1-1} D_{p_1-2} \\ + (2^{r_2-1} 1^{r_1}) D_{p_1} D_{p_1-2} + (2^{r_2-1} 1^{r_1-1}) D_{p_1-1} D_{p_1-2} \\ \qquad\qquad\qquad\qquad\qquad + (2^{r_2-2} 1^{r_1}) D_{p_1-2} D_{p_2-2} \end{array} \right\} U.$$

Thence writing symbolically

$$(2^{r_2} 1^{r_1}) = a^{r_1} b^{r_2},$$

so that symbolically

$$D_{p_1}(2^{r_2} 1^{r_1})\; U = a^{r_1} b^{r_2} (D_{p_1} + a^{-1} D_{p_1-1} + b^{-1} D_{p_1-2})\; U,$$

$$D_{p_1}^2 (2^{r_2} 1^{r_1})\; U = a^{r_1} b^{r_2} (D_{p_1} + a^{-1} D_{p_1-1} + b^{-1} D_{p_1-2})^2\; U,$$

and generally

$$D_{p_1}^{\pi_1}(2^{r_2}1^{r_1})\,U = a^{r_1}b^{r_2}(D_{p_1} + a^{-1}D_{p_1-1} + b^{-1}D_{p_1-2})^{\pi_1}\,U,$$

and more generally

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots(2^{r_2}1^{r_1})\,U$$
$$= a^{r_1}b^{r_2}(D_{p_1} + a^{-1}D_{p_1-1} + b^{-1}D_{p_1-2})^{\pi_1}(D_{p_2} + a^{-1}D_{p_2-1} + b^{-1}D_{p_2-2})^{\pi_2}$$
$$(D_{p_3} + a^{-1}D_{p_3-1} + b^{-1}D_{p_3-2})^{\pi_3}\ldots U.$$

By reasoning similar to that advanced above we have here the coefficient of a^0b^0 as the effective part in the development of the operator, and denoting this by T_{ab}, we find that

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots(2^{r_2}1^{r_1})\,U$$
$$= T_{ab}\,\{a^{r_1}b^{r_2}(D_{p_1} + a^{-1}D_{p_1-1} + b^{-1}D_{p_1-2})^{\pi_1}$$
$$\times (D_{p_2} + a^{-1}D_{p_2-1} + b^{-1}D_{p_2-2})^{\pi_2}$$
$$\times (D_{p_3} + a^{-1}D_{p_3-1} + b^{-1}D_{p_3-2})^{\pi_3}\ldots\}\,U,$$

and introducing as above the symbolism

$$\pi_k\,(\pi_k - 1)\ldots(\pi_k - s + 1) = \pi_k^s,$$

we find finally

$$\frac{1}{r_1!\,r_2!}\,D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots\left(\pi_1\frac{D_{p_1-1}}{D_{p_1}} + \pi_2\frac{D_{p_2-1}}{D_{p_2}} + \ldots\right)^{r_1}$$
$$\left(\pi_1\frac{D_{p_1-2}}{D_{p_1}} + \pi_2\frac{D_{p_2-2}}{D_{p_2}} + \ldots\right)^{r_2}U$$

as the symbolic form of the operator which when performed upon U produces the coefficients of the term

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$$

in the development of

$$(2^{r_2}1^{r_1})\,U.$$

6. We now readily proceed to the general result

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots(\ldots 3^{r_3}2^{r_2}1^{r_1})\,U$$
$$= T_{abc}\ldots\{a^{r_1}b^{r_2}c^{r_3}\ldots(D_{p_1} + a^{-1}D_{p_1-1} + b^{-1}D_{p_1-2} + c^{-1}D_{p_1-3} + \ldots)^{\pi_1}$$
$$\times (D_{p_2} + a^{-1}D_{p_2-1} + b^{-1}D_{p_2-2} + c^{-1}D_{p_2-3} + \ldots)^{\pi_2}$$
$$\times (D_{p_3} + a^{-1}D_{p_3-1} + b^{-1}D_{p_3-2} + c^{-1}D_{p_3-3} + \ldots)^{\pi_3}$$
$$\times \ldots\ldots\ldots\ldots\ldots\ldots\ldots\}\,U,$$

and this, adopting the π symbolism as above, is

$$\left\{ \frac{1}{r_1!\, r_2!\, r_3!\, \dots} D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots \right.$$

$$\times \left(\pi_1 \frac{D_{p_1-1}}{D_{p_1}} + \pi_2 \frac{D_{p_2-1}}{D_{p_2}} + \pi_3 \frac{D_{p_3-1}}{D_{p_3}} + \dots \right)^{r_1}$$

$$\times \left(\pi_1 \frac{D_{p_1-2}}{D_{p_1}} + \pi_2 \frac{D_{p_2-2}}{D_{p_2}} + \pi_3 \frac{D_{p_3-2}}{D_{p_3}} + \dots \right)^{r_2}$$

$$\times \left(\pi_1 \frac{D_{p_1-3}}{D_{p_1}} + \pi_2 \frac{D_{p_2-3}}{D_{p_2}} + \pi_3 \frac{D_{p_3-5}}{D_{p_3}} + \dots \right)^{r_3}$$

$$\left. \times \; \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \right\} U,$$

the value of the coefficients of the term

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots),$$

in the development of $(\dots 3^{r_3} 2^{r_2} 1^{r_1})\, U$.

7. I will now apply this result to the cases dealt with by Cayley (*loc. cit.*).

The first multiplication he took was

$$(2^\alpha)\, (2^\beta),$$

where I think it best to retain his original notation. I take the precisely equivalent case

$$(1^\alpha)\, (1^\beta).$$

Here U is (1^β) and the general form of monomial term in the product is

$$(2^A 1^{\alpha+\beta-2A}).$$

To find the coefficients of this term we have, as above,

$$D_2^A D_1^{\alpha+\beta-2A} (1^\alpha) . (1^\beta)$$

$$= T_a \{ a^\alpha (D_2 + a^{-1} D_1)^A (D_1 + a^{-1})^{\alpha+\beta-2A} \} . (1^\beta)$$

$$= T_a \{ a^{\alpha-A} D_1^A (D_1 + a^{-1})^{\alpha+\beta-2A} \} . (1^\beta)$$

$$= \binom{\alpha+\beta-2A}{\alpha-A} D_1^\beta (1^\beta) = \binom{\alpha+\beta-2A}{\alpha-A}.$$

So that $\quad (1^\alpha)\, (1^\beta) = \sum_A \binom{\alpha+\beta-2A}{\alpha-A} (2^A 1^{\alpha+\beta-2A})$

(cf. *loc. cit.* and *Collected Papers*, Vol. XII, p. 244).

It will be observed that the present process is not concerned with 'frequencies' and 'multiplicities' but is altogether algebraical.

Cayley's second case is

$$(3^\alpha 2^\beta)\, (3^\gamma 2^\delta)$$

because he was only concerned with non-unitary forms.

Here $U = (3^\gamma 2^\delta)$,

and the general form of monomial in the product is

$$(6^A 5^B 4^C 3^D 2^E),$$

where $D = \alpha + \gamma - 2A - B$, $E = \beta + \delta - B - 2C$;

to find the coefficient of this general term we have

$$T_{bc} \{ b^\beta c^\alpha (D_6 + b^{-1} D_4 + c^{-1} D_3)^A (D_5 + b^{-1} D_3 + c^{-1} D_2)^B$$
$$\times (D_4 + b^{-1} D_2 + c^{-1} D_1)^C (D_3 + b^{-1} D_1 + c^{-1})^D$$
$$\times (D_2 + b^{-1})^E \} . (3^\gamma 2^\delta).$$

Since the operand is annihilated by each of the four operators D_6, D_5, D_4, D_1 we put each of these equal to zero and find

$$T_{bc} \{ b^{\beta-C} c^{\alpha-A} D_3^A (b^{-1} D_3 + c^{-1} D_2)^B D_2^C$$
$$(D_3 + c^{-1})^D (D_2 + b^{-1})^E \} . (3^\gamma 2^\delta),$$

whence we observe at once that the sought coefficient is expressible as a linear function of terms each of which is a product of these binomial coefficients (cf. *loc. cit.* p. 246, the final conclusion).

Before calculating we put

$$D = \alpha + \gamma - 2A - B, \quad E = \beta + \delta - B - 2C,$$

and if (with Cayley) we take

$$y, z, r \text{ to be integers}$$
$$\ll B, \ll D, \ll E \text{ respectively,}$$

we find as the coefficient sought

$$\Sigma \binom{B}{y} \binom{D}{z} \binom{E}{r},$$

the summation being controlled by the relations

$$-y + r = \delta - B - C$$
$$y + z = \gamma - A.$$

8. Having thus verified the results of Cayley by pure algebra I now resume, from a general point of view, by considering the next case in order to $(1^\alpha) (1^\beta)$, viz.

$$(2^{\alpha_2} 1^{\alpha_1}) (2^{\beta_2} 1^{\beta_1}).$$

Here $U = (2^{\beta_2} 1^{\beta_1})$ and

$$D_4^A D_3^B D_2^C D_1^D (2^{\alpha_2} 1^{\alpha_1}) . (2^{\beta_2} 1^{\beta_1})$$
$$= T_{ab} \{ a^{\alpha_1} b^{\alpha_2} (D_4 + a^{-1} D_3 + b^{-1} D_2)^A (D_3 + a^{-1} D_2 + b^{-1} D_1)^B$$
$$(D_2 + a^{-1} D_1 + b^{-1} \quad)^C (D_1 + a^{-1} \quad)^D \}$$
$$(2^{\beta_2} 1^{\beta_1}).$$

Since here $D_4 = D_3 = 0$ this is

$$T_{ab} \{a^{a_1} b^{a_2 - A} D_2^A (a^{-1} D_2 + b^{-1} D_1)^B$$
$$\times (D_2 + a^{-1} D_1 + b^{-1})^C (D_1 + a^{-1})^D\} (2^{\beta_2} 1^{\beta_1}),$$

and we observe that the coefficient sought is a linear function of terms each of which is a product of a binomial, a trinomial and a binomial coefficient, in the order named.

We say that the structure of the coefficient is

232.

Observe that in the case $(1^a) (1^\beta)$ the coefficient is a single binomial coefficient so that its structure is denoted by

2.

9. It is not necessary to carry this case further and I proceed to the product

$$(3^{a_3} 2^{a_2} 1^{a_1}) (3^{\beta_3} 2^{\beta_2} 1^{\beta_1}).$$

We have to consider

$$D_6^A D_5^B D_4^C D_3^D D_2^E D_1^F (3^{a_3} 2^{a_2} 1^{a_1}) . (3^{\beta_3} 2^{\beta_2} 1^{\beta_1}),$$

where U is $(3^{\beta_3} 2^{\beta_2} 1^{\beta_1})$.

Merely writing down the D suffixes which occur in the operator factors, viz.

(6543) (5432) (4321) (3210) (210) (10),

and striking out the numbers greater than 3 we obtain

(3) (32) (321) (3210) (210) (10),

and we observe that the character of the sought coefficient is

23432.

Again for the case

$$(4^{a_4} 3^{a_3} 2^{a_2} 1^{a_1}) . (4^{\beta_4} 3^{\beta_3} 2^{\beta_2} 1^{\beta_1}),$$

we have first of all

(87654) (76543) (65432) (54321) (43210) (3210) (210) (10),

and thence

(4) (43) (432) (4321) (43210) (3210) (210) (10),

shewing that the character of the coefficient is

2345432.

10. It is now clear that in the case of the product

$$(k^{a_k} \ldots 3^{a_3} 2^{a_2} 1^{a_1}) (k^{\beta_k} \ldots 3^{\beta_3} 2^{\beta_2} 1^{\beta_1}),$$

the character of the coefficient is

234 ... $k, k + 1, k$ ... 432.

11. The same principle is available for the product of three functions through the composition of numbers into three parts.

Consider the simplest case

$$(1^\alpha)(1^\beta)\,U,$$

where U is any given function of weight w.

$$D_{p_1}(1^\alpha)(1^\beta)\,U = [(1^\alpha)(1^\beta)\,D_{p_1} + \{(1^{\alpha-1})(1^\beta) + (1^\alpha)(1^{\beta-1})\}\,D_{p_1-1} \\ + (1^{\alpha-1})(1^{\beta-1})\,D_{p_1-2}]\,U,$$

wherein writing $(1^\alpha) = a^\alpha$, $(1^\beta) = b^\beta$ symbolically

$$D_{p_1}(1^\alpha)(1^\beta)\,U = a^\alpha b^\beta \left\{ D_{p_1} + \left(\frac{1}{a} + \frac{1}{b}\right) D_{p_1-1} + \frac{1}{ab}\,D_{p_1-2}\right\}\,U,$$

leading to

$$D_{p_1}^{\pi_1}(1^\alpha)(1^\beta)\,U = a^\alpha b^\beta \left\{ D_{p_1} + \left(\frac{1}{a} + \frac{1}{b}\right) D_{p_1-1} + \frac{1}{ab}\,D_{p_1-2}\right\}^{\pi_1}\,U,$$

and if

$$(p_1^{\pi_1}\,p_2^{\pi_2}\,p_3^{\pi_3}\,...)$$

be a partition of $w + \alpha + \beta$, representative of a symmetric function which appears in the development of

$$(1^\alpha)(1^\beta)\,U,$$

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} ... (1^\alpha)(1^\beta)\,U$$

$$= T_{ab}\left[a^\alpha b^\beta \left\{ D_{p_1} + \left(\frac{1}{a} + \frac{1}{b}\right) D_{p_1-1} + \frac{1}{ab}\,D_{p_1-2}\right\}^{\pi_1} \right.$$

$$\times \left\{ D_{p_2} + \left(\frac{1}{a} + \frac{1}{b}\right) D_{p_2-1} + \frac{1}{ab}\,D_{p_2-2}\right\}^{\pi_2}$$

$$\times \left\{ D_{p_3} + \left(\frac{1}{a} + \frac{1}{b}\right) D_{p_3-1} + \frac{1}{ab}\,D_{p_3-2}\right\}^{\pi_3}$$

$$\left. \times \,... \right]\,U.$$

The operator which survives the process T_{ab} is of weight w and when performed upon U produces the coefficient of the term

$$(p_1^{\pi_1}\,p_2^{\pi_2}\,p_3^{\pi_3}\,...)$$

in the development of $(1^\alpha)(1^\beta)\,U$.

The reader will have no difficulty in proceeding to the result which follows from introducing the further symbolism in regard to $\pi_1,\ \pi_2,\ \pi_3,\ ....$

12. Take as an example $U = (1^\gamma)$ and, without loss of generality, we may suppose

$$\alpha \geqslant \beta \geqslant \gamma.$$

Let the general term in the developed product involve

$$(3^A\,2^B\,1^{\alpha+\beta+\gamma-3A-2B}).$$

We have (writing $\alpha + \beta + \gamma - 3A - 2B = C$ for short)

$$D_3^A D_2^B D_1^C (1^\alpha)(1^\beta)(1^\gamma)$$

$$= T_{ab}\left[a^\alpha b^\beta \left\{ D_3 + \left(\frac{1}{a} + \frac{1}{b}\right) D_2 + \frac{1}{ab}\, D_1 \right\}^A \right.$$

$$\times \left\{ D_2 + \left(\frac{1}{a} + \frac{1}{b}\right) D_1 + \frac{1}{ab} \quad \right\}^B$$

$$\left. \cdot \left\{ D_1 + \left(\frac{1}{a} + \frac{1}{b}\right) \qquad \right\}^C \right] (1^\gamma)$$

$$= T_{ab}\left[a^{\alpha-A} b^{\beta-A} D_1^A \left\{ \left(\frac{1}{a} + \frac{1}{b}\right) D_1 + \frac{1}{ab} \right\}^B \left\{ D_1 + \left(\frac{1}{a} + \frac{1}{b}\right) \right\}^C \right] (1^\gamma).$$

If y, z be integers $\ll B$, $\ll C$ respectively, we find

$$\Sigma T_{ab}\left[a^{\alpha-A} b^{\beta-A} D_1^A \binom{B}{y} \left(\frac{1}{a} + \frac{1}{b}\right)^y D_1^y \left(\frac{1}{ab}\right)^{B-y} \binom{C}{z} D_1^z \left(\frac{1}{a} + \frac{1}{b}\right)^{C-z} \right](1^\gamma)$$

or $\Sigma \binom{B}{y}\binom{C}{z} D_1^{A+y+z} T_{ab} \left[a^{\alpha-A-B+y} b^{\beta-A-B+y} \left(\frac{1}{a} + \frac{1}{b}\right)^{y+C-z} \right] (1^\gamma).$

If x be an integer $\ll y + C - z$, we find

$$\Sigma \binom{B}{y}\binom{C}{z}\binom{y+C-z}{x} D_1^{A+y+z}\, T_{ab}\, (a^{\alpha-A-B+y-x}\, b^{\beta-A-B-C+z+x})\,(1^\gamma),$$

or $\qquad\qquad \Sigma \binom{B}{y}\binom{C}{z}\binom{y+C-z}{x},$

where $\qquad\qquad\qquad A + y + z = \gamma,$

$$\alpha - A - B + y - x = 0,$$

$$\beta - A - B - C + z + x = 0,$$

three relations to control the summation, reducing to the two

$$y + z = \gamma - A,$$

$$y - x = -\alpha + A + B.$$

The coefficient is thus a linear function of terms each of which is a product of three binomial coefficients.

In particular, the coefficient of $(1^{\alpha+\beta+\gamma})$ reduces to

$$\binom{\alpha+\beta+\gamma}{\gamma}\binom{\alpha+\beta}{\alpha},$$

or $\qquad\qquad\qquad \dfrac{(\alpha+\beta+\gamma)!}{\alpha!\,\beta!\,\gamma!},$

which is obviously correct.

We may similarly treat the product

$$(2^{\alpha_2} 1^{\alpha_1})(2^{\beta_2} 1^{\beta_1})\, U.$$

$$D_{p_1} (2^{a_2} 1^{a_1}) (2^{\beta_2} 1^{\beta_1}) \, U$$

is in fact dealt with by considering the composition of p_1

$$0, \, 0, \, p_1; \; 0, \, 1, \, p_1 - 1; \; 1, \, 0, \, p_1 - 1; \; 0, \, 2, \, p_1 - 2; \; 1, \, 1, \, p_1 - 2;$$
$$2, \, 0, \, p_1 - 2; \; 2, \, 1, \, p_1 - 3; \; 1, \, 2, \, p_1 - 3; \; 2, \, 2, \, p_1 - 4;$$

yielding, symbolically,

$$a_2^{a_2} a_1^{a_1} b_2^{\beta_2} b_1^{\beta_1} \left\{ D_{p_1} + \left(\frac{1}{a_1} + \frac{1}{b_1} \right) D_{p_1-1} + \left(\frac{1}{a_2} + \frac{1}{a_1 b_1} + \frac{1}{b_2} \right) D_{p_1-2} \right.$$
$$\left. + \left(\frac{1}{a_2 b_1} + \frac{1}{a_1 b_2} \right) D_{p_1-3} + \frac{1}{a_2 b_2} D_{p_1-4} \right\} U,$$

and we proceed in the usual manner to express

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots (2^{a_2} 1^{a_1}) (2^{\beta_2} 1^{\beta_1}) \, U.$$

Finally, the procedure is clear in the case of the product of any number of symmetric functions.

The process is valuable when we require the calculation of the successive powers of a symmetric function.

13. Suppose, for example, that we have calculated the development of $\qquad (1^2)^{m-1},$

and we require that of $\qquad (1^2)^{m}.$

$$D_{p_1}^{\pi_1}(1^2) . (1^2)^{m-1}$$
$$= a^2 \left(D_{p_1} + \frac{1}{a} D_{p_1-1} \right)^{\pi_1} . (1^2)^{m-1}$$
$$= a^2 \left\{ D_{p_1}^{\pi_1} + \binom{\pi_1}{1} \frac{1}{a} D_{p_1}^{\pi_1-1} D_{p_1-1} + \binom{\pi_1}{2} \frac{1}{a^2} D_{p_1}^{\pi_1-2} D_{p_1-1}^2 \right\} (1^2)^{m-1},$$

since powers of $\dfrac{1}{a}$ greater than the second cannot contribute to the final result.

Thence

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots (1^2) . (1^2)^{m-1}$$
$$= T_a \left[a^2 \left\{ D_{p_1}^{\pi_1} + \binom{\pi_1}{1} \frac{1}{a} D_{p_1}^{\pi_1-1} D_{p_1-1} + \binom{\pi_1}{2} \frac{1}{a^2} D_{p_1}^{\pi_1-2} D_{p_1-1}^2 \right\} \right.$$
$$\therefore \left\{ D_{p_2}^{\pi_2} + \binom{\pi_2}{1} \frac{1}{a} D_{p_2}^{\pi_2-1} D_{p_2-1} + \binom{\pi_2}{2} \frac{1}{a^2} D_{p_2}^{\pi_2-2} D_{p_2-1}^2 \right\}$$
$$\therefore \left\{ D_{p_3}^{\pi_3} + \binom{\pi_3}{1} \frac{1}{a} D_{p_3}^{\pi_3-1} D_{p_3-1} + \binom{\pi_3}{2} \frac{1}{a^2} D_{p_3}^{\pi_3-2} D_{p_3-1}^2 \right\}$$
$$\times \, \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \left. \right] (1^2)^{m-1}$$

$$= \left\{ \Sigma \binom{\pi_1}{2} D_{p_1}^{\pi_1 - 2} D_{p_1 - 1}^2 D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots \right\} (1^2)^{m-1}$$

$$+ \left\{ \Sigma \binom{\pi_1}{1} \binom{\pi_2}{1} D_{p_1}^{\pi_1 - 1} D_{p_1 - 1} D_{p_2}^{\pi_2 - 1} D_{p_2 - 1} D_{p_3}^{\pi_3} \dots \right\} (1^2)^{m-1},$$

so that if we multiply certain coefficients of $(1^2)^{m-1}$ by numbers $\binom{\pi_1}{2}$ and certain others by numbers $\binom{\pi_1}{1}\binom{\pi_2}{1}$ and all these numerical results together we obtain the coefficients of the term

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \dots)$$

in the development of $(1^2)^m$.

Ex. gr. We know that

$$(1^2)^3 = (3^2) + 3\,(321) + 6\,(31^3) + 6\,(2^3)$$
$$+ 15\,(2^2 1^2) + 36\,(21^4) + 90\,(1^6),$$

and we find for the coefficient of $(32^2 1)$ in $(1^2)^4$

$$T_a \left\{ a^2 \left(D_3 + \frac{1}{a} D_2 \right) \left(D_2^2 + \frac{2}{a} D_2 D_1 + \frac{1}{a^2} D_1^2 \right) \left(D_1 + \frac{1}{a} \right) \right\} (1^2)^3$$
$$= (D_3 D_1^3 + 2 D_3 D_2 D_1 + D_2^3 + 2 D_2^2 D_1^2)(1^2)^3$$
$$= \quad 6 \quad + \quad 2.3 \quad + \quad 6 \quad + \quad 2.15$$
$$= \quad 48.$$

14. The general result

$$\left\{ \Sigma \binom{\pi_1}{2} D_{p_1}^{\pi_1 - 2} D_{p_1 - 1}^2 D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \dots \right\} (1^2)^{m-1}$$

$$+ \left\{ \Sigma \binom{\pi_1}{1} \binom{\pi_2}{1} D_{p_1}^{\pi_1 - 1} D_{p_1 - 1} D_{p_2}^{\pi_2 - 1} D_{p_2 - 1} D_{p_3}^{\pi_3} \dots \right\} (1^2)^{m-1}$$

can be put into an interesting and suggestive form by writing

$$\pi_k (\pi_k - 1)(\pi_k - 2) \dots (\pi_k - s + 1) = \pi_k^s \text{ symbolically,}$$

for it then appears in the symbolic form

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi} \dots \frac{1}{2} \left(\pi_1 \frac{D_{p_1 - 1}}{D_{p_1}} + \pi_2 \frac{D_{p_2 - 1}}{D_{p_2}} + \pi_3 \frac{D_{p_3 - 1}}{D_{p_3}} + \dots \right)^2 (1^2)^{m-1}.$$

So that in the particular case exemplified we have

$$D_3 D_2^2 D_1 \frac{1}{2} \left(1 \frac{D_2}{D_3} + 2 \frac{D_1}{D_2} + 1 \frac{D_0}{D_1} \right)^2 (1^2)^3,$$

where, by the symbolism, the squared bracket is to be read

$$1.0 \frac{D_2^2}{D_3^2} + 2.1.2. \frac{D_1}{D_3} + 2.1 \frac{D_1^2}{D_2^2} + 2.1.1. \frac{D_2}{D_3 D_1} + 2.2.1. \frac{1}{D_2} + 1.0 \frac{1}{D_1^2},$$

or
$$4\frac{D_1}{D_3} + 2\frac{D_1^2}{D_2^2} + 2\frac{D_2}{D_3 D_1} + 4\frac{1}{D_2},$$

and thence $\quad (2D_2^2 D_1^2 + D_3 D_1^3 + D_2^3 + 2D_3 D_2 D_1)\,(1^2)^3.$

15. Consider next $\qquad (1^n)\,.\,(1^n)^{m-1},$

$$D_{p_1}(1^n)\,.\,(1^n)^{m-1}$$
$$= \{(1^n)\,D_{p_1} + (1^{n-1})\,D_{p_1-1}\}\,(1^n)^{m-1}$$
$$= a^n\left(D_{p_1} + \frac{1}{a}\,D_{p_1-1}\right)(1^n)^{m-1} \text{ symbolically,}$$

and thence

$$(D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3}\,\ldots)\,(1^n)\,.\,(1^n)^{m-1}$$

$$= T_a\left[a^n\left(D_{p_1} + \frac{1}{a}D_{p_1-1}\right)^{\pi_1}\left(D_{p_2} + \frac{1}{a}D_{p_2-1}\right)^{\pi_2}\left(D_{p_3} + \frac{1}{a}D_{p_3-1}\right)^{\pi_3}\ldots\right](1^n)^{m-1}$$

$$= T_a\left[a^n\left\{D_{p_1}^{\pi_1} + \binom{\pi_1}{1}\frac{1}{a}D_{p_1}^{\pi_1-1}D_{p_1-1} + \binom{\pi_1}{2}\frac{1}{a^2}D_{p_1}^{\pi_1-2}D_{p_1-1}^2 + \ldots\right.\right.$$
$$\left.+ \binom{\pi_1}{n}\frac{1}{a^n}D_{p_1}^{\pi_1-n}D_{p_1-1}^n\right\}$$

$$\times\left\{D_{p_2}^{\pi_2} + \binom{\pi_2}{1}\frac{1}{a}D_{p_2}^{\pi_2-1}D_{p_2-1} + \binom{\pi_2}{2}\frac{1}{a^2}D_{p_2}^{\pi_2-2}D_{p_2-1}^2 + \ldots\right.$$
$$\left.+ \binom{\pi_2}{n}\frac{1}{a^n}D_{p_2}^{\pi_2-n}D_{p_2-1}^n\right\}$$

$$\times\left\{D_{p_3}^{\pi_3} + \binom{\pi_3}{1}\frac{1}{a}D_{p_3}^{\pi_3-1}D_{p_3-1} + \binom{\pi_3}{2}\frac{1}{a^2}D_{p_3}^{\pi_3-3}D_{p_3-1}^2 + \ldots\right.$$
$$\left.+ \binom{\pi_3}{n}\frac{1}{a^n}D_{p_3}^{\pi_3-n}D_{p_3-1}^n\right\}$$

$$\times\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Bigg](1^n)^{m-1}.$$

Introducing the symbolism
$$\pi_k\,(\pi_k - 1)\,(\pi_k - 2)\,\ldots\,(\pi_k - s + 1) = \pi_k^s$$

this is

$$T_a\left[a^n\,D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots\exp\left(\frac{\pi_1}{a}\frac{D_{p_1-1}}{D_{p_1}} + \frac{\pi_2}{a}\frac{D_{p_2-1}}{D_{p_2}} + \frac{\pi_3}{a}\frac{D_{p_3-1}}{D_{p_3}} + \ldots\right)\right](1^n)^{m-1}$$

$$= \frac{1}{n!}\,D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}D_{p_3}^{\pi_3}\ldots\left(\pi_1\frac{D_{p_1-1}}{D_{p_1}} + \pi_2\frac{D_{p_2-1}}{D_{p_2}} + \pi_3\frac{D_{p_3-1}}{D_{p_3}} + \ldots\right)^n(1^n)^{m-1},$$

a noteworthy generalization of the ordinary multinomial theorem.

In fact we have established the formula

$$(1^n)^m$$

$$= \Sigma \frac{1}{n!} D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \cdots \left(\pi_1 \frac{D_{p_1-1}}{D_{p_1}} + \pi_2 \frac{D_{p_2-1}}{D_{p_2}} + \pi_3 \frac{D_{p_3-1}}{D_{p_3}} + \ldots \right)^n (1^n)^{m-1},$$

and the ordinary multinomial emerges as

$$(1)^m = \Sigma D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} D_{p_3}^{\pi_3} \cdots \left(\pi_1 \frac{D_{p_1-1}}{D_{p_1}} + \pi_2 \frac{D_{p_2-1}}{D_{p_2}} + \pi_3 \frac{D_{p_3-1}}{D_{p_3}} + \ldots \right) (1)^{m-1},$$

wherein the summations are in respect of all symmetric functions

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots),$$

which present themselves in the developments.

The formula above for $n = 1$ is readily seen to be equivalent to the formula

$$(1)^m = \Sigma \frac{m!}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} (p_3!)^{\pi_3} \ldots} (p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots).$$

16. In the discussion of American Tournaments which has appeared recently* the present author has dealt with the symmetric function algebra involved in an analogous manner—but in that case the applications were of a particular and not of a general character.

However, the study of those Tournaments is responsible for bringing to light the advance in symmetrical algebra which is brought forward in this paper.

* 'An American Tournament treated by the Calculus of Symmetric Functions,' *The Quarterly Journal of Pure and Applied Mathematics,* Vol. XLIX, No. 193, 1920.

Fluctuations in an Assembly in Statistical Equilibrium. By
Mr C. G. DARWIN and Mr R. H. FOWLER.

§ 1. *Introduction.* In a series of papers* we have developed a
general method of calculating the average statistical state of
certain assemblies of molecular systems. At the same time we have
shewn by the calculation of the fluctuations that this average state
of the system is effectively its normal state. The examples of
fluctuations which we have so far given are, however, of a pre-
liminary nature, though sufficient for the purpose in hand. The
method of calculation can be systematized and pushed much
further, so as to enable us to evaluate asymptotically a large variety
of fluctuations.

We have previously spoken of the fluctuation of any quantity
P, whose average value is $\overline{P}$, meaning thereby $\overline{(P - \overline{P})^2}$. We shall
here be concerned, however, with general fluctuations such as
$\overline{(P - P)^n}$ and shall distinguish the simpler type above by the name
second order fluctuations. It is, of course, these second order
fluctuations which are primarily of interest and importance, but
the general results are often elegant, almost as easily obtained, and
not without a certain interest of their own. When this is so we
give them in full in this paper. Many of them will be seen to
generalize corresponding results due to Gibbs, obtained by him on
his assumption of canonical distribution-in-phase.

In § 2 we establish formulae for the general fluctuations in the
energy partition in an assembly containing two types of systems;
the extensions to any number of types are obvious and need not be
detailed. In § 3 we do similar calculations for the number of
systems of one type which have a specified energy or (in the case
of classical systems) lie in a specified cell of the phase-space. In
§ 4 we evaluate the second order fluctuations in the reaction of the
assembly on external bodies such as the walls of its containing
vessel. General fluctuations are here too cumbersome to be worth
calculating. In §§ 2–4 we deal with assemblies in which the numbers
of each type of system are fixed. In § 5 we consider assemblies in
which dissociation and association of atoms and molecules are pro-
ceeding and calculate general fluctuations of the degree of association.
In § 6 we conclude by extending the results of §§ 2–4 to the more
general assemblies.

* Darwin and Fowler, *Phil. Mag.* Vol. XLIV, pp. 450, 823, papers 1 and 2; *Proc.
Camb. Phil. Soc.* Vol. XXI, part 3; Fowler, *Phil. Mag.* Vol. XLV, papers 3 and 4.
(Not yet published.)

§ 2. *General fluctuations in the energy partition.* It is sufficient to consider an assembly of two types of quantized systems. The results can be extended at once to more general assemblies (with or without classical systems) in which no dissociation takes place. In the notation of our previous papers we have

$$C\overline{E_A^{\ n}} = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z^{E+1}} \left\{ \left(z\frac{d}{dz} \right)^n [f(z)]^M \right\} [g(z)]^N. \quad ...(2\cdot1)$$

In this equation $f(z)$ and $g(z)$ are the partition functions of the two types of system, M and N in number, E is the total energy of the assembly, and E_A the energy on the first group of systems. C is the total number of weighted complexions representing the assembly with energy E. The contour γ is any circle with its centre at the origin and its radius less than unity. In order to evaluate $(2\cdot1)$ etc. asymptotically (M, N, E large) we choose that circle for which $|z| = \vartheta$, where ϑ is the unique positive fractional root of the equation

$$E = M\vartheta \frac{d}{d\vartheta} \log f(\vartheta) + N\vartheta \frac{d}{d\vartheta} \log g(\vartheta). \quad ......(2\cdot11).$$

It is our purpose to evaluate the dominant term in expressions such as $(\overline{E_A - \overline{E_A}})^n$. This could be done by expansion and a repeated use of $(2\cdot1)$, but as many terms then cancel the dominant term in $(2\cdot1)$ is insufficient and a more or less complete asymptotic expansion would be required. We avoid this difficulty by constructing first an exact integral for $C\,\overline{(E_A - \overline{E_A})^n}$ analogous to $(2\cdot1)$, a direct evaluation of which leads at once to the dominant term required.

A change of notation is expedient. Put

$$z = e^u, \quad f(z) = e^{F(u)}, \quad g(z) = e^{G(u)}.$$

We shall usually omit the arguments of F and G. Then

$$C\overline{E_A^{\ n}} = \frac{1}{2\pi i} \int_{\gamma'} e^{NG-Eu} \left\{ \left(\frac{d}{du} \right)^n e^{MF} \right\} du. \quad ......(2\cdot2).$$

The contour γ' is now the straight line from $\log \vartheta - i\pi$ to $\log \vartheta + i\pi$. To form the integral for $C\,\overline{(E_A - \overline{E_A})^n}$ we use Leibnitz' theorem and the equation $E = \overline{E_A} + \overline{E_B}$. Thus

$$C\overline{(E_A - \overline{E_A})^n} = C\,[\overline{E_A^{\ n}} - {}_nC_1\overline{E_A^{\ n-1}}\,\overline{E_A} + {}_nC_2\overline{E_A^{\ n-2}}\,(\overline{E_A})^2 - ...],$$

$$= \frac{1}{2\pi i} \int_{\gamma'} e^{NG-Eu} du \left\{ \left(\frac{d}{du} \right)^n - {}_nC_1\overline{E_A} \left(\frac{d}{du} \right)^{n-1} \right.$$
$$\left. + {}_nC_2\,(E_A)^2 \left(\frac{d}{du} \right)^{n-2} - ... \right\} e^{MF},$$

$$= \frac{1}{2\pi i} \int_{\gamma'} e^{NG-\overline{E_B}u} \left\{ \left(\frac{d}{du} \right)^n e^{MF-\overline{E_A}u} \right\} du. \quad ......(2\cdot21).$$

Now we know that effectively the whole value of this integral is provided by a small strip of the contour near $u = \log \Im$. We therefore write $u = \log \Im + i\zeta$ and F for $F(\log \Im)$, etc. We recall also the result of our previous papers, that

$$E_A = M\Im \frac{d}{d\Im} \log f = \left[M \frac{dF}{du} \right]_{\log \Im} = MF'. \quad \ldots\ldots(2\cdot22)$$

Then

$$MF - \overline{E_A}u = M(F - \log \Im F') - \tfrac{1}{2}MF''\zeta^2 - \tfrac{1}{6}iMF'''\zeta^3 + \ldots.$$

There is a similar expression for $NG - E_B u$. The integration now is with respect to ζ, and while ζ is still small we may suppose that $(MF'' + NG'')^{\frac{1}{2}} \zeta$ ranges effectively from $-\infty$ to $+\infty$ while all the other terms such as $MF'''\zeta^3$ remain small. Thus

$$C \, \overline{(E_A - E_A)^n} = \frac{(i)^{-n}}{2\pi} e^{M(F - \log \Im F') + N(G - \log \Im G')}$$
$$\times \int_{-\infty}^{+\infty} e^{-\frac{1}{2}NG''\zeta^2} \{1 + O(N\zeta^3)\} \left(\frac{d}{d\zeta}\right)^n [e^{-\frac{1}{2}MF''\zeta^2} \{1 + O(M\zeta^3)\}] \, d\zeta.$$

The O-terms may be differentiated. If we take the special case $n = 0$ we obtain C, and it is clear that the O-terms cannot then contribute to the leading term in the integral which is

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(MF'' + NG'')\zeta^2} \, d\zeta = \frac{2\pi}{(MF'' + NG'')^{\frac{1}{2}}}.$$

Thus

$$\overline{(E_A - E_A)^n} = (i)^{-n} \left(\frac{MF'' + NG''}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}NG''\zeta^2} \{1 + O(N\zeta^3)\}$$
$$\times \left(\frac{d}{d\zeta}\right)^n [e^{-\frac{1}{2}MF''\zeta^2} \{1 + O(M\zeta^3)\}] \, d\zeta. \quad \ldots\ldots(2\cdot3)$$

The further approximation to (2·3) depends on whether n is even or odd. It is clear in any case that the first differentiation of the bracket $\{1 + O(M\zeta^3)\}$ does not alter the order of the integral; while every time we differentiate $e^{-\frac{1}{2}MF''\zeta^2}$ we increase the order of the resulting integral by $\sqrt{M}$. Thus the highest order term arises from differentiating $e^{-\frac{1}{2}MF''\zeta^2}$ n times, and the O-terms are both irrelevant *provided that this highest order term does not vanish on integration*. This is the case when n is even. When n is odd it does vanish, and further consideration is required to which we return later. If we put $n = 2v$ we therefore find for the required asymptotic form of $\overline{(E_A - E_A)^{2v}}$,

$$\overline{(E_A - E_A)^{2v}} =$$
$$(-)^v \left(\frac{MF'' + NG''}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}NG''\zeta^2} \left\{\left(\frac{d}{d\zeta}\right)^{2v} e^{-\frac{1}{2}MF''\zeta^2}\right\} d\zeta. \quad \ldots(2\cdot31)$$

To evaluate (2·31) let us write

$$I_v = (-)^v \int_{-\infty}^{+\infty} e^{-\frac{1}{2}NG''\zeta^2} \left\{ \left(\frac{d}{d\zeta} \right)^{2v} e^{-\frac{1}{2}MF''\zeta^2} \right\} d\zeta$$

$$= (-)^v \int_{-\infty}^{+\infty} \phi_2 \phi_1^{(2v)} d\zeta = (-)^{v+r} \int_{-\infty}^{+\infty} \phi_2^{(r)} \phi_1^{(2v-r)} d\zeta,$$

the last equation following by integration by parts. Then, since $\phi_1' = -MF''\zeta\phi_1$, $\phi_2' = -NG''\zeta\phi_2$, we find that

$$\frac{MF'' + NG''}{MF''NG''} I_v$$

$$= (-)^{v+1} \left\{ \frac{1}{NG''} \int_{-\infty}^{+\infty} \phi_2'\phi_1^{(2v-1)} d\zeta + \frac{1}{MF''} \int_{-\infty}^{+\infty} \phi_1'\phi_2^{(2v-1)} d\zeta \right\}$$

$$= (-)^v \int_{-\infty}^{+\infty} \zeta \left(\phi_2 \phi_1^{(2v-1)} \div \phi_1 \phi_2^{(2v-1)} \right) d\zeta.$$

If we integrate again by parts we find

$$\frac{MF'' + NG''}{MF''NG''} I_v = 2I_{v-1} + (-)^{v-1} \int_{-\infty}^{+\infty} \zeta \left(\phi_2'\phi_1^{(2v-2)} + \phi_1'\phi_2^{(2v-2)} \right) d\zeta,$$

and by continued repetition

$$\frac{MF'' + NG''}{MF''NG''} I_v = 2vI_{v-1} + \int_{-\infty}^{+\infty} \zeta \left(\phi_2^{(v)}\phi_1^{(v-1)} + \phi_1^{(v)}\phi_2^{(v-1)} \right) d\zeta$$

$$= 2vI_{v-1} - \int_{-\infty}^{+\infty} \zeta d \left(\phi_2^{(v-1)}\phi_1^{(v-1)} \right)$$

$$= (2v-1) I_{v-1}.$$

Thus

$$I_v = (2v-1) \dots 3.1. \left(\frac{MF''NG''}{MF'' + NG''} \right)^v I_0, \quad \left\{ I_0 = \left(\frac{2\pi}{MF'' + NG''} \right)^{\frac{1}{2}} \right\},$$

$$\dots (2\cdot 4)$$

$$\overline{(E_A - \overline{E_A})^{2v}} = (2v-1) \dots 3.1. \left(\frac{MF''NG''}{MF'' + NG''} \right)^v$$

$$(2v-1) \dots 3.1. \left\{ \overline{(E_A - \overline{E_A})^2} \right\}^v, \dots (2\cdot 41)$$

and finally, with the help of (2·22) differentiated,

$$\overline{(E_A - \overline{E_A})^2} = \frac{MF''NG''}{MF'' + NG''} = \Im \frac{d\overline{E_A}}{d\Im} \left(1 - \frac{\Im d\overline{E_A}/d\Im}{\Im dE/d\Im} \right). \quad \dots (2\cdot 5)$$

Fluctuations of the energy of all even orders are thus completely determined.

We now return to the odd orders. To an equivalent approximation these all vanish. Actually they are thus all of order lower by $\sqrt{M}$ or $\sqrt{N}$ than the corresponding even orders. We must retain

the exact $M\zeta^3$ and $N\zeta^3$ terms in (2·3), the $M\zeta^4$ and $N\zeta^4$ terms now being negligible. We find after reduction

$$
\overline{(E_A - \overline{E_A})^{2v-1}}
$$
$$
= \left(\frac{MF'' + NG''}{2\pi}\right)^{\frac{1}{2}} \frac{(-)^v}{6} \left[MF''' \int_{-\infty}^{+\infty} \zeta^3 e^{-\frac{1}{2}NG''\zeta^2} \left\{ \left(\frac{d}{d\zeta}\right)^{2v-1} e^{-\frac{1}{2}MF''\zeta^2} \right\} d\zeta \right.
$$
$$
\left. + NG''' \int_{-\infty}^{+\infty} e^{-\frac{1}{2}NG''\zeta^2} \left\{ \left(\frac{d}{d\zeta}\right)^{2v-1} \zeta^3 e^{-\frac{1}{2}MF''\zeta^2} \right\} d\zeta \right].
$$

But

$$
\zeta^3 e^{-\frac{1}{2}MF''\zeta^2} = -\frac{1}{M^3F''^3} \left(\frac{d}{d\zeta}\right)^3 e^{-\frac{1}{2}MF''\zeta^2} - \frac{3}{M^2F''^2} \frac{d}{d\zeta} e^{-\frac{1}{2}MF''\zeta^2}.
$$

Therefore

$$
\overline{(E_A - \overline{E_A})^{2v-1}} = \left(\frac{MF'' + NG''}{2\pi}\right)^{\frac{1}{2}} \frac{(-)^{v-1}}{6}
$$
$$
\times \left\{ MF''' \int_{-\infty}^{+\infty} \left(\frac{1}{M^3F''^3} \phi_2^{(3)}\phi_1^{(2v-1)} + \frac{3}{M^2F''^2} \phi_2'\phi_1^{(2v-1)} \right) d\zeta \right.
$$
$$
\left. + NG''' \int_{-\infty}^{+\infty} \left(\frac{1}{N^3G''^3} \phi_2\phi_1^{(2v+2)} + \frac{3}{N^2G''^2} \phi_2\phi_1^{(2v)} \right) d\zeta \right\},
$$
$$
= (2v-1)\ldots.3.1.\left(\frac{MF''NG''}{MF'' + NG''}\right)^v \left\{ \frac{G'''}{6NG''^2}\left(3 - (2v+1)\frac{MF''}{MF'' + NG''}\right) \right.
$$
$$
\left. - \frac{F'''}{6MF''^2}\left(3 - (2v+1)\frac{NG''}{MF'' + NG''}\right) \right\}. \quad \ldots(2\cdot6)
$$

This expression is of order $v - 1$ in M or N, i.e. $\frac{1}{2}(2v-1) - \frac{1}{2}$, while $(E_A - \overline{E_A})^{2v}$ is of order $\frac{1}{2}(2v)$ which is relatively greater, as stated above, by $\sqrt{M}$ or $\sqrt{N}$.

When the systems of type A are in a bath of B's, that is when N is very large compared to M, (2·6) reduces to

$$
\overline{(E_A - \overline{E_A})^{2v-1}} = \tfrac{1}{3}(v - 1)(2v - 1)\ldots.3.1.(MF'')^{v-2}MF'''.
$$
$$
\ldots\ldots(2\cdot61)
$$

The corresponding formula in the even case is

$$
\overline{(E_A - \overline{E_A})^{2v}} = (2v - 1)\ldots.3.1.(MF'')^v. \ldots\ldots(2\cdot62)
$$

We recall that

$$
MF'' = \vartheta \frac{d\overline{E_A}}{d\vartheta}, \quad MF''' = \left(\vartheta \frac{d}{d\vartheta}\right)^2 \overline{E_A}. \ldots\ldots(2\cdot63)
$$

These formulae give the dominant terms of the formulae given by

Gibbs*, which alone are relevant when M is large. The general formulae (2·5) and (2·6) refer on the other hand to the case when the systems of type A are not in a temperature bath.

Finally, we may observe that the formula

$$C\,\overline{(E_A - \overline{E_A})^v\,(E_B - \overline{E_B})^w}$$
$$= \frac{1}{2\pi i} \int_{\gamma'} \left\{ \left(\frac{d}{du}\right)^v e^{MF - \overline{E_A}u} \right\} \left\{ \left(\frac{d}{du}\right)^w e^{NG - \overline{E_B}u} \right\} du \;\ldots(2\cdot7)$$

can be derived and evaluated in an exactly similar way.

§ 3. *General fluctuations of $\bar{a}_r$.* A similar, but slightly more complicated, investigation will provide us with the values of $\overline{(a_r - \overline{a_r})^n}$, where a_r is the number of systems of type A in the assembly which are in the rth possible state or cell. By the general formulae of our first paper

$$C\,(a_r - \overline{a_r})^n = \Sigma_{a,\,b}\, \frac{M\,!}{a_1!\,a_2!\,\ldots}\, p_1{}^{a_1} p_2{}^{a_2} \ldots \frac{N\,!}{b_1!\,b_2!\,\ldots}\, q_1{}^{b_1} q_2{}^{b_2} \ldots (a_r - \overline{a_r})^n,$$
$$\ldots\ldots(3\cdot1)$$

where the summation $\Sigma_{a,\,b}$ is to be taken over all zero or positive values of the a's and b's consistent with

$$\Sigma_r a_r = M, \quad \Sigma_s b_s = N, \quad \Sigma_r a_r \epsilon_r + \Sigma_s b_s \epsilon_s = E. \;\ldots\ldots(3\cdot11)$$

There are only a finite number of terms in (3·1). Therefore $C\,(a_r - \overline{a_r})^n$ is $(n\,!)$ times the coefficient of x^n in the expression obtained by replacing $(a_r - \overline{a_r})^n$ in (3·1) by $e^{x(a_r - \overline{a_r})}$. This is conveniently expressed by the notation

$$C\,\overline{(a_r - \overline{a_r})^n} = \mathrm{Coef}_n \Sigma_{a,\,b}\, \frac{M\,!}{a_1!\,a_2!\,\ldots}\, p_1{}^{a_1} p_2{}^{a_2} \ldots \frac{N\,!}{b_1!\,b_2!\,\ldots}\, q_1{}^{b_1} q_2{}^{b_2} \ldots e^{x\,(a_r - \overline{a_r})}.$$
$$\ldots\ldots(3\cdot12)$$

But the series $\Sigma_{a,\,b}$ is now of the usual type with p_r replaced by $p_r e^x$. Therefore in the usual integral $f(z)$ is replaced by

$$f(z) + p_r z^{\epsilon_r}(e^x - 1).$$

On changing to the variable u and the notation of § 2, we find

$$C\,(a_r - \overline{a_r})^n = \mathrm{Coef}_n \frac{1}{2\pi i} \int_{\gamma'} e^{-Eu + MF + NG - \overline{a_r}x} \{1 + p_r e^{\epsilon_r u - F}(e^x - 1)\}^M\, du.$$
$$\ldots\ldots(3\cdot2)$$

The terms containing x in (3·2) can be written

$$\exp\left[-\overline{a_r}x + M \log\{1 + p_r e^{\epsilon_r u - F}(e^x - 1)\}\right]; \;\ldots\ldots(3\cdot21)$$

* *Elementary Principles in Statistical Mechanics*, p. 78.

in (3·21) x can be fixed as small as we please, and $u = \log \mathfrak{S} + i\zeta$, where ζ is small on the only effective part of the contour. It is clear that the presence of the x-terms does not affect the choice of the u-contour when x is sufficiently small. Moreover

$$\overline{a_r} = M p_r e^{\epsilon_r \log \mathfrak{S} - F(\log \mathfrak{S})}.$$

Thus (3·21) becomes after reduction

$$\exp \{i\zeta x \overline{a_r} (\epsilon_r - F') + \tfrac{1}{2} x^2 (\overline{a_r} - \overline{a_r}^2/M) + O(M\zeta^2 x) \\ + O(M\zeta x^2) + O(Mx^3)\}.$$

When $n = 0$ we can at once put $x = 0$ in (3·2) and obtain C. Using this fact and approximating as in § 2 we find that

$$\overline{(a_r - \overline{a_r})^n}$$

$$= \operatorname{Coef}_n \left(\frac{MF'' + NG''}{2\pi} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\zeta^2 (MF'' + NG'') + i\zeta x \overline{a_r}(\epsilon_r - F') + \frac{1}{2} x^2 (\overline{a_r} - \overline{a_r}^2/M)}$$

$$\times \{1 + O(M\zeta^3) + O(M\zeta^2 x) + O(M\zeta x^2) + O(Mx^3)\} \, d\zeta$$

$$= \operatorname{Coef}_n \exp \left\{ \tfrac{1}{2} x^2 \left[\overline{a_r} - \frac{\overline{a_r}^2}{M} - \frac{\overline{a_r}^2 (\epsilon_r - F')^2}{MF'' + NG''} \right] \right\}$$

$$\times [1 + O(M^{-\frac{1}{2}}) + O(x) + O(M^{\frac{1}{2}}x) + O(Mx^3)]. \ldots(3·3)$$

We require the highest order term in (3·3) and it is obvious that when n is even, $n = 2v$, all the O-terms are negligible. Thus

$$\overline{(a_r - \overline{a_r})^{2v}} = \operatorname{Coef}_{2v} \exp \left\{ \tfrac{1}{2} x^2 \left[\overline{a_r} - \frac{\overline{a_r}^2}{M} - \frac{\overline{a_r}^2 (\epsilon_r - F')^2}{MF'' + NG''} \right] \right\}$$

$$= \frac{2v!}{2^v v!} \left[\overline{a_r} - \frac{\overline{a_r}^2}{M} - \frac{\overline{a_r}^2 (\epsilon_r - F')^2}{MF'' + NG''} \right]^v$$

$$= (2v - 1) \ldots 3 . 1 . \left[\overline{a_r} - \frac{\overline{a_r}^2}{M} \left(1 + \frac{M (\epsilon_r - \overline{E_A}/M)^2}{\mathfrak{S} dE/d\mathfrak{S}} \right) \right]^v$$

$$\ldots\ldots(3·4)$$

$$= (2v - 1) \ldots 3 . 1 . \left[\overline{a_r} - \frac{\overline{a_r}^2}{M} - \frac{(\mathfrak{S} d\overline{a_r}/d\mathfrak{S})^2}{\mathfrak{S} dE/d\mathfrak{S}} \right]^v \ldots(3·41)$$

$$= (2v - 1) \ldots 3 . 1 . \{(a_r - \overline{a_r})^2\}^v. \qquad \ldots\ldots(3·42)$$

When n is odd, the approximation must be carried one stage further to obtain $\overline{(a_r - a_r)^{2v-1}}$. The actual formula we omit; it can be easily obtained if desired. By an obvious extension we can also calculate all such expressions as

$$\overline{(a_r - \overline{a_r})^v (a_s - \overline{a_s})^w} \text{ and } \overline{(a_r - \overline{a_r})^v (b_s - \overline{b_s})^w}$$

by evaluating the coefficients of $x^v y^w$ in the integrals

$$\frac{1}{2\pi i}\int_{\gamma'} du\, e^{-Eu+MF+NG-\overline{a_r}x-\overline{a_s}y}$$
$$\times [1 + p_r e^{\epsilon_r u - F}(e^x - 1) + p_s e^{\epsilon_s u - F}(e^y - 1)]^M,$$

$$\frac{1}{2\pi i}\int_{\gamma'} du\, e^{-Eu+MF+NG-\overline{a_r}x-\overline{b_s}y}$$
$$\times [1 + p_r e^{\epsilon_r u - F}(e^x - 1)]^M [1 + q_s e^{\eta_s u - G}(e^y - 1)]^N,$$

respectively. We shall content ourselves with giving the results required in calculating the second order fluctuations in the external reactions of the assembly, such as its partial pressures on the walls. We find without difficulty that

$$\overline{(a_r - \overline{a_r})(a_s - \overline{a_s})} = -\,\overline{a_r}\overline{a_s}\left[\frac{1}{M} + \frac{(\epsilon_r - F')(\epsilon_s - F')}{MF'' + NG''}\right]$$
$$= -\frac{\overline{a_r}\overline{a_s}}{M} - \frac{(\Im\, d\overline{a_r}/d\Im)(\Im\, d\overline{a_s}/d\Im)}{\Im\, dE/d\Im},\ ...(3\cdot5)$$

$$\overline{(a_r - \overline{a_r})(b_s - \overline{b_s})} = -\,\overline{a_r}\overline{b_s}\frac{(\epsilon_r - F')(\eta_s - G')}{MF'' + NG''}$$
$$= -\frac{(\Im\, d\overline{a_r}/d\Im)(\Im\, d\overline{b_s}/d\Im)}{\Im\, dE/d\Im}.\qquad ......(3\cdot51)$$

§ 4. *Fluctuations in the external reactions.* The generalized reaction of the systems of type A on an external body is given by the equation*

$$Y = \Sigma_r\left(-\frac{\partial\epsilon_r}{\partial y}\right)a_r,\qquad ......(4\cdot1)$$

where y is the corresponding coordinate defining the position of the external body. The average reaction is

$$\overline{Y} = \Sigma_r\left(-\frac{\partial\epsilon_r}{\partial y}\right)\overline{a_r} = M\Sigma_r\left(-\frac{\partial\epsilon_r}{\partial y}\right)p_r\Im^{\epsilon_r}/f(\Im)\ ...(4\cdot11)$$
$$= \frac{M}{\log 1/\Im}\frac{\partial}{\partial y}\log f(\Im).\qquad ......(4\cdot12)$$

The second order fluctuation of Y is

$$\overline{(Y - \overline{Y})^2} = \Sigma_r\left(\frac{\partial\epsilon_r}{\partial y}\right)^2\overline{(a_r - \overline{a_r})^2} + 2\Sigma_{r,s}\frac{\partial\epsilon_r}{\partial y}\frac{\partial\epsilon_s}{\partial y}\overline{(a_r - \overline{a_r})(a_s - \overline{a_s})}.$$

Applying the formulae $(3\cdot41)$ and $(3\cdot5)$ we find that

$$\overline{(Y - \overline{Y})^2} = \Sigma_r\left(\frac{\partial\epsilon_r}{\partial y}\right)^2\overline{a_r} - \frac{1}{M}\left\{\Sigma_r\left(\frac{\partial\epsilon_r}{\partial y}\right)\overline{a_r}\right\}^2$$
$$- \frac{1}{\Im\,\partial E/\partial\Im}\left\{\Sigma_r\left(\frac{\partial\epsilon_r}{\partial y}\right)\Im\frac{\partial\overline{a_r}}{\partial\Im}\right\}^2.$$

* Second paper, § 7.

This expression can be simplified, for on differentiating (4·11) we find

$$\Sigma_r \left(\frac{\partial \epsilon_r}{\partial y}\right)^2 \overline{a_r} - \frac{1}{M} \left\{\Sigma_r \left(\frac{\partial \epsilon_r}{\partial y}\right) \overline{a_r}\right\}^2 = \left(\frac{\partial \overline{Y}}{\partial y} - \frac{\overline{\partial Y}}{\partial y}\right) \Big/ \log 1/\vartheta. \ldots(4\cdot2)$$

Therefore by (4·2) and (4·11)

$$(Y - \overline{Y})^2 = \left(\frac{\partial \overline{Y}}{\partial y} - \frac{\overline{\partial Y}}{\partial y}\right) \Big/ \log 1/\vartheta - \frac{\{\vartheta \, \partial \overline{Y}/\partial \vartheta\}^2}{\vartheta \, \partial E/\partial \vartheta} \ . \ \ldots(4\cdot3)$$

Gibbs* gives a formula for $\overline{(Y - \overline{Y})^2}$ equivalent to (4·3) without the last term, which is properly negligible under bath conditions (N large compared to M).

With the help of (3·51) it is easy to shew that (4·3) is formally unaltered if Y refers to the total reaction due to all groups of systems instead of to the partial reaction of a single group.

In formula (4·3) all the terms except $\overline{\partial Y/\partial y}$ or $\Sigma_r \, \overline{a_r} \, (-\, \partial^2 \epsilon_r/\partial y^2)$ have an obvious interpretation. This term lies deeper and is compared by Gibbs to an elasticity. In illustration we shall apply these results to the limiting case in which the reaction is a pressure. It is sufficient to consider the reaction with a part of the wall only, which may be taken to be plane and represented by a moveable piston in a cylinder of cross-section A which completes the enclosure. We cannot progress without *some* definite assumption as to the field of force near the wall. We shall suppose its potential is D/d^s, where d is the distance of the molecule from the wall and D and s are constants. If D is small and $s > 4$ or so, this will adequately represent an intense local field of force. If y is the length of the cylinder and x_1, x_2, x_3 rectangular cartesian coordinates, x_1 along the cylinder, then

$$\epsilon_r = \frac{1}{2m}(p_1{}^2 + p_2{}^2 + p_3{}^2)_r + [D/(y - x_1)^s]_r, \ \ldots\ldots(4\cdot4)$$

$$f(\vartheta) = \frac{(2\pi m)^{\frac{3}{2}} A}{h^3 (\log 1/\vartheta)^{\frac{3}{2}}} \int_0^y \exp\left\{-\log 1/\vartheta \, \frac{D}{(y - x_1)^s}\right\} dx_1 . \ \ldots(4\cdot41)$$

When the field is sufficiently local, or D nearly zero, we have effectively

$$f(\vartheta) = \frac{(2\pi m)^{\frac{3}{2}} Ay}{h^3 (\log 1/\vartheta)^{\frac{3}{2}}}, \qquad\qquad \ldots\ldots(4\cdot42)$$

in agreement with our first paper § 12. Further, by differentiating (4·41) with respect to y we find effectively (D nearly zero)

$$\frac{1}{f} \frac{\partial f}{\partial y} = \frac{1}{y}. \qquad\qquad \ldots\ldots(4\cdot5)$$

* *Loc. cit.* p. 81.

This gives us at once by (4·12)

$$\overline{Y} = M/(y \log 1/\mathfrak{S}), \qquad \ldots\ldots(4\cdot51)$$

or the usual result

$$\overline{Y}y = PV = MkT. \qquad \ldots\ldots(4\cdot52)$$

It appears that we must arrive at (4·52) whatever the law of force. But when we come to calculate $\overline{\partial Y/\partial y}$ we find that it depends essentially on the form of the law. Thus

$$\overline{\frac{\partial Y}{\partial y}} = \Sigma_r \left(-\frac{\partial^2 \epsilon_r}{\partial y^2}\right) \overline{a_r} = -\frac{M}{y} \int_0^y \frac{Ds(s+1)}{(y-x_1)^{s+2}} \exp\left\{-\log 1/\mathfrak{S} \frac{D}{(y-x_1)^s}\right\} dx_1$$

$$= -\frac{(s+1)M}{y} \int_{D/l^s}^\infty \left(\frac{q}{D}\right)^{1/s} e^{-q\log 1/\mathfrak{S}} dq.$$

Now D/l^s is the potential of the wall at the other end of the cylinder, and in any case must be indistinguishable from zero. Thus

$$\overline{\frac{\partial Y}{\partial y}} = -\frac{(s+1)M}{yD^{1/s}} \frac{\Gamma(1+1/s)}{(\log 1/\mathfrak{S})^{1+1/s}}. \qquad \ldots\ldots(4\cdot6)$$

This depends essentially on D and s, and moreover must tend to infinity if $D \to 0$ or $s \to \infty$. The largeness of $\partial Y/\partial y$ and therewith the fluctuation $\overline{(Y-\overline{Y})^2}$ is, however, to be expected in this case, for in the limit the momentary reaction of the boundary with a single molecule of finite energy must itself become infinite, although $\overline{Y}$ retains its ordinary value. This by itself is sufficient to permit of an infinity in $(Y-\overline{Y})^2$. In spite of this, however, if we calculate roughly the order of $\overline{(Y-\overline{Y})^2}/(\overline{Y})^2$ for reasonable values of D and s we find that it is still negligibly small.

Other second order fluctuations involving reactions, such as

$$\overline{(Y-\overline{Y})(Z-\overline{Z})} \text{ and } \overline{(Y-\overline{Y})(E_A - \overline{E_A})},$$

both mentioned by Gibbs, may be calculated in a similar manner. We find

$$\overline{(Y-\overline{Y})(Z-\overline{Z})} = \left(\frac{\partial \overline{Y}}{\partial z} - \overline{\frac{\partial Y}{\partial z}}\right)\bigg/\log 1/\mathfrak{S} - \frac{(\mathfrak{S}\,\partial\overline{Y}/\partial\mathfrak{S})(\mathfrak{S}\,\partial\overline{Z}/\partial\mathfrak{S})}{\mathfrak{S}\,\partial E/\partial\mathfrak{S}},$$

$$\ldots\ldots(4\cdot7)$$

$$\overline{(Y-Y)(E_A - \overline{E_A})} = \frac{\mathfrak{S}\,\partial\overline{Y}}{\partial\mathfrak{S}}\left(1 - \frac{\mathfrak{S}\,\partial\overline{E_A}/\partial\mathfrak{S}}{\mathfrak{S}\,\partial E/\partial\mathfrak{S}}\right). \qquad \ldots\ldots(4\cdot71)$$

§ 5. *Fluctuations of concentration in a dissociating assembly.* It has been shewn in § 9 of the third paper of this series that in a gaseous assembly formed of dissociating and associating atoms and

molecules the average number of molecules of type 1 is given by the formulae

$$C = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z^{E+1}} [g_1(z)]^{X_1} \dots [g_s(z)]^{X_s} \Sigma_{(N)} \frac{X_1! \dots X_s! \, \beta_1^{N_1} \dots \beta_j^{N_j}}{M_1! \dots M_s! \, N_1! \dots N_j!},$$
$$\dots\dots(5\cdot 1)$$

$$C\overline{N_1} = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z^{E+1}} [g_1(z)]^{X_1} \dots [g_s(z)]^{X_s} \beta_1 \frac{\widehat{c}}{\widehat{\partial}\beta_1} \Sigma_{(N)}. \qquad \dots\dots(5\cdot 11)$$

The X's are total numbers of each sort of atom, the N's the numbers of each sort of molecule, the M's the numbers of atoms remaining uncombined, and the g's the partition functions for their translational motion. The β's are functions of the partition functions which we need not specify further here. Reference should be made to the paper quoted for fuller details.

It is expedient to make a change of notation similar to that of § 2. We write

$$z = e^u; \quad \log\beta_1(z) = B_1(u) - B_1, \text{ etc.}; \quad \log g_1(z) = G_1(u) - G_1, \text{ etc.};$$

$$G_1' = \frac{dG_1}{du}, \text{ etc.}; \quad \log\Sigma_{(N)} = \sigma(B_1 \dots B_j) = \sigma; \quad \frac{\partial\sigma}{\partial B_1} = \sigma_1, \text{ etc.}$$

Then
$$C\overline{N_1} = \frac{1}{2\pi i} \int_{\gamma'} du \, e^{-Eu + X_1 G_1 + \dots + X_s G_s} \frac{\partial}{\partial B_1} e^{\sigma}. \quad \dots(5\cdot 12)$$

It is found that $\overline{N_1}$, etc. are determined by equations equivalent to $\overline{N_1} = \sigma_1$, etc. provided that $u = \log\vartheta$ and that ϑ is determined by an equation equivalent to

$$E = [X_1 G_1' + \dots + X_s G_s' + \overline{N_1} B_1' + \dots + \overline{N_j} B_j']_{u = \log\vartheta}. \dots(5\cdot 121)$$

By an obvious extension of the arguments of § 9 of the third paper and of § 2 here

$$C\overline{N_1{}^n} = \frac{1}{2\pi i} \int_{\gamma'} du \, e^{-Eu + X_1 G_1 + \dots + X_s G_s} \left(\frac{\widehat{c}}{\widehat{\partial} B_1}\right)^n e^{\sigma}, \quad \dots(5\cdot 13)$$

$$C\overline{(N_1 - \overline{N_1})^n} = \frac{1}{2\pi i} \int_{\gamma'} du \, e^{-Eu + X_1 G_1 + \dots + X_s G_s + \overline{N_1} B_1} \left(\frac{\widehat{c}}{\widehat{\partial} B_1}\right)^n e^{\sigma - \overline{N_1} B_1}.$$
$$\dots\dots(5\cdot 14)$$

We can treat $(5\cdot 14)$ just as we treated $(2\cdot 21)$; for this purpose we write the integrand in the form

$$\exp\left[-Eu + X_1 G_1 + \dots + X_s G_s + \overline{N_1} B_1 + \dots + N_j B_j\right]$$
$$\times \left(\frac{\partial}{\partial B_1}\right)^n \exp\left[\sigma - \overline{N_1} B_1 - \dots - \overline{N_j} B_j\right].$$

Put $u = \log \Im + i\zeta$, and assume that $\log \Im$ is the argument of every function that occurs. The integrand becomes

$$\exp\left[-E\log\Im + X_1 G_1 + \ldots + X_s G_s + \sigma\right.$$
$$- \tfrac{1}{2}\zeta^2\{X_1 G_1'' + \ldots + X_s G_s'' + \overline{N_1}B_1'' + \ldots + \overline{N_j}B_j''\} + O\left(X\zeta^3\right)\right]$$
$$\times \left(\frac{\partial}{\partial(\delta B_1)}\right)^n \exp\left[\tfrac{1}{2}\{\sigma_{11}(\delta B_1)^2 + \ldots + 2\sigma_{12}\delta B_1\delta B_2 + \ldots\}\right.$$
$$\left. + O\left(\frac{\partial^3\sigma}{\partial B^3}(\delta B)^3\right)\right].$$

The coefficient of $i\zeta$ in the first exponential vanishes by (5·121). The coefficients of δB_1, etc. in the second all vanish in virtue of the equations $\overline{N}_1 = \sigma_1$, etc. Further

$$\delta B_1 = i\zeta B_1' + O\left(\zeta^2 B_1''\right), \text{ etc.}$$

By arguments exactly similar to those of § 2, we find that the leading term in $(N_1 - \overline{N}_1)^n$ is determined by the equation

$$(N_1 - \overline{N_1})^n = \left(\frac{\Im\,\partial E/\partial\Im}{2\pi}\right)^{\frac{1}{2}}(i)^{-n}\int_{-\infty}^{+\infty}d\zeta\,e^{-\frac{1}{2}\zeta^2\{X_1 G_1'' + \ldots + \overline{N_j}B_j''\}}$$
$$\times \frac{1}{\zeta^n}\left(\frac{\partial}{\partial B_1'}\right)^n e^{-\frac{1}{2}\zeta^2\{\sigma_{11}B_1'^2 + \ldots + 2\sigma_{12}B_1'B_2' + \ldots\}}. \quad \ldots(5\cdot2)$$

This integral will of course be found to vanish if n is odd, and further approximations are required, which we shall omit.

The integral in (5·2) can be cast into the same form as the integral in (2·31). For we can write the coefficient of the second $-\tfrac{1}{2}\zeta^2$ in (5·2) in the form $\sigma_{11}(B_1' + \mu)^2 + \lambda$, where λ and μ do not depend on B_1'. Thus the integral takes the form

$$\frac{1}{(B_1' + \mu)^n}\int_{-\infty}^{+\infty}d\zeta\,e^{-\frac{1}{2}\zeta^2\{X_1 G_1'' + \ldots + N_j B_j'' + \lambda\}}\left(\frac{\partial}{\partial\zeta}\right)^n e^{-\frac{1}{2}\sigma_{11}(B_1'+\mu)^2\zeta^2}.$$

It follows at once by the same reasoning as in § 2 that

$$(N_1 - N_1)^{2v} = (2v-1)\ldots\ldots 3.1.\left[\frac{\sigma_{11}\{X_1 G_1'' + \ldots + N_j B_j'' + \lambda\}}{\Im\,\partial E/\partial\Im}\right]^v$$
$$= (2v-1)\ldots\ldots 3.1.\left[\sigma_{11} - \frac{\sigma_{11}^2(B_1' + \mu)^2}{\Im\,\partial E/\partial\Im}\right]^v$$
$$(2v-1)\ldots\ldots 3.1.\left[\sigma_{11} - \frac{(\sigma_{11}B_1' + \sigma_{12}B_2' + \ldots + \sigma_{1j}B_j')^2}{\Im\,\partial E/\partial\Im}\right]^v.$$
$$\ldots\ldots(5\cdot31)$$

The relation between $(N_1 - \overline{N}_1)^{2v}$ and $(N_1 - \overline{N}_1)^2$ is therefore the

same as in all the similar previous cases. The various terms in $\overline{(N_1 - \overline{N_1})^2}$ can be to some extent interpreted. We find that

$$\sigma_{11} = \frac{\partial^2 \sigma}{\partial B_1{}^2} = \frac{\partial \overline{N_1}}{\partial B_1} = \beta_1 \frac{\partial \overline{N_1}}{\partial \beta_1}; \quad \sigma_{12} = \beta_2 \frac{\partial \overline{N_1}}{\partial \beta_2};$$

$$B_1' = \frac{\vartheta}{\beta_1} \frac{\partial \beta_1}{\partial \vartheta}, \text{ etc.};$$

so that

$$\overline{(N_1 - \overline{N_1})^2} = \beta_1 \frac{\partial N_1}{\partial \beta_1} - \left\{ \left[\vartheta \frac{\partial \beta_1}{\partial \vartheta} \frac{\partial}{\partial \beta_1} + \ldots + \vartheta \frac{\partial \beta_j}{\partial \vartheta} \frac{\partial}{\partial \beta_j} \right] \overline{N_1} \right\}^2 \Big/ \vartheta \frac{\partial E}{\partial \vartheta}$$

$$= \beta_1 \frac{\partial \overline{N_1}}{\partial \beta_1} - \left(\vartheta \frac{\partial \overline{N_1}}{\partial \vartheta} \right)^2 \Big/ \vartheta \frac{\partial E}{\partial \vartheta}. \qquad \ldots\ldots(5\cdot32)$$

The method by which such mean values as $\overline{(N_1 - N_1)^v (N_2 - N_2)^w}$ are to be calculated is sufficiently obvious from the foregoing, but the complete results would be complicated. It is clear that we get in the simplest case

$$\overline{(N_1 - \overline{N_1})(N_2 - \overline{N_2})} = \beta_2 \frac{\partial \overline{N_1}}{\partial \beta_2} - \left(\vartheta \frac{\partial \overline{N_1}}{\partial \vartheta} \right) \left(\vartheta \frac{\partial \overline{N_2}}{\partial \vartheta} \right) \Big/ \vartheta \frac{\partial E}{\partial \vartheta} \ldots(5\cdot33)$$

§ 6. *Other fluctuations in dissociating assemblies.* In conclusion we will indicate shortly how the analysis of §§ 2–4 must be modified in order to extend it to deal with dissociating assemblies. If we consider one particular example of the assembly with $M_1, \ldots$ free atoms and $N_1, \ldots$ molecules of each sort, the number of weighted complexions representing it is found to be

$$\frac{1}{2\pi i} \int_{\gamma'} du\, e^{-Eu + M_1 G_1 + \ldots + M_s G_s + N_1 (H_1 + R_1 - \chi_1 u) + \ldots + N_j (H_j + R_j - \chi_j u)} \ldots(6\cdot1)$$

The H-terms and the R-terms arise from the partition functions for the translational and internal motions of the molecules respectively and the χ's from their heats of dissociation. If we wish to calculate the contribution of this example of the assembly to such an expression as, say, $C (E_{R_1} - E_{R_1})^n$ we have to replace the factor $e^{N_1 R_1}$ in (6·1) by $e^{E_{R_1} u} \left(\dfrac{d}{du} \right)^n e^{N_1 R_1 - E_{R_1} u}$. We have then to sum (6·1) or its analogue for all examples of the assembly. We thus get in the integrand, instead of the factor, which we have called $e^{\sigma (B_1, \ldots B_j)}$, an expression which may be written

$$e^{\overline{E_{R_1} u}} \left(\frac{d}{du} \right)^n e^{\sigma - \overline{E_{R_1} u}},$$

provided that the differentiations are only applied to a single term, the R_1-term in the function B_1 the first argument of σ, and to $e^{\overline{E_{R_1}} u}$. With these restricted differentiations we find

$$C\,(E_{R_1} - \overline{E_{R_1}})^n = \frac{1}{2\pi i} \int_{\gamma'} du\, e^{X_1 G_1 + \ldots + X_s G_s + (\overline{E_{R_1}} - E)\,u} \left(\frac{d}{du}\right)^n e^{\sigma - \overline{E_{R_1}} u}.$$
$$\ldots\ldots(6\cdot2)$$

The argument then proceeds on just the same lines as before, and we arrive finally at results which can be cast into exactly the same form as equations $(2\cdot41)$ and $(2\cdot5)$ of the non-dissociating case.

We can obtain similar extensions of the results for $(a_r - \overline{a_r})^n$, where a_r refers (say) to the number of molecules of type 1 whose rotations are in the rth quantum state. We find

$$C\,(a_r - \overline{a_r})^n = \mathrm{Coef}_n \frac{1}{2\pi i} \int_{\gamma'} du\, e^{-Eu + X_1 G_1 + \ldots + X_s G_s - \overline{a_r} x + \sigma}, \ \ldots(6\cdot3)$$

where now

$$\sigma = \sigma\,(B_1 + D_1,\, B_2,\, \ldots,\, B_j), \quad D_1 = \log\,[1 + p_r e^{\epsilon r u - R}\,(e^x - 1)].$$
$$\ldots\ldots(6\cdot31)$$

From $(6\cdot31)$ we can proceed with obvious modifications of former arguments to equations of the same form as $(3\cdot4)$, $(3\cdot41)$ and $(3\cdot42)$.

On some α-Ray tracks. By C. T. R. Wilson, F.R.S.

(Plates III, IV.)

[*Read* 27 November 1922.]

I have recently been applying the cloud method mainly to the study of X-rays and β-rays. A number of stereoscopic pictures of α-ray tracks were however taken to test the working of the apparatus. Some of these are of interest in themselves and as illustrating points regarding the applications of the method and the interpretation of such photographs.

The pictures here reproduced (on four times the scale of the original objects) were all obtained by spreading a small quantity of Thorium oxide along a narrow strip across the middle of the floor of the cloud-chamber* (which was about 18 cms. in diameter and 3 cms. in height) and covering the oxide with black paper. Thorium emanation is continually diffusing through the paper into the air of the cloud-chamber. From time to time an emanation atom ejects an α-particle and within a small fraction of a second an α-particle is in turn ejected by the resulting Thorium-A atom. A potential difference was maintained between the roof and floor of the cloud-chamber, the roof being negative so that free positive ions travelled upwards, negative downwards. The clouds condensed on the ions as a result of sudden expansion of the air were photographed through the side of the cloud-chamber.

In the left half of Fig. 1, Plate III are visible two inclined diffuse cloud tracks, *b* and *b′*. An α-particle had passed through the air before the expansion occurred and had left a trail of free positive and negative ions; these have been separated by the electric field before being rendered immobile by condensed water. The α-particle had traversed the whole height of the cloud chamber along a path *aa* (whether from roof to floor or floor to roof it is impossible to say); the vertical displacement of the ions (positive upwards, negative downwards) has exceeded half the height of the cloud-chamber. The potential difference maintained between the roof and floor amounted in this case to 100 volts; from the amount of the vertical separation of the tracks, it follows that the α-particle traversed the air about $\frac{1}{30}$ of a second before the expansion.

The other events of which the picture gives a record occurred after the expansion. Near the extreme right of the picture a thorium emanation atom at *d* has ejected an α-particle along *de* downwards and away from the camera, and the ions liberated, including

* *Roy. Soc. Proc.* A, 87, p. 277, 1912.

the Th A atom, have been at once fixed by condensation of water so that a sharply defined cloud track is formed. (The track goes out of focus as it recedes from the camera and ultimately passes out of the illuminated region.) The Th A atom in the head of this cloud track has within a small fraction of a second ejected an α-particle in a nearly horizontal direction along df. Over the greater part of the length of its path condensation has at once occurred giving a sharply defined and straight cloud track. But in the immediate neighbourhood of the lower end of the diffuse cloud b, already condensed on the positive ions of the older track described above, the new α-particle finds the air robbed of a considerable portion of its water vapour; in consequence no condensation here takes place on the ions liberated by it until they have been carried by the electric field into a region where the necessary supersaturation still exists. In the present case the negative ions moving down under the action of the field have fairly soon reached a region where the water vapour is still sufficiently supersaturated to condense upon them; the V-shaped diversion in the otherwise straight track is thus easily explained. The positive ions have been carried further into the old cloud by the action of the field and have thus left no track, except for a short distance above the two upper ends of the V.

Again, ions formed along the initial portion of the α-ray from Th A (as well as by the recoil atom) were liberated within or very near the cloud which had already condensed along the track of the original emanation α-particle and of the recoiling Th A atom. Condensation on these ions, positive or negative, could thus only take place when they had been carried sufficiently far up or down to enter regions in which the critical supersaturation was still exceeded. The initial portion of the cloud track actually left by the α-particle from the Th A atom is therefore pincer-shaped, and the head of the original α-ray track from the emanation atom lies midway between the jaws of the pincers.

The sharply defined portions of the α-ray track from the Th A atom in Fig. 1 show here and there little projections—the tracks of slow-speed β-rays. Photographs showing such rays originating from α-ray tracks in hydrogen were obtained by Bumstead[*]; they were called by him fast δ-rays.

A good example of the perfectly straight beaded track of a very fast β-particle is visible towards the right of the picture.

On the other hand the large deviations from straightness which characterise the last few centimetres of the path of a β-particle, when its velocity has been largely reduced by encounters, are well shown in the threadlike track visible in Fig. 2, Plate IV.

Fig. 2, Plate IV shows the initial portions of two α-ray tracks, g and g', each of which starts from an enlarged head—the track of

[*] *Physical Review*, VIII, p. 715, 1916.

the recoil atom. The potential difference between the roof and floor was again 100 volts. The short β-ray tracks (Bumstead's δ-ray tracks) radiating from the path of the α-particle are well shown. A number of pictures showing similar features have been obtained. The maximum range of the δ-ray tracks in air at $\frac{2}{3}$ of normal density is between ·4 and ·5 mm. This range greatly exceeds that of the β-rays excited in air by the K-radiation of aluminium and amounts to about one-fifth of that of the longest β-rays excited by copper K-radiations. (The copper and aluminium ranges were obtained also by the cloud method in the course of experiments which will be described elsewhere.) If we assume the range to vary as the cube of the velocity of the β-particle, we find for the δ-rays of maximum range a velocity of about $3 \cdot 5 \times 10^9$ cms. per second—about twice that of the α-particle. We should expect a β-particle which had been expelled from among the relatively slowly moving electrons of the *outer* level of an atom, with a velocity comparable with twice that of the α-particle, to have a large forward component in its velocity. There is however no indication of a preponderating forward component in the pictures; the β-particles generally appear to be emitted nearly at right angles to the α-ray tracks. Accordingly it seems probable that some at least come from the K levels of the atoms.

In none of the photographs do any δ-rays appear on the last two centimetres of the α-ray tracks. This is well illustrated by the stereoscopic picture reproduced in Fig. 3, Plate IV. The potential difference between the roof and floor of the cloud chamber was 20 volts. Here are seen side by side the initial portion of one α-ray track, *h*, and the final portion of another, *i*. The initial portion of a third track, *k*, is also shown crossing the lower part of the picture from left to right.

If we suppose that the range of the δ-rays is proportional to the cube of their velocity like that of the α-ray and that the velocity of the δ-particle is proportional to the velocity of the α-particle which ejects it, then the maximum range of the δ-particle (which is about ·5 mm. at 5 cms. from the end of the α-ray) will only be about 0·1 mm. at 1 cm. from the end. Now a sharply defined α-ray track has generally a radius of about this magnitude, so that near the end of the α-ray track the δ-ray would not project beyond the general cloud track.

It is unlikely however that the distribution of the δ-rays is to be wholly explained in this way. It may be connected with the fact that the velocity of the α-particle near the middle of its course passes through values which on Bohr's theory are comparable with the velocities of the electrons in the K orbits of Oxygen and Nitrogen.

If electrons are expelled from the K level of some of the atoms

traversed by the α-particles, we should expect the corresponding characteristic X-radiations to result from the falling in of outer electrons into the vacancies thus created. In Fig. 2 there are in fact visible, at radial distances of 0·5 to 1·5 mm. from the early portion of the α-ray track, minute globular cloudlets—the tracks probably of very short-range β-rays ejected from atoms of the gas by K-radiations which originate in the α-ray track.

Kapitza has found* that the loss of energy of an α-particle per ion liberated is greater in the first part of its path than in the last few centimetres. He suggests that this excess may be due to a certain proportion of the electrons liberated coming from the K level of the atoms. The present experiments tend to confirm that view; they indicate that some of the missing energy may escape as X-rays which are absorbed outside the track.

A much more detailed study of the range and distribution of the δ-rays along a large number of α-ray tracks is obviously required.

The separated positive and negative α-ray cloud tracks of Fig. 1 and similar pictures are sufficiently nearly resolvable into their separate drops to suggest that it may be possible to apply the cloud method to the direct determination of the ionisation along the path of an α-ray or even of a recoil atom. Some of the pictures which have been obtained show in a striking way the much greater density of the ionisation in the recoil tracks.

Another application suggested by some of the photographs is the measurement of the mobilities of the ions and in particular the separation of ions of different mobilities. This is a problem of considerable interest in view of recent work by J. J. Nolan† and others. Some multiple diffuse cloud tracks, e.g. the quadruple track which appears in the background of Fig. 3, are most naturally interpreted as indicating the existence of ions of different mobilities. It is possible, however, that under certain conditions some of the ions escape immediate capture at the moment of expansion and so lose their chance of finding the necessary supersaturation of water vapour till they have been carried by the electric field beyond the influence of the first formed drops.

There is one ion of which it is of especial interest to observe the displacement—i.e. the Th A or similar atom left behind by the expulsion of an α-particle from an emanation atom. The point at which it takes its origin by the expulsion of an α-ray from the emanation atom as well as the point where its life comes to an end by the expulsion of a second α-particle are both marked in a cloud track picture. If the first of these events occurs before the ions are fixed by condensation of water, the second after, we can compare the displacement of the Th A atom with that of the other ions,

* *Roy. Soc. Proc.* A, 102, p. 48, 1922.
† *Proc. Roy. Irish Acad.* 36, A, p. 81, 1922.
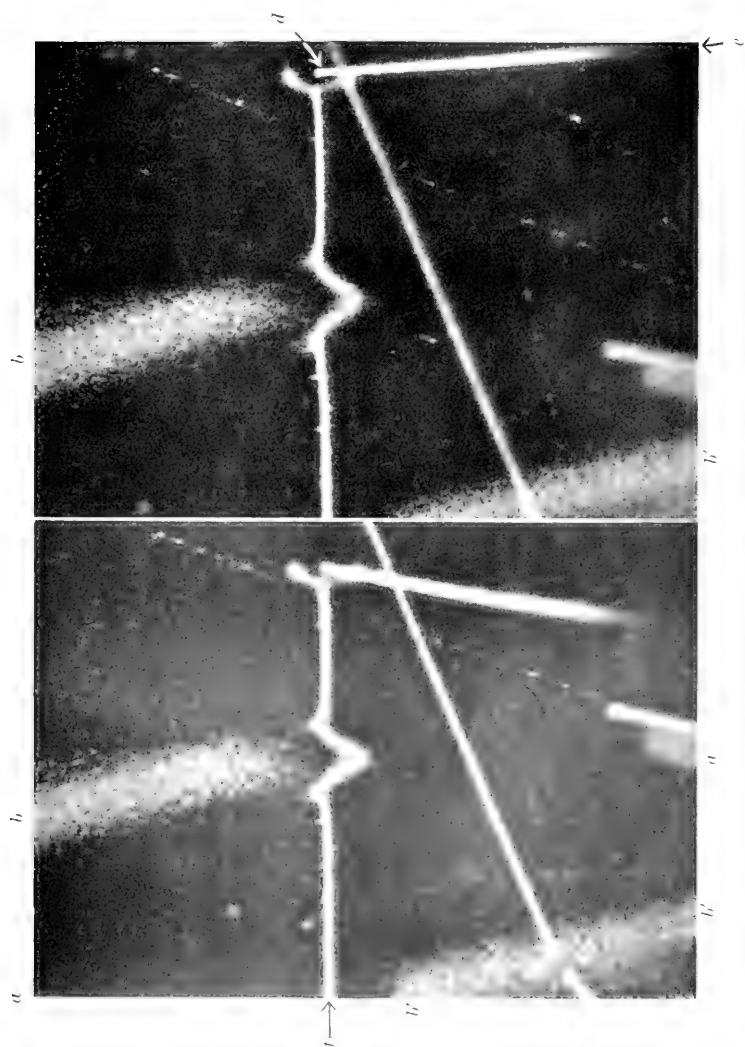
Fig. 1. 4-fold enlargement of original objects
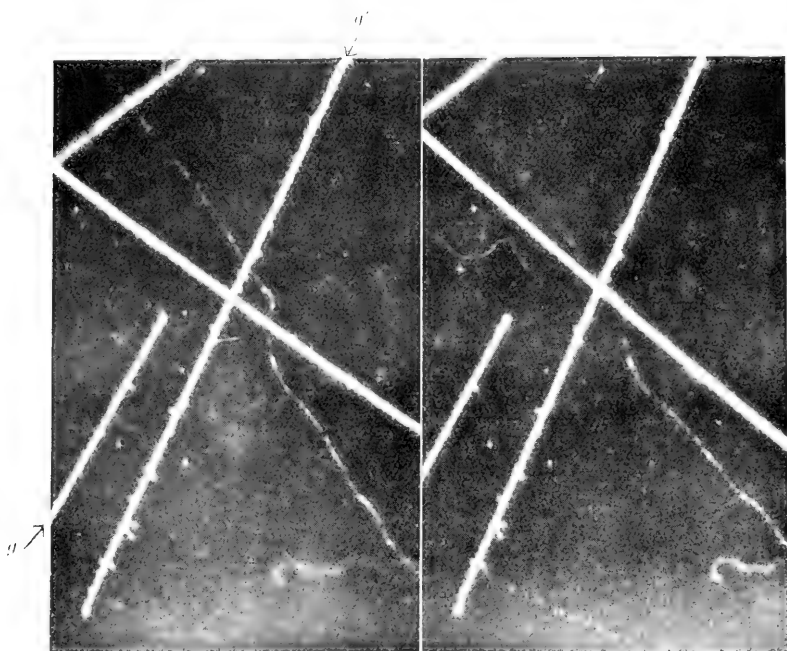
Fig. 2. 4-fold enlargement of original objects
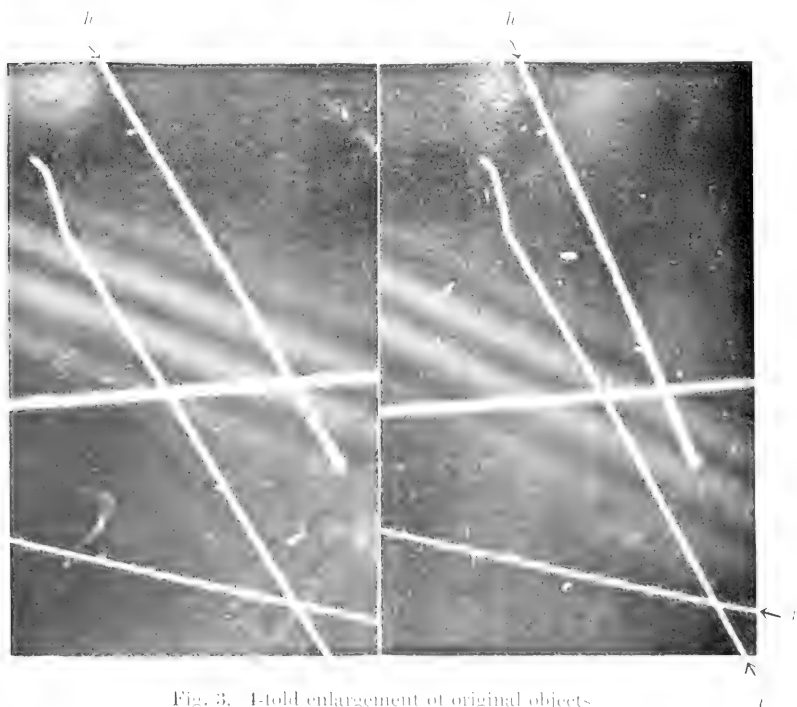
Fig. 3. 4-fold enlargement of original objects

and thus obtain their relative mobilities. In the only two pictures that have thus far admitted of this comparison being made, the Th A atom appeared to have been displaced at least twice as far as the ordinary ions. This double mobility might be interpreted as indicating a double charge, but this is unlikely in view of previous work. A more probable explanation would be that the charged Th A atom less readily attaches to itself water molecules than do charged molecules of Nitrogen and Oxygen (Nolan, *l.c.*).

The work has been carried out at the Solar Physics Observatory, and I have to thank the Director, Professor Newall, for providing me with all facilities for carrying on the research. I am indebted to Mr Manning for the enlargements of the photographs which are here reproduced.

The stellate appendages of telescopic and entoptic diffraction. By
Sir Joseph Larmor, Lucasian Professor.

The sections which follow set out the proof of statements made
in abstract in a paper communicated to the Mathematical Congress
at Strasbourg in September 1920. See *Report*, p. 589.

Let the coordinates of a point on the aperture-plane, which is
a wavefront of incident light, be (x, y), and those of a point in the
parallel image-plane be (a, b), the rays converging as in a telescope
to a point-image at the origin of coordinates. Let these planes be
at a large distance e apart. Then an element of polarised disturbance,
of area of front δS at (x, y), contributes to the light-vector at (a, b),
on the Fresnel theory sufficiently valid for small obliquities, an
amount varying as

$$\delta S \cos n (ct - R), \quad \text{where} \quad R^2 = e^2 + (x - a)^2 + (y - b)^2,$$

in which a, b are usually small compared with x, y, and all are
small compared with e.

Thus $R = e + \dfrac{a^2 + b^2}{2e} - \dfrac{ax + by}{e} + \dfrac{x^2 + y^2}{2e} - \ldots,$

so that the light-vector at (a, b) in the image-plane for the case of
a uniform incident beam is

$$\int dS \cos n \left(ct + A - \frac{ax + by}{e} \right),$$

in which A is a constant of phase, and n is $2\pi/\lambda$.

Well-known theorems of correspondence here arise. The dis-
tributions on the aperture-plane and image-plane correspond in all
cases in such manner as maintains $ax + by$ invariant. Thus if the
form of aperture is altered so that x, y become

$$x' = px, \quad y' = qy,$$

then the pattern in the image-plane is altered so that a, b become

$$a' = p^{-1}a, \quad b' = q^{-1}b.$$

Taking the axis of x through the point (a, b) on the focal plane,
thereby making b zero, the integral becomes

$$\int dx\, F(x) \cos n \left(ct + A - \frac{a}{e} x \right),$$

where $F(x)$ is the transverse breadth of the aperture at distance x

from the projection of the image-point on its plane. The diagram shows it as the length of an infinitesimal slice, for the case of a triangular aperture.

If the aperture is of any form bounded by straight lines, the graph of $F(x)$ the length of the differential element will be a succession of straight lines, with abrupt changes of inclination as it passes each corner of the boundary. This suggests the process of integration by parts, leading to

$$\left| -\frac{e}{na} F(x) \sin + \frac{e^2}{n^2 a^2} F'(x) \cos \right|$$
$$- \int dx \, \frac{e^2}{n^2 a^2} F''(x) \cos n \left(ct + A - \frac{a}{e} x \right).$$

This result may be interpreted in alternative ways. If the corners are regarded as sharp angles, the final integral vanishes over each straight segment: and of the terms at the limits the first vanishes except in a crucial special case, while the second provides a contribution from each corner of the aperture. If, however, the corners are regarded as rounded, on account of the continuity thereby introduced the terms at the limits provide nothing, while the integral term gives a contribution spread over each rounding off, on account of the great local magnitude of $F''(x)$. The two modes will agree only when the rounding off is completed in a small fraction of a wave-length. But in either case the diffraction pattern may be regarded as due to a distribution of coherent point-sources at the corners, though the relative magnitudes assumed for them will have to alter with the direction of the diffraction.

The exceptional case is when the direction of diffraction is exactly perpendicular to one edge of the aperture, say of length l; then the first term above at the limits gives, for focal length e, a contribution

$$\mp \frac{el}{na} \sin n \left(ct + A - \frac{a}{e} x \right)$$

with choice of sign according as it is a forward or rearward edge. This is to be compared with the contribution from a corner which is of order $\frac{e^2}{n^2 a^2}$. Its ratio to the latter is $\frac{2\pi a}{\lambda} \alpha$, where α is the angle subtended by the edge l of the aperture at the focus. Now the mean radius a' of the diffraction pattern around the focus is readily seen to be of order $\frac{\lambda}{\alpha'}$, where α' is the angle subtended at
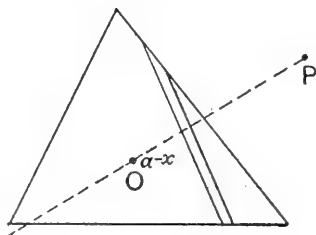
the focus by the mean diameter of the aperture: thus the ratio is $2\pi \dfrac{a}{a'} \dfrac{a}{a'}$, in which the first factor is the ratio of the distance of the point P on the ray to the radius of the central ring and the second is the ratio of the length of the straight edge to the radius of the aperture. The edge thus strongly preponderates over each corner, while the various corners counteract each other owing to differences of phase. It is only in a very definite direction that this preponderance of the edge-effect over the area-effect exists; at an angle $\dfrac{\lambda e}{la}$ or $\dfrac{a'}{l}$ with the direction normal to the edge it entirely disappears, while exactly along that direction the diffraction would be sensible to a much greater distance than near it. This would mean that in the diffraction pattern a prolonged sharp ray emerges from the focus perpendicular to each *straight* edge of the aperture. An edge not quite straight will show the effect if its curvature is of smaller order than $\dfrac{a'}{l^2}$ or $\dfrac{\theta}{l}$, where θ is the angle the central diffraction pattern subtends at the aperture.

These rays form a familiar feature in optical images: in Sir John Herschel's drawings (*Encyc. Metropolitana*) of telescopic diffraction with an equilateral triangular aperture, they constitute the hexagonal star diverging from the central rings. The present general mode of argument may be compared with the detailed special integration for this triangular case, as carried out by Airy in his *Tract* on the undulatory theory of light. The present reasoning would predict irregular stars for any triangular or polygonal aperture: in fact it would demand a ray at right angles to each straight part of an aperture elsewhere curved.

The diffraction patterns for two apertures whose forms are related after the manner of geometrical projection should correspond linearly according to the relation above stated: it is easily verified that if for one of the related apertures a ray is perpendicular to an edge it will be so for the other, as ought to be the case. For example, the diffraction pattern for a scalene triangular aperture would have the rays shooting out at right angles in both directions from the centres of its sides. The best focus would be their meeting point.

The very copious illustrations of telescopic diffraction given long ago by Schwerd in his treatise *Beugungserscheinungen* are, it may be observed, in many cases, including the scalene triangle, examples of this simple principle of the ray-effect of straight edges.

It has been stated (by de Haas and Lorentz) that Helmholtz ascribed the rays that appear, in normal focused vision of a distant small bright point, to the irregular boundary of the iris diaphragm

of the eye. As above, straight segments would be required to produce long sharp rays.

In short-sighted vision (with one eye) such that the focal plane is in front of the retina, extremely definite ray patterns, it may be radiating from one or two small concentric central rings, are found to develop with years, and could be due, in so far as they are not shadows of the internal structure of the lens, to patches of irregular refraction forming in the lens, a hexagonal star with ringed centre corresponding to a triangular patch, the same in fact as described by Herschel for his telescope. It would be of interest for the sake of comparison to examine the telescopic appearances out of focus.

The effect of locally increased refractive power over a limited region of the lens would be to alter the effective length of path of the ray say by l so that the vibration changes from

$$\cos n \, (ct - x) \text{ to } \cos n \, (ct - x - l)$$

the increase of the vibration-vector being thus their difference $2 \sin \frac{1}{2} nl \sin n \, (ct - x - \frac{1}{2}l)$. This added distribution of vibration is imposed on the regular refraction which would produce normal focusing. Thus if l were a constant say $k\lambda$ over an area of the lens, it would be the same as superposing a vibration of relative amplitude $2 \sin k\pi$ integrated over that area but with phase increased by $\frac{1}{2}k$ of a period. If k were $\frac{7}{6}$ it would nearly be equivalent to blocking out that area of the lens: and by Babinet's principle, the bands of diffraction are the same for a blocking screen as for an aperture.

If there are two parallel edges on the aperture there will be interferences along the ray of diffraction that shoots out normal to them, producing alternations of bright and dark places with homogeneous light merging into the colours of Newton's rings with ordinary light. This appearance is shown, for example, with a square aperture, along the rays of its four-rayed star: and the beaded pattern of the hexagonal rays figured and described by Herschel for a tesselated triangular aperture made up of alternate smaller equilateral triangles would be elucidated on similar lines.

[Compare the (merely descriptive) section on astigmatism in Helmholtz, 'Physiologische Optik': also the treatise of Donders 'On accommodation and refraction of the eye.']

Can gravitation really be absorbed into the frame of space and time?
By Sir JOSEPH LARMOR, Lucasian Professor.

[*Read* 22 January 1923.]

The more obvious local effects of gravitation can be masked by
motion of the system concerned, with appropriate changing velocity.
If the room containing the observer and his apparatus were travel-
ling in free space, without extraneous contacts or constraint, the
attraction towards the Earth would be annulled relative to the
observer and all the masses in it would lose completely their
weights. Cf. Newton's *Principia*, Leges Motus, Lex iii, cor. 6. As
then the effects of gravitation can be to this degree annulled by
mere motion of the observer and his system, may we adopt as a
provisional guide the postulate that it can thus be wholly obliter-
ated? This is the primary postulate of recent schemes of general
relativity. It would mean that gravitation should be wholly
absorbed into a scheme of moving space and time, which, as we
have seen, can always be done in each locality so far as regards
the main overt effect: and it would, of course, require that the
local partial schemes of space and time shall cohere into a con-
tinuous spatial whole. So far, this mode of representation seems
possible. But as regards effects not so overt: is it true that a field
of gravitation, say downward, also affects light from an outside
source, say from above, merely as if it were replaced by that
accelerated motion of the observer upward which would produce
for him the same sensations of weight and mechanical force? That
hypothesis is the 'principle of equivalence' of gravitation and mere
acceleration, made fundamental as an exploring clue by Einstein;
it implies that bodies and the rays by which alone we trace them
are essentially projectiles flying across space, and that there is
nothing else there. This is a hard saying in face of the developed
science of optics and electrics. It will merit attention as an alter-
native to that domain of knowledge, only in so far as it can itself
produce a coherent scheme fortified by verified predictions. Such
a theory has been advanced, and has become famous: but is it
coherent? It has been suggested in regard to it, now for a con-
siderable time, that the only adequate test of its coherence is that
it must come into line with the principle of minimal Action, which
is installed as the one relentless all-embracing criterion of coherence
between the different domains of physical science: their supplanter
can hardly demand a more lenient test. Answer has nevertheless
had to be made that in this ultimate domain the finality of any
such principle cannot be recognised. It remains open then to

construct a theory mathematically cognate on the basis of the principle of Action, and examine how it compares. Such a theory was put together last autumn, formally completed on October 17: the more essential parts have now been published in the *Philosophical Magazine* for January 1923. It appears as the main tangible result that a field of gravitation should not affect light in the way sketched above, as if space were empty, but to only half the extent that such a view would necessitate, that is to half the extent which the solar eclipse expeditions from Greenwich and Cambridge in 1919 are usually held to have confirmed. The development from minimal Action introduces however (as is readily verified) the same planetary perturbations, most prominently an advance of the perihelion of Mercury, as the other. Thus as regards physical astronomy no practical difference arises: the two spatial schemes are alike successful in accounting for the small planetary outstanding discrepancy. One of them, built up tentatively by tensorial adaptations, claims to have predicted the astronomical deflection of light as confirmed afterwards by observation. The other is not amenable to any adaptation, except the fundamental one to Kepler's Laws of Planetary Motion; but it gives only half this deflection for the rays of light*. It came at first as a surprise that they do not agree. Can there be any discrepancy in either—e.g. any contradiction lurking in the successive adaptations of a purely spatial formulation?

A recent critique by an eminent analyst, M. Jean Le Roux, Professor at Rennes, seems to shed decisive light on this question. He recalls that in the quasi-spatial theory the gravitational orbit of a body is postulated to be given by $\delta \int ds = 0$; and he remarks simply that this ds, as worked out in special examples, cannot be an element of path in any fourfold space; for in addition to dx, dy, dz, dt, the local differentials of the path, and its coordinates x, y, z, t, its expression involves a fourfold series of variables x_r, y_r, z_r, t_r, one set for each of the other influencing bodies in the field. If these were unvaried parameters, the path would be a geodesic curve in a fourfold expanse defined in terms of x, y, z, t: but they are the variables of other paths concurrent with x, y, z, t, and therefore ds has no claim to express a self-contained element of path or interval in any fourfold at all. The orbits are, in fact, inter-related.

'On a bien ainsi une forme quadratique de quatre différentielles, mais, en vertu de la variation solidaire de toutes les coordonnées, cette forme n'est plus un élément linéaire à quatre dimensions; les mouvements ne sont plus définis par les géodésiques d'univers, les considérations relatives à la courboure de l'univers n'ont plus de sens, et enfin, fait plus grave, l'explication quasi géométrique de

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* Absence of any deflection would perhaps exclude all tampering with the uniform spatial frame.

la gravitation disparaît' (*Comptes rendus*, Dec. 4, p. 1136). In fact, if the curves determined by $\delta \int ds_r = 0$ are not geodesics, they are (in default of being proved to be expressible in terms of geodesics) changed essentially by mere change of the frame of reference for the same spatial domain; therefore they cannot be orbits of bodies in the physical world that underlies all such accidental frames.

It is emphasised by M. Le Roux that in working out the scheme the coordinate with the character of time, t_r, is to be taken the same, in fact the necessary universal working time of the analysis, for all the bodies: then the other coordinates x_r, y_r, z_r represent simultaneous positions. This means only that approximations are best carried out by aid of sections of the fourfold determined by t constant: for then ordinary notions of distance and time can assist, and potentials approximately Newtonian. For development is feasible only for an auxiliary fourfold nearly flat, a condition to which actual astronomy conforms; and these coordinates enter into the approximate adaptation of Newtonian theory through the simultaneous mutual distances r_{pq} measured as if it were exactly flat. If the velocities of the bodies were considerable in relation to c, it might rather be thought to be mutual distances at an earlier time $t - r/c$*. But if the fourfold were not nearly flat, it is almost no use contemplating any such formula for the Action of the masses at all, so complex would the entanglement between the orbits be when among other complications r_{pq} has to be measured on a curved geodesic in an undetermined expanse, instead of with sufficient approximation on one virtually straight. Actual orbits, however curved they be in ordinary space, are nearly straight lines in the fourfold space-time expanse.

Reversal of the procedure perhaps places in clearer light its scope and limitations. The single equation of Action which consolidates into one formula the Newtonian mutual dynamical influences of the n bodies can be written down. It is necessarily invariant for the Newtonian group of frames of reference defined by $dx^2 + dy^2 + dz^2$ invariant, t universal. Therefore it cannot be invariant for the group conserving invariance of optical radiation which is defined by $dx^2 + dy^2 + dz^2 - c^2 dt^2$ invariant. It can, however, provided certain very small terms are added to this Newtonian Action, be changed to the form

$$A = - \Sigma \int m_q c\, ds_q, \qquad ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

which now is invariant in the latter group of frames, but not in the former. Gravitation is not yet in it, so orbits are all straight

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* The striking general result that electric, and presumably also gravitational, potentials between bodies prove to be practically instantaneous and not retarded, and that this is the reason why there is no Laplacian aberration concerned in them, will be discussed elsewhere.

unless the masses are electric ions: after Einstein gravitation is to be introduced by making the fourfold expanse variable instead of uniform. If this latter is assumed to be the true and exact form of nature, being in accord with the fundamental electrodynamic group, and it is interpreted back with its new terms that contravene Newtonian invariance, into the special Newtonian uniform frame that is used in practical physical astronomy, the slight new terms involve the Einstein modifications in the planetary motions.

Having thus submitted to the restrictions confining the problem to the feasible approximate form effective for actual physical astronomy, the expression $-\int m_1 c\, ds_1$ involves local position through $x_1,\ y_1,\ z_1$ and their differentials in addition to dt, but involves also variables such as $x_q,\ y_q,\ z_q$, entering through r_{1q}: this makes up for an orbital problem of n bodies, $3n$ concurrent variables in all, in addition to the single independent variable t. To minimise one such integral involves $3n$ differential equations of condition. To minimise all the integrals of type $-\int m_q c\, ds_q$ separately thus would involve $3n^2$ equations between $3n$ independent variables. It cannot be effected at all within a fourfold extension, just as M. Le Roux insists.

But cosmic extension in fourfold, and no higher, type is a postulate. In it only one Action function can be minimised, and that is the conjoint Action of all the interacting bodies. In the previous paper the invariance of an additive linear form of Action was guaranteed by its being derived from a fourfold integral of Action inherent in the medium and so made everywhere locally invariant over the whole expanse: the crucial importance of this guarantee was noted, and also the purely formal nature of its demonstration. It would be of no avail to propose to restrict the minimising of each component $-\int m_q c\, ds_q$ to variation with respect to its own coordinates $x_q,\ y_q,\ z_q$: such a partial operation would carry no significance, for the essential criterion, invariance of the orbits as regards mere change of frame, would be entirely absent.

It is true that the fourfold cosmos exists merely statically, as a spread-out history of the universe: but an orbit is to be determined by some process of variation, and one orbit cannot be varied in the relatively adjusted expanse without affecting the whole cosmos and thus changing the other orbits which in turn react upon it. Nothing less than complete concomitant variation of all the orbits in their adjusted physical expanse can be self-consistent.

It is the presence of a pervading medium alone, whether as the arena of dynamical Action or the spatial seat of formal tensors, a medium for which the particles are essential singularities in its structure, that retains the problem always as one of a 4-fold extension, instead of for n bodies one in a $3n$-fold variational extension as counting of the orbital variables would indicate. And it is

the slowness of the motions of the celestial bodies compared with the velocity of adjustment by waves across the medium, that permits an analysis attributing to each body a statical field of force and so renders practical discussion of inter-related orbits feasible.

This destructive criticism of a purely spatial scheme seems to be fundamental and complete. Can then an amended scheme be formed? The alternative determination of orbits by minimal conjoint Action has already been set out formally as regards its general plan in the previous paper: it seemed to satisfy automatically the criterion of invariance, provided however we are assured that the form of the material part of the Action

$$A = -\int m_1 c\, ds_1 - \int m_2 c\, ds_2 - \ldots$$

is immune from the present dilemma. For if ds_r is not, as urged above, an element of a path in any fourfold at all, what right can we have to fix on this as the one suitable invariant form for the Action? The paradox at first sight still remains. The element ds for each body cannot help being an element in the fourfold map of the history of the world, which stands as the postulated final and absolute representation that has shed off all relative features. It ought thus to be determined locally, implying an expression in terms of its own varying coordinates alone: yet the expression obtained for it. which is to be subjected to the process of variation, involves the varying coordinates of the orbits of all the other masses in the field. If we take the fourth coordinate to be common to all the masses, to be the universal time-coordinate necessary to any mathematical analysis, one would say that ds depends on the simultaneous positions of all the other masses. Can the conclusion be evaded that these two conditions for specifying ds here also involve a contradiction? So far as this inference may be inevitable, all such formulations in terms of a changing space would lapse.

May the paradox be connected with the very inconvenient property of the pseudo-space, that length and time range without limit within the same compacted differential element of fourfold extension determined by finite range of ds? The property of locality, inherent in ordinary space, is thus absent. If so, the pseudo-spatial analogy may be a bridge leading not merely to mystical but to false notions, unless it is restricted to purely algebraic uses*. The aim would then be transferred to saving as much of the algebraic analysis as may be feasible. The analogy of the analytic process, with an extended fourfold quasi-Euclidean in its differential elements, could remain: but the purely geometrical idea of a geodesic or locally shortest path would go. Some universal scheme of

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* This is in agreement with a recent critique by A. A. Robb: as regards the rest of this paragraph compare A. N. Whitehead's recent book on *Relativity*.

coordinates is essential: one of the four would be sharply marked
off as a universal time: the Action as distributed in three-fold
heterogeneous space, integrated in one-fold heterogeneous time,
would be expressed by a fourfold integral: there would be local
invariance as regards radiation which demands (*Phil. Mag.* Jan.
1923, p. 249) the group of frames of the Lorentz transformation.
It is legitimate to separate off the finite part of the Action that is
close around, and associated with, each particle, by integration
into a material term $\int k\,dt$, which transferred to a local Lorentz
frame in which the particle is moving changes into $\int kc^{-1}ds$ which
the criterion of locally minimal Action identifies with $-\int mc\,ds$ as

previously. The material part of the Action is now $-\Sigma\int mc\,\dfrac{ds}{dt}\,dt$,

integration being along each orbital path in threefold space. But
now ds/dt is an analytical expression and ds has nothing to do with
a possibly misleading fourfold pseudo-space for which the idea of
distance breaks down: it is now an expression arising analytically
out of the transformation of a fourfold integral in space and time
into a single integral in time. The analysis might thus become one
of Action, purely dynamical, utilising a fourfold expanse only to
visualise domains of integration with regard to space and time.

It is not necessary, however, to pursue the possibilities of such
a mode of reconstruction. The dynamical procedure by Action seems
to provide a direct escape from the paradox which besets the quasi-
geometrical theory. To recognise this, it is necessary to scrutinise
more closely what the nature of the analytical process of mini-
mising would be, in cases where it might be feasible to carry it
through. In the problem of n bodies one would begin by assuming
a Newtonian approximation to the orbits and adapting it into the
fourfold frame. This assumed specification of the orbits determines
ds throughout the fourfold and also the Action-density in that
expanse, e.g. by utilising the methods of approximate calculation
developed by Einstein (and Hilbert) on the basis of a foundation
expanse nearly flat. Integration over the whole expanse gives the
total Action. If the assumed configuration of the orbits is slightly
changed, a different value of the total Action is obtained. The
problem is to adapt this configuration so that slight variation of it
does not sensibly alter the Action. For each configuration of the
orbits the Action, adjusted throughout the field, consolidates into
an expression involving a single integration along the orbits, the
integrands ds_r involving the local coordinates of position in the
space which the orbits determine: for the integral is restricted as
usual by invariance in the fourfold to the form $-\Sigma\int m_r c\,ds_r$. It is
true ds_r involves the concurrent coordinates of all the orbits, as
well as its own coordinates as a point of the expanse. But there is
here no question of going outside a fourfold expanse so long as the

configurations of the orbits are not changed; and when the configurations are varied the expanse which they determine is in consequence altered, and the new orbits lie in the new expanse.

To resume: Each configuration of the orbits determines a fourfold expanse in which they exist: and the total Action is determined as a line integration in this expanse along them: then the configuration is to be varied so as to minimize this Action: but it is not implied at all that the ds^2 for the varied configuration continues to belong to the same spatial expanse.

If things stand as here claimed it is not involved that the beautiful mathematical theory of general relativity so-called becomes derelict. But an inference does lie that the metaphysic which would absorb gravitation, and even electric fields, into a physical space and time belonging to the masses, and which postulates that space is otherwise mere emptiness and nullity, has been founded on a faulty analysis inconsistent with itself. It is when the theory tries to come into contact with the ordinary material world, with fields of activity produced by atoms, that misfit seems to arise as regards its metaphysical postulations. But the elegant mathematical analysis of fields would still provide an effective method of exploring the inner connections of a field of gravitation with the other primary physical agencies: and if, as is here claimed, closer scrutiny compels a reversion to the view that space is a *plenum*, is an aether in the historical and ultimate British sense held ever since Maxwell and even long ago by MacCullagh, not a crude material substance but a coherent scheme of local physical relations, then further mathematical and observational development may have gained rather than lost in interest.

Flying-fishes and soaring flight. By E. H. HANKIN, Sc.D.

[*Received* 20 November 1922.]

If the span of the more efficient soaring animals is plotted against the loading, a remarkably regular curve is obtained with one striking irregularity. As shown in Fig. 1 there is an increase of loading with increase of span in the case of inland birds. The loading of flying-fishes, on the other hand, is very much greater than would be expected in birds of similar size[1].
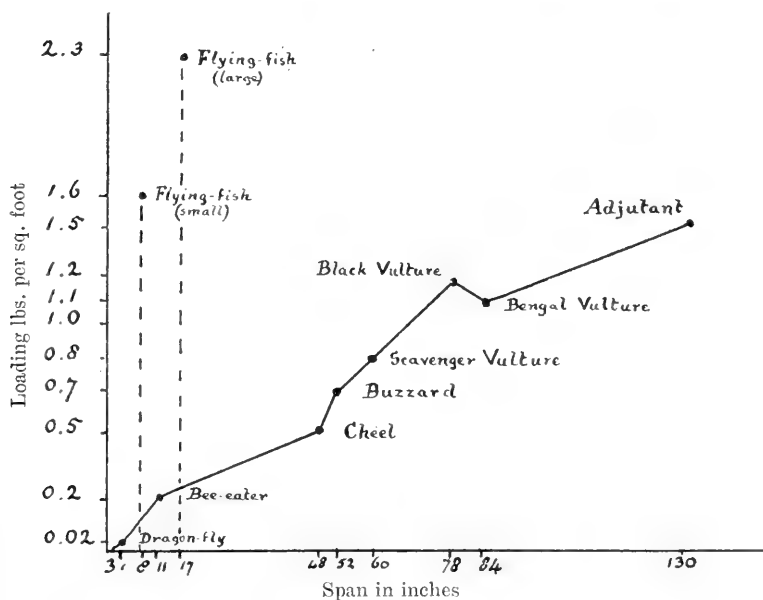
Fig. 1. Relation of span and loading of soaring animals.

Generally their flights are less than 50 metres in length. Occasionally they make longer flights extending from 200 to 400 metres. Rarely they have been seen, both by me and by other observers, to glide at a uniform height above the water till they were out of sight.

[1] The loading of the small flying-fish shown in the figure was obtained from two freshly-caught specimens of *Exocœtus oxycephalus* measured by Major R. B. Seymour Sewell of the Zoological Survey of India, to whom I am much obliged. Their weights were 44·84 and 57·60 grammes respectively. Their span was about 21 to 22 centimetres. Their loading worked out at 1·73 and 1·98 lb. per square foot if the hind wings (pelvic fins) were not taken into account. Including these hind wings the loadings were 1·57 and 1·67.

Though flapping may occur at starting, ample proof exists that the continued flight of these fishes is not due to movements of their singularly feeble wing muscles[1]. Consequently the flying-fish is by far the most efficient of existing soaring animals[2] in respect of power of carrying weight in a horizontal direction.

Though longer flights and flights at high speed only occur in the presence of wind, flights at low speed may occur in the apparent absence of wind but then only in the presence of sunshine. During a recent voyage (February 1922) flights estimated at between 50 and 100 metres in length were seen by me when the sea had a glassy surface. At the time, at a distance of a few miles, isolated cat's paws were visible. Once when the surface was smooth and glassy, it was noticed that the funnel smoke was not being left exactly astern. Hence, on this occasion, a wind (estimated from other data at 2 metres per second) was present at the level of the top of the funnel. In calm, in the absence of sunshine, flying-fishes attempting a flight have been seen by me to fall back into the water immediately the propelling action of the tail came to an end. In calm, if the sun is obscured by cirrus cloud, so that sunshine is lessened but not abolished, gliding flying-fishes, at intervals of about two seconds, lower the tail momentarily into the water and by wagging it to and fro gain an increase of speed that is easily observed. On one occasion it was noticed that flying-fishes, gliding in the apparent absence of wind, began to show lateral instability in the afternoon. In this respect their flight resembles their flight in the presence of wind and also the soaring flight of inland birds and dragon-flies in all of which lateral instability occurs more often in the afternoon than at other times of the day.

It seems probable that where the sea is smooth with a glassy surface, air movements immediately over it are gentler and less complicated than they are under all other conditions in which soaring flight has been observed. Hence the study of movements in the air over a glassy sea surface in the presence and in the absence of sunshine may perhaps reveal the nature of the condition on which soaring flight depends.

In view of the efficiency of the flight of the flying-fish, the

[1] See correspondence in *Nature*, April to August 1921, pp. 233, 267, 376 and 714, and also my paper, "Observations on the flight of flying-fishes" (*Proc. Zool. Soc. London*, December 1920).

[2] It has been asserted that when flying up and down as it glides at a uniform height above the water, the flying-fish is merely lifted and dropped by the movement of the air in which it is supported. No doubt a piece of thistledown floating near the surface of a rough sea would behave in this way. But it is equally certain that a rifle bullet would not show any appreciable deviation from its course from this cause. From its speed and heavy loading, it may be concluded with certainty that the flying-fish resembles the bullet in this respect far more than it does the thistledown. Its following the water surface, therefore, can only be due to some active adjustment for steering up and down. It is probable that this adjustment consists in changes of position of the pelvic fins.

structure of its wings is of interest. They consist of a thin membrane supported on fin-rays. Sections taken near the base of the wing are shown in Fig. 2. In the case of a soaring bird the bones of the wing stand out on the under surface forming a ridge transverse to the line of flight. Such transverse ridges are present on the wings of all the more efficient soaring animals. It is of interest to notice that they are more numerous in the case of the flying-fish than with other soaring animals.

Occasionally a flying-fish makes a flight, not in a straight line, but in a curve of several hundred metres radius. Such flight on a
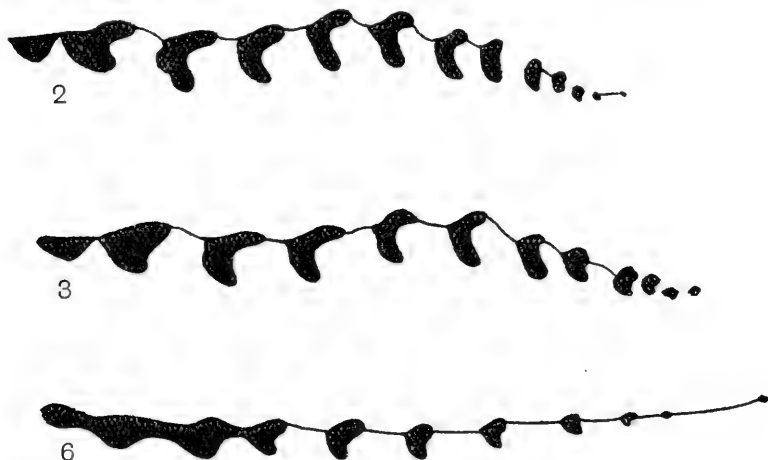
Fig. 2. Sections at right angles to long axis of wing of flying-fish showing fin-rays projecting as ridges on underside of wing. The numbers indicate distance in centimetres from base of wing which was 18 cms. long.

curved course probably is due to an accidental lack of identity in the dispositions of the two wings. The regularity of the curve furnishes a presumption that during its course balancing or steering movements did not occur. An inherently stable model aeroplane described by Bairstowe[1] has an arrangement of supporting and rudder surfaces not very unlike those of the flying-fish. For these two reasons it is probable that the flying-fish has a higher degree of inherent stability than have other soaring animals.

The facts mentioned in this paper indicate that the flying-fish is likely to be a useful guide in attempts to achieve artificial soaring flight.

[1] "The stability of Aeroplanes" (*Aeronautical Journal*, vol. XVIII, p. 68, April 1914).

On the air brake used by vultures in high speed flight. By E. H. HANKIN, Sc.D.

If a vulture is gliding at slow speed and wishes to come to earth, it gradually rotates its wings so that, just before perching, their surface lies almost at right angles to the direction of flight. This method would be quite unsuitable for use in high speed flight. Such flight may result in an air speed of 40 or more metres per second. At such high velocity any attempt to rotate the wings would, with little doubt, lead to disaster. Another method is used when flying at high velocity which checks speed gradually and without exposing the wing to undue strain. It was observed by me many years ago and described in detail in my book *Animal Flight* (chapter VII, p. 121).

The ligaments attached to the secondary quill feathers are so arranged that camber is at a maximum when the wing is fully extended. If the wing is flexed slightly at the carpal joint, camber decreases or vanishes and such flexure with absence of camber is used in high speed horizontal flight. If the flexing occurs not at the carpal joint, but at the metacarpal joint, then camber persists. This metacarpal flexing occurs when the bird is checking speed in fast flight. It is accompanied by a reduction of the angle of incidence apparently to zero. As much as a minute or more may be needed before this disposition of the wings brings the speed down to such a point that descent is possible by one or other of the methods that have been described by me in my book. Why or how this wing disposition should check speed was not known to me at the time of my observations. An investigation carried out by Gustav Lilienthal appears to throw a light on the question.

He made models of the wings of soaring birds and also planes having sections similar to those of birds' wings. These were exposed to currents of air either at the end of a whirling arm or in a wind. Numbers of small flags were attached to the models. These were made of thin cardboard mounted on needles. The attachment was somewhat stiff so that the flags would only turn when the air currents reached a certain strength and the flags would retain the position thus impressed on them when the wing was brought to rest. By numerous experiments carried out in this way, Lilienthal discovered that an eddy current is formed under the wing. Part of this eddy is shown in Fig. 1 which is copied from Lilienthal. The whole eddy had a spiral or ram's horn shape and the air streaming from the eddy is shot out at the wing-tip in a direction parallel to the long axis of the wing. At the other end of the spiral eddy, near the body of the bird, a backward current is formed.

That such an eddy may exist in flight is suggested by an observation of my own. A feather that was floating in the air was seen to pass just under the wing of a cheel. Instantly it was shot out sideways with almost explosive suddenness to a distance of three or four metres from the bird.

Lilienthal concludes that this eddy, which is caused by the bird's passage through the air, is the source of the energy of soaring flight. For reasons that it is unnecessary to specify his opinion is not likely to gain many converts.

Another and an entirely different conclusion may be drawn from these experiments. On looking at Lilienthal's figures, it is clear that his wing sections had a degree of camber that may be found in the dead bird but that never occurs in flight except when the bird is checking speed in high speed flight as above described. Lilienthal's figures show that his heavily cambered wing sections

Fig. 1. Diagram of Lilienthal's eddy.

were given a zero angle of incidence. This also is the case when the vulture is checking speed. When in flight with wings cambered, but probably to a lesser degree than in Lilienthal's models, as happens in circling flight with gain of height, there is a large positive angle of incidence.

It is obvious that much energy must be absorbed in forming this complicated eddy. It is only likely to be formed under the conditions described above. In the absence of camber it would be less or non-existent. Hence it may reasonably be suggested that its formation is the reason why the above described wing disposition is used as an air brake in high speed flight.

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Soaring flight of gulls following a steamer. By E. H. HANKIN, Sc.D.

[*Received* 3 December 1922.]

If a solid object projects from a flat surface over which a wind is blowing, then the air current rises on the upwind side and descends on the downwind or leeward side of the object. There is a descending current of this nature near the stern of a steamer. If a wind is present of such strength and direction that the smoke from the funnel is being left behind over the stern or over the quarter, then, generally but not always, gulls are able to soar in a limited space near the stern—the so-called "soarable area[1]." Outside this space gulls near sea-level can, as a rule, only remain in flight with the aid of flapping. Sometimes, especially if the wind is directly ahead, a small ascending current exists immediately astern and the descending current is situated from 10 to 50 yards farther to leeward. Contrary to what one might expect the soarable area is not situated in this ascending current.

The frequenting of the soarable area by gulls is not due to their finding it a convenient position for waiting for food to be thrown out astern, for, on one occasion, in the Great Bitter Lake on the Suez Canal, when the steamer went astern at slow speed, it was noticed that the soarable area at once shifted to a position on the leeward side of the ship between the bow and the bridge, though the majority of the birds were using the windward ascending current.

Generally when the steamer is at rest, gulls may be seen gliding in the ascending current on the windward side of the ship. Rarely when the steamer is under way, some gulls also glide in the ascending current to windward while others are soaring in the soarable area to leeward of the stern. There are important differences in the mode of flight under these two conditions.

In the windward ascending current gain of height is gradual and with the long axis of the body horizontal so far as can be seen. In the soarable area gain of height is rapid. Steep upward glides occur in which the long axis of the body is not horizontal but inclined upwards in the direction in which the bird is going. The direction of the glide may make an angle of 50 or more degrees with the horizon. If a dozen gulls remain behind to settle on refuse thrown overboard, then, as they leave it, they come after the ship always in flapping flight at a height of 3 or 4 feet above the water.

[1] As described in my paper "On the flight of sea gulls," *Aeronautical Journal,* vol. XIX, July–September 1915, p. 84, and *Animal Flight*, chapter XIV.

They reach the soarable area at this height, usually one by one; then sharply turning they glide up, commonly to the level of the top of the stern flagstaff. Such observations prove that the conditions that permit steep upward gliding are constantly present: they are not due to an occasional gust or eddy. They also prove that the upward glide is not due to momentum gained by a downward swoop. When crossing from Dover to Calais on one occasion, when the steamer had a high speed and the wind was light, the soarable area was restricted to a space between the level of the deck and the level of the top of the stern flagstaff.

Secondly, gliding flight in an ascending current, in my experience, is very stable and in this respect differs greatly from what generally happens in the soarable area, where the gulls show various forms of instability described by me elsewhere[1]. Instability is not a constant characteristic of flight in the soarable area. Gustav Lilienthal has informed me that on one occasion in the North Sea he observed a gull gliding astern which, during a period of 45 minutes, did not change its position relatively to the steamer by more than a metre or two.

Another difference between flight in the soarable area and in the windward ascending current lies in the appearance of a faint colour on the wings of gulls in the former position. It has been observed by me that the underside of the wing of a soaring bird often shows a power of reflecting colour which power is usually absent when the bird is gliding with loss of height or in an ascending current of air. Both on my last voyage and on a previous voyage, it was noticed that gulls in the ascending current to windward showed little or no colour of this nature as a rule, while gulls gliding in the soarable area showed a faint tint of green or blue or yellow according to whether they were gliding over green or blue water or over the deck of the ship. On one occasion gulls in flapping flight showed no colour while gulls in the ascending current to windward showed yellow on the nearer wing (due to reflection from the deck) and blue on the further wing (due to reflection from the sea). The colours were faint but readily recognised by an artist friend when they were pointed out to him.

In my paper on the flight of sea gulls[2], reasons were given for suspecting that the soarable area was situated in a descending current. Definite evidence that this is the case is contained in the following letter from Mr F. Clark:

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[1] *Loc. cit.* A very interesting form of instability was noticed on the above quoted Channel steamer. The wings and body of the bird were, during most of the voyage, trembling so rapidly that the whole outline of the gull appeared blurred. Other gulls a few feet away outside the soarable area appeared sharp-cut. On another occasion momentary trembling of the wings occurred and it was noticed that it often preceded a steep upward glide.

[2] *Loc. cit.*

"While bound from Bombay to London on the P. & O. s.s. *Delta*, I performed the following experiment to test the assertion that their soaring is due to their taking advantage of an ascending current at the stern of the steamer. At about 5.0 p.m. on February 19th, 1922, the ship was about 70 miles north of the Daedalus reef in the Red Sea and proceeding towards Suez at a speed of about 14 knots. There was a moderate breeze (force 3) blowing from right ahead and a moderate short sea from the same direction. A large number of gulls were present. A reel of cotton was unwound and the thread allowed to stream astern. Its end was carried upwards at an angle of about 60 degrees with the horizon to a height of about 15 feet above the rail of the ship. On unreeling more cotton, the end gradually assumed a more horizontal position until it reached a point about 30 feet from the stern, when it began to be carried downwards. The thread now took the form of a parabola rising upwards from the stern and about 30 feet away descending at about the same angle. Thus it was proved that an ascending current existed just aft of the stern and that farther to leeward there was a descending current. Contrary to what one might expect, it was only in this descending current that the gulls indulged in soaring flight. On one occasion the thread at the point where it was in the descending current became entangled with a gull and about 10 feet of it was carried away by the bird. The gulls seemed carefully to avoid the ascending current. Sometimes a gull did get within its range. When this happened there was much flapping until the bird regained the descending current farther to leeward. The zone of the descending current did not seem to extend very far. It was noticed that when a gull went down to water level after food, on again starting, it had to flap till it reached the region of the descending current, when it at once resumed its effortless soaring flight. My observations were continued for about half an hour, during the whole of which time the behaviour of the gulls was as above described[1]."

This experiment was performed in my presence and it was noticed by me that the free end of the thread while it was passing through the ascending current showed movements that proved the presence of a high degree of turbulence. It would have been impossible for the birds to change the disposition of their wings sufficiently rapidly for them to be able to adapt them to the changing air currents. This is the probable reason for their avoidance of this area.

A similar instance was observed by me on another occasion when some grass thrown overboard at the stern proved the presence of an ascending current there which was avoided by the birds. Farther to leeward, about 50 yards away from the ship, there was

[1] *Flight*, March 9th, 1922.

a descending current, as shown by movements of pieces of grass and here, and here only, the gulls were in continued gliding flight.

More often the presence of the descending current in the soarable area is shown by movements of smoke. Masses of smoke in a highly-diluted form separate from the main smoke trail and descend to sea-level some distance away from the ship. Gulls may frequently be seen gliding steeply upwards without loss of speed while enveloped in this diluted smoke. Should a mass of thicker smoke come into the soarable area, the gulls may glide to one side to avoid it. If, when doing this, they glide out of the probable region of the descending current, they are at once reduced to flapping flight.

A further proof that the soarable area is a region of descending air is furnished by the fact that on entering it, and on no other occasion in my experience, gulls may show instability round the transverse axis. Inland birds of different species have been observed by me to show this form of instability under two conditions only: first, on leaving an ascending current for comparatively still air, and secondly, on leaving still air for a descending current. They do not show it on entering an ascending current, as I have definitely observed. No evidence is known to me that they show it on leaving a descending current.

Soaring flight of gulls has also been observed by me in the probable position of descending gusts of wind to leeward of Gibraltar[1] and, on my last voyage, to leeward of Aden. On the latter occasion it was noticed that the wings of these gulls showed a bluish tint. Three minutes later gulls were observed gliding in an ascending current. The undersides of their wings appeared white.

My observations make it probable that, in some cases at least, the soarable area is confined to the central part of the stern descending current.

Thus the evidence goes to show that near sea-level, as a rule, gulls can only soar in a descending current of air. That a gull should soar in an apparently descending current is, perhaps, not more surprising than that a vulture should glide with gain of height in an apparently horizontal wind. What is perhaps more surprising is that gulls generally[2] are unable to soar near sea-level while they are frequently able to do so at a height of two or three hundred metres above sea-level as I have observed not only abroad but also off the south coast of England and occasionally in London near Westminster Bridge.

[1] "On the flight of sea gulls," *loc. cit.* p. 98.
[2] For an exception to the rule see *Animal Flight*, p. 260.

L Series of Tungsten and Platinum. By J. S. Rogers. (Communicated by Sir E. Rutherford.)

[*Read* 5 February 1923.]

Introduction. Recently many new lines have been discovered by Dauvillier* in the L spectra of the elements of atomic numbers lying between those of tungsten and uranium. Experiments have been carried out to verify these and, if possible, to discover new lines for the elements tungsten and platinum. The results have to a very large extent substantiated the work of Dauvillier and there is evidence that several new faint lines of these elements have been measured.

Apparatus. The spectrometer was the same as that previously described by the writer†. The protection against scattered radiation was, however, modified, as it was found that the protecting channel of lead was sufficient. Again, as the X-rays under investigation were much softer, it was found necessary to remove the intensifying screen on the crystal side of the photographic film. A Coolidge "medium focus" tube was used for the tungsten rays, and the Gundelach tube, described in the above-cited paper, for the platinum rays. The width of the tube slit was usually 0·1 mm.

Experiment. The Coolidge tube was worked at a potential of 40,000 volts with the current through the tube about 6 milliamperes. The Gundelach tube could not be kept permanently at any one potential, but exposures were taken with potentials varying between 30,000 and 60,000 volts, the tube currents being respectively 10 and 6 milliamperes. It was possible to keep the tube running continuously between these limits for about 3 hours before a softening of the tube was necessary. The exposures, for each tube, were as long as 18 hours when the whole of the spectrum from the α_2 line to the short wave length limit was covered.

Measurement of lines. As the spectrometer was not designed as an absolute one, the wave lengths of the lines were evaluated by reference to standard lines. These standard lines were the four intense lines (α_1, β_1, β_2, γ_1) of the L spectrum of tungsten and the values assumed were the mean of those obtained by Duane and Patterson ‡ and by Siegbahn§. The corresponding lines of platinum were obtained by photographing on the same film the lines of the tungsten and platinum spectra. The remaining lines of each spectrum were then evaluated by reference to their own intense lines.

        * *Comp. rend.* 173 (1921), p. 647.
        † *Proc. Roy. Soc. of Victoria,* 34 [N.S.] (1922), p. 196.
        ‡ *Phy. Rev.* 16 (1920), p. 526.   § *Phil. Mag.* 38, Nov. (1919).

The lines were projected by means of a lantern on to a vertical board carried by the moving platform of a dividing engine. On this board were two vertical lines 1·5 mm. apart, and each spectral line was brought in turn between these lines and the reading of the engine noted. The angles at which the reference lines were "reflected" at the crystal were calculated from their wave lengths, taking "d" for calcite as 3·02904 A.U. The value in minutes per mm. of the projection was then calculated by dividing the angular differences $(\alpha_1 - \beta_2)$, $(\beta_1 - \gamma_1)$ by their respective distances apart as measured by the dividing engine, the mean of the two values being taken. From this, the value of the angle of reflection of any line at the crystal could be obtained by noting its distance from any reference line. The value of the wave length was then calculated by Bragg's formula $(n\lambda = 2d \sin \theta)$.

Results. The values of the wave lengths, in Angström units, are given in the Table. The agreement between these values and those given by Dauvillier* and Coster† is generally good, although there are discrepancies for some of the fainter lines. The values are the average ones obtained, usually, from at least three films. The spectral regions on the long wave length side of the α_2 lines have not been examined. The K absorption edge of bromine, due to the silver bromide in the photographic emulsion, was obtained and the wave length is ·9178 A.U. (cf. Duane and Blake's‡ value ·9179).

Discussion. Before the lines, given in the Table as new lines, can be definitely allotted to their respective spectra, it is necessary to consider what other agencies would possibly cause their presence. In the first place any impurities in the anticathode materials would emit their own characteristic lines and secondly some of the lines may be due to the secondary radiations excited in the slits and other parts of the apparatus exposed to the primary beam. Several lines which occurred on the films have not been included in the Table because their wave lengths could be accounted for by this means.

Although there appeared to be no trace of impurities in the anticathodes, the K series of antimony used in the tube slit was found in the second and third orders. The $L\alpha_1$ line of lead was also recognized in both spectra, but as this was excited by the general radiation issuing from the X-ray tubes, it was a very faint line.

If, on the other hand, the new lines actually belong to the spectra as given in the Table, it should be possible to show the electron passages within the atom from which they originate. Such a classification is here limited as two substances only have been investigated, but comprehensive schemes of the energy levels

* *Loc. cit.*
† *Zeit. f. Phy.* 4 (1921), p. 178.
‡ *Phy. Rev.* 10 (1917), p. 697.

within the atom have been developed by Coster* and Dauvillier†. In order to obtain a measure of the various levels of energy within the atom the wave number, the reciprocal of the wave length of the line expressed in A.U., will be taken as a measure of its energy.

TABLE.

Wave Lengths in Angström Units of L Lines of Tungsten and Platinum.

Tungsten				Platinum		
Line	Intensity	Wave Length		Line	Intensity	Wave Length
a_2	m.s.	1·4843		a_2	m.s.	1·3213
a_1	v.s.	1·47327	Value assumed	a_1	v.s.	1·3100
a_3	v.f.	1·4503		a_3	f.	1·3038
η	f.	1·4173		η	f.	1·2403
	v.f.	1·3735		β_{11}	v.f.	1·1658
β_{11}	v.f.	1·3212		β_4 } β_6 }	m.s.	1·1399
β_4	m.s.	1·2987				
β_6	m.f.	1·2876		β_1	v.s.	1·1172
β_1	v.s.	1·27905	Value assumed	β_2'	v.f.	1·1060
β_3	m.s.	1·2601		β_3 } β_2 }	v.s.	1·0998
β_2'	v.f.	1·2487				
β_2	v.s.	1·24192	Value assumed	β_2''	v.f.	1·0936
β_2''	v.f.	1·2355			v.f.	1·0803
β_7'	f.	1·2300		β_7'	f.	1·0772
β_7	f.	1·2206		β_7	f.	1·0752
β_5	v.f.	1·2126		β_5	f.	1·0697
β_8	f.	1·2021		β_{10}	v.f.	1·0660
γ_5	f.	1·1292		β_9	v.f.	1·0599
	v.f.	1·1138		β_8	f.	1·0520
γ_1	v.s.	1·09553	Value assumed		v.f.	1·0375
γ_{10}	v.f.	1·0862		γ_5	m.f.	·9855
γ_6	f.	1·0780		γ_1	v.s.	·9552
γ_7	v.f.	1·0715		γ_{10}	v.f.	·9446
γ_2	s.	1·0650		γ_6	f.	·9383
γ_3	s.	1·0590		γ_7 } γ_2 }	s.	·9318
γ_9	f.	1·0433				
γ_4	m.s.	1·0256		γ_3	s.	·9252
				γ_9	f.	·9156
				γ_4	m.s.	·8942

v.s., very strong; s., strong; m.s., moderately strong; m.f., moderately faint; f., faint; v.f., very faint.

In the tungsten spectrum the line 1·4503 has been called a_3 and the line 1·2487, β_2'. According to Dauvillier the former line is due to the passage M_1L_1. The energy of the line is ·6895, while the differences of energy between L_1 and M_1 is ·6788. The wave number

* *Phil. Mag.* 43 (1922), p. 1070; 44 (1922), p. 546.
† *Journ. de Phy.* 3 (1922), p. 221.

of β_2' is ·8008, while the energy difference between N_3 and L_1 is ·8052. The line 1·3735 (wave number ·7281) appears to be due to the passage N_3L_1 (energy difference ·7258) and the line 1·1138 (·8978) to the passage M_5L_3 (·9001).

In the platinum spectrum the line 1·0660 has been called β_{10}. The line 1·0803 appears to be due to a passage from an O orbit to the L_1 orbit, and finally the line 1·0375 (·9639) may be due to the passage M_1L_3 (·9506). The agreement is, on the whole satisfactory, since the values of the M, N, O energy levels have been obtained by subtracting the energy value of an L line from that of the particular L level to which the electron, giving rise to the line, passes.

Further evidence as to the correct naming of some of the new lines is obtained by drawing the Moseley graphs. The line 1·4503 of tungsten falls close to the graph of the α_3 lines obtained by Dauvillier, and the same occurs for the line 1·2487 of tungsten with regard to the graph of the β_2' lines. The line 1·0660 of platinum also falls very close to the graph of the β_{10} lines.

PROCEEDINGS

On the Intersection of Constructs in Space of Three or Four Dimensions, with special reference to the Matrix Representation of Curves and Surfaces. By C. G. F. JAMES, Trinity College, Cambridge.

[*Read* 5 February 1923.]

The questions, with which this paper is concerned, arose in connection with the representation of constructs, other than those given by a single equation, by means of matrices. The discussion is restricted to spaces of three and four dimensions (S_3 and S_4, respectively), but the methods are clearly capable of extension. In S_3 we consider what curves can be represented by matrices of l rows and $l + 1$ columns, whose elements are quite general. We then proceed, in the case $l = 2$, to consider the introduction of one or more redundant curves, our object being to obtain formulae for the characteristics of such curves. We include this in the more general problem of *the genus of the curve of section of two surfaces residual to a system of multiple curves* with mutual intersections. We also obtain *the number of points of intersection of three surfaces residual to such a group.*

In S_4 we consider briefly the characteristics of the surface given by a matrix of 2 rows and three columns, and of the curve given by a matrix of two rows and four columns, in each case, when the elements are general, and shew how to obtain the characteristics. We then consider the characteristics of the curve when the elements are satisfied identically by a redundant curve or surface. We obtain, in fact, a formula for *the genus of a curve, intersection of three three-dimensional forms, residual to a group of curves and surfaces in arbitrary position.* We can readily extend these to the case when the curves have intersections with themselves, or with the surfaces, but the case of surfaces with mutual intersections along curves requires analysis of a more fundamental nature than that employed here.

The base formulae required are given in papers of Severi cited in § 9, wherein the intersection of constructs residual to a single simple construct of lower dimension is solved in all its generality. He also gives a formula for the curve of intersection of constructs residual to a single curve multiple on each, and one for the number of intersections of n forms[*] in S_n residual to an

[*] Following Severi I use this term for $(n-1)$ dimensional constructs in S_n in preference to the term *hypersurface*.

h-dimensional construct, which is at most *double* on one or more of the forms. These formulae are the only ones which overlap the results of the present paper.

Throughout the work we suppose the elements quite general of their type. We exclude throughout the case when the elements represent forms with a common multiple point. In S_3 (for example) this effects the genus, but not the order or number of apparent nodes* of a curve. It merely introduces an actual node, for whose effect we can easily allow. The same is true for the number of points of intersection of forms. We shall denote by c_n^p a curve of order n and genus p, by c_n^R a curve of order n and first rank† R, and by c_n^h a curve of order n with h apparent nodes, according as to which characteristic it is convenient to employ. The upper suffixes will be used exclusively for these characteristics. Expressions a_c^n will denote homogeneous polynomials in $x_1 \ldots \ldots x_r$ of degree n, and these in turn will generally be denoted simply by (n), etc. We shall write $2n + R = l$, this being a frequently occurring combination.

§ 1. Confining ourselves to ordinary space until the contrary is stated, the matrix in l rows and $l + 1$ columns,

$$\| a_x^{n_{i,j}} \| = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

plainly represents a curve, whose order and genus are calculable by known rules. The elements being considered general of their type, and taking the elements of the first column as being of order $n_1 \ldots \ldots n_\rho$, and those of the i-th row‡ as n_i, $n_i + h_1$, $n_i + h_2$, $\ldots\ldots$; the curve represented is known to be of order

$$\Sigma n_i^2 + \Sigma n_i n_j + (\Sigma n_i)(\Sigma h_i) + \Sigma h_i h_j.$$

If then we wish to determine whether any given curve is representable by a matrix of l rows, it is clearly sufficient to consider the case§ when n_1, n_2, $\ldots\ldots n_\rho$ are all greater than zero, for otherwise we could reduce the matrix to one with fewer rows. Conversely if the curve has first a representation with l rows it has at least one for each case with more than one row. We may call a matrix with no elements mere constants *normal*. Thus *normal matrices of l rows represent curves whose orders exceed* $\frac{1}{2}l(l + 1)$. It will be abundantly clear, that there are curves, which can in no way be represented by a matrix of the type considered.

§ 2. Let us consider first curves representable by a general matrix of two rows. These include complete intersections, and

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* Number of chords through a point (S_3) or meeting a line (S_4).

† Or simply rank.

‡ It is not essentially restrictive to suppose $h_i \leqslant h_{i+1}$.

§ It must not be supposed that other representations are of no value. They are extremely useful for representing systems of curves.

curves represented by normal matrices of two rows. *The latter are thus curves which are the complete intersection of two surfaces, residual to a curve which is a complete intersection of two other surfaces.* Taking the matrix as

$$\left\| \begin{matrix} (m) & (m+h) & (m+k) \\ (n) & (n+h) & (n+k) \end{matrix} \right\| = 0,$$

we can shew that the curve has for order, and number of apparent nodes, respectively

$$M = m^2 + mn + n^2 + (k+h)(m+n) + hk, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$
$$H = \tfrac{1}{2}mn(m-1)(n-1) + \tfrac{1}{2}(M-mn)(m+n+h-1)(m+n+k-1)^*.$$
$$\ldots\ldots\ldots(3)$$

The curves in question must include those of minimum† H of given order, for it is known that such lie on a quadric, either as a complete intersection or residual to a line‡. It is not difficult for a given M to determine by exhaustion what curves occur, and the representations which are possible. *The number of such representations is necessarily finite* (this holds for any value of l). Excluding plane curves, as far as $M=7$, exclusive, only curves of minimum H occur. For $M=7$, $H=9$, 10; for $M=8$, $H=[12]$, 14; for $M=9$, $H=16$, $[18]$, etc.; are found to be the only possible cases, the cases in brackets being complete intersections.

Corollaries.

(1) The $c_4^3, c_5^6, c_5^7, c_6^{10}, c_6^9, c_6^8$, the upper suffixes referring to H, cannot occur in matrices with general elements§.

(2) The following matrices are the only ones for the elliptic quartic

$$\left\| \begin{matrix} 2 & 2 & 2 \\ 0 & 0 & 0 \end{matrix} \right\|; \ \left\| \begin{matrix} 0 & 2 & 2 \\ 0 & 2 & 2 \end{matrix} \right\|; \ \left\| \begin{matrix} 2 & 2 & 1 \\ 1 & 1 & 0 \end{matrix} \right\|;$$

so that these would be of use in the study of systems of these curves.

(3) Conversely *all curves which satisfy the condition at the beginning of this paragraph are representable by a matrix with two rows.* We may pass on to state the general principle, of which the truth is self-evident. *Curves represented by a normal matrix of l rows are complete intersections residual to a curve representable by*

* Salmon, *Geometry of Three Dimensions*, 5th Edn. 1914, vol. I. § 345, p. 358. We use H, in place of the genus p, to avoid difficulties of proper nodes.
† Or those curves which if they had no proper nodes would be of maximum genus.
‡ Noether, "Zur Grundlegung der Theorie der algebraischen Curven," *Berl. Abhand.* 1882, p. 42.
§ The sextics since the only matrix normal for $l=3$ giving $M=6$ gives $H=7$.

a normal matrix of $(l-1)$ *rows.* The converse is not true for $l > 2$; it depends on the possibility of drawing through the second curve a surface of sufficiently low order. This may be illustrated by the c_4^3 and c_6^7, which, when they have suitable intersections from the complete intersection of a quadric and quintic surface, and the latter has a matrix representation, but not the former.

§ 3. Having then seen that it is impossible in, in fact, the majority of cases, to represent a curve of S_3 by a matrix of general type, let us consider under what conditions it is possible to obtain a representation at all. It is clearly necessary to introduce redundant curves of simpler character[*]. It is sufficient from now onwards to confine ourselves to the case $l = 2$, though a parallel theory could be generally developed. We consider the matrix equation

$$\left\| \begin{array}{ccc} (m)_\alpha & (m+h)_{\alpha+\rho} & (m+k)_{\alpha+\sigma} \\ (n)_\beta & (n+h)_{\beta+\rho} & (n+k)_{\beta+\sigma} \end{array} \right\| = 0,$$

where the elements denote expressions of the order stated, which if equated to zero would represent surfaces passing through a certain c_μ the number of times indicated by the suffixes. We must have m, n, h, k, α, β all positive or zero. In addition we have a number of *essential* restrictions. To express that, when we develop in a definite manner, the final surfaces, whose partial intersection gives the curve in question, do not degenerate, we have

$$\binom{m+n+h-1}{2} \geqslant \mu \binom{\alpha+\beta+\rho}{2}$$

$$\binom{m+n+k-1}{2} \geqslant \mu \binom{\alpha+\beta+\sigma}{2}.$$

Clearly also we must express that it is possible for elements of these forms to exist. Thus for a line ($\mu = 1$), we must have

$$\alpha \leqslant m, \quad \beta \leqslant n, \quad \alpha + \rho \leqslant m + h, \quad \ldots\ldots, \quad \beta + \sigma \leqslant n + k.$$

Further $0 \leqslant \alpha + \rho$, $0 \leqslant \beta + \sigma$ in all cases. Lastly, in order that the introduction of c_μ may be significant,

$$\alpha + \beta + \rho > 0, \quad \alpha + \beta + \sigma > 0.$$

Further we have a series of *unessential* restrictions such as $0 \leqslant h$, k; $h \leqslant k$; $m \leqslant n$; and others in special cases (such as the equality of h and k when we may take $\rho \leqslant \sigma$, etc.).

It is easily shewn that the order of the curve represented by the vanishing of our matrix is

$$M = m^2 + mn + n^2 + (k+h)(m+n) + hk - \mu\{\alpha^2 + \alpha\beta + \beta^2 + (\rho+\sigma)(\alpha+\beta) + \rho\sigma\}.$$

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[*] For a second method of overcoming this difficulty see an interesting memoir, *Algèbre à deux Dimensions*, M. Stuyvaert, Ghent, 1920, p. 63, and Chap. XII. p. 212.

For a given value of M we can determine solutions, for successive choices of m and n, etc., by a process of exhaustion. The ultimate solution depends on those of equations of the type $A(k+h)+hk=x$ (A and x being given integers), which are easily determined. The general problems involved however are not so easy to handle as before. In the first place we may state as before geometrical conditions which the matrix expresses. *Thus curves represented by matrices of two rows are clearly complete intersections of two surfaces residual to a c_μ multiple on each, and another curve which is the complete intersection of two surfaces residual merely to a c_μ multiple on each, and so on.* It is clear that the converse, as before, ceases to hold for $l > 2$.

It would be interesting to determine whether for any given curve c_M, restricting μ to be less than M, the number of possible (normal) matrices, is finite or not. It is certainly not, in some cases at least, if we remove the restriction of normality. Thus the assemblage of all curves of a given order, which lie on any surface, residual sections by a surface having one of its plane sections (as multiple), has an infinite number of representations, and hence at least one such curve has (and, in general, all).

We propose later to develop methods for calculating the genera of these curves. For the present let us remark that an obvious generalization is to the case, when there are more than one redundant curves, and pass on to devote a few words to the special case of the rational quartic*, this being perhaps the most important example we can select.

Taking first the case of a single redundant line, l, and assigning successive values to m and n, we find up to $m=2$, $n=2$ inclusive, the following matrices giving $M=4$,

$$\left\| \begin{matrix} 0 \cdot 2_1 \cdot 3_2 \\ 0 \cdot 2_1 \cdot 3_2 \end{matrix} \right\|, \ \left\| \begin{matrix} 0 \cdot 1_0 \cdot 2_1 \\ 1_1 \cdot 2_1 \cdot 3_2 \end{matrix} \right\|, \ \left\| \begin{matrix} 1 \cdot 1_1 \cdot 2_1 \\ 1 \cdot 1_1 \cdot 2_1 \end{matrix} \right\|; \ \left\| \begin{matrix} 1 \cdot 1 \cdot 2 \\ 1_1 \cdot 1_1 \cdot 2_1 \end{matrix} \right\|;$$

$$\left\| \begin{matrix} 1_0 \cdot 1_0 \cdot 3_2 \\ 1_1 \cdot 1_1 \cdot 3_3 \end{matrix} \right\|, \ \left\| \begin{matrix} 1_0 \cdot 2_1 \cdot 2_1 \\ 1_1 \cdot 2_2 \cdot 2_2 \end{matrix} \right\|;$$

$$\left\| \begin{matrix} 2_1 \cdot 2_2 \cdot 2_2 \\ 2_1 \cdot 2_2 \cdot 2_2 \end{matrix} \right\|, \ \left\| \begin{matrix} 1_0 \cdot 1_1 \cdot 2_2 \\ 2_1 \cdot 2_2 \cdot 3_3 \end{matrix} \right\| \dagger.$$

Of these the first three in the first row are of the second species, the fourth of the first species. These require no comment. Those in the second row are both of the second species, as can be shewn easily by a degeneration argument or directly from the formulæ

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* Necessarily of the 2nd species, intersection of a quadric and cubic with two skew lines in common. That with an actual node we have excluded.

† It is not claimed that these are exhaustive up to $m=2$, $n=2$. An attempt was made to determine all types, but in the absence of a theory of the equations involved it is impossible to ensure that this is the case.

of § 7. Those in the last column each represent four lines, all
skew. Thus the first matrix represents the intersection of two
ruled quartic surfaces* with the same triple line l, and a common
cubic $c_3 \equiv l^2$. We may allow each to break into a quadric K (or K')
passing through c_3 and l, and planes ϖ_1, ϖ_2; $(\varpi_1' \varpi_2')$ through l (§4).
The residual quartic breaks into four lines each meeting l and c_3,
but no two meeting. Such a figure is underivable from a proper
quartic curve. Thus *the section of two ruled quartics with the same
triple line and a common cubic curve having l as chord necessarily
consists of four common generators.*

From the second matrix it is seen that *a similar theorem holds
for a cubic surface* $C^3 \equiv (l)^2 c_2$†, *and a quartic surface* $Q^4 \equiv (l)^3 c_2$,
where c_2 is a conic meeting l. These are both included in *a theorem
of much more general character.* The matrix

$$\left\| \begin{array}{ccc} n_{n-1} & (n+h)_{n+h} & (n+k)_{n+k} \\ m_{m-1} & (m+h)_{m+h} & (m+k)_{m+k} \end{array} \right\| = 0$$

represents the intersection of scrolls R^{N+h}, S^{N+k} residual to the
common multiple directrix l, $N+h-1$ fold on the first and
$N+k-1$ fold on the second, and a rational $c_{N-1} \equiv l^{N-2}$, itself the
intersection of two scrolls, and *this intersection consists of $N+k+h$
common generators.* Here $N = n+m \geqslant 2$, and the curve c lies on an
infinite number of scrolls of order $N \equiv l^{N-1} l'$, locus of lines meeting
simultaneously l, l' and c_{N-1}. Let us break R^{N+h} into one of these,
and into h planes through l, and S^{N+k} similarly into another, and
k planes through l. Then the theorem will follow if we can prove
it for two of these N-tic scrolls, and for these it is obviously true,
for if l'' be the simple directrix of the second scroll, their
intersection is simply the N lines meeting c_{N-1}, l, l', and l''‡.

Let us consider in what ways we can extend this theorem. Consider first
two scrolls $R^N \equiv (c_\tau)^\mu (c_\mu)^\tau$ generated by lines meeting l', c_τ, c_μ; and l'', c_τ, c_μ;
respectively, where c_μ and c_τ have σ intersections. Then the conditions
require, in the first place,

$$N^2 - N = \mu\tau^2 + \mu^2\tau,$$

where N, the order of the scroll, is equal to $2\tau\mu - \sigma$; and hence

$$\sigma^2 - \sigma(4\tau\mu - 1) + \mu\tau(4\mu\tau - \mu - \tau - 2) = 0,$$

so that a first condition is

$$1 + 4\mu\tau(\mu + \tau) = \xi^2, \text{ a perfect square,}$$

when $\sigma = \frac{1}{2}(4\tau\mu - 1 - \xi)$. It seems possible to get an infinitude of solutions of

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* Salmon, *op. cit.* vol. II. § 546, p. 203.

† The notation $R^n \equiv (c_\mu)^r P^s$ denotes that the surface R^n passes r times through
c_μ and has P as an s-ple point. $c_\mu \equiv (c_\nu)^r P^s$ denotes that the curve c_μ meets c_ν r
times, (outside P if $c_\nu \equiv P$), and has P as an s-ple point.

‡ Given by the intersection of c_{N-1} and the regulus of lines $\equiv l l' l''$, removing
the lines through the $(N-2)$ points of c_{N-1} on l, meeting l' and l''.

this problem by the following process. Select a definite value of μ, and suppose first τ or $\tau + \mu$ prime. Then we may have one of the following possibilities

$$\begin{matrix} (\xi - 1) = a\tau(\mu + \tau) \\ (\xi + 1) = \beta \end{matrix} \Big\}, \quad \begin{matrix} \xi + 1 = a\tau(\mu + \tau) \\ \xi - 1 = \beta \end{matrix} \Big\}, \text{ where } a\beta = 4\mu$$

$$\begin{matrix} (\xi - 1) = a\tau \\ (\xi + 1) = \beta(\mu + \tau) \end{matrix} \Big\}, \quad \begin{matrix} \xi + 1 = a\tau \\ \xi - 1 = \beta(\mu + \tau) \end{matrix} \Big\} \qquad " \qquad "$$

Taking the various possibilities for a and β it is possible to exhaust all such solutions for an assigned μ. Let τ' be any one such corresponding to ξ'. Then we may rewrite our equation:

$$4\mu(\tau - \tau')(\tau + \tau' + \mu) = \xi^2 - \xi'^2,$$

and proceed, in precisely the same manner, to exhaust all cases when $\tau - \tau'$ or $\tau + \tau' + \mu$ is prime, and so on. However we can shew that, if this is to give a permissible value of σ, either μ or τ must be unity. Let N be the order of the surface of least order passing through c_μ. Then, excluding the case when c_μ and c_τ lie on such a surface, $\sigma \leqslant \tau N$, and exceeds this value, regarded as a function of τ, except when

$$0 \leqslant \tau \leqslant \bar{\tau} = (\mu^2 + 2\mu - N)/[4\mu^2 - \mu(4N + 1) + N^2],$$

giving a definite upper limit to τ for a given μ (supposed > 1). Consider now all curves for which N has a definite value. Then it is immediately shewn that $\bar{\tau} \leqslant 1$ except for

$$\tfrac{1}{3} N \leqslant \mu \leqslant N + 1,$$

and except for $\mu = 2, 3$ there are no such curves. The excluded case is treated similarly. In the case $\mu = 1$ (or $\tau = 1$) we may have any value for τ, but $\sigma = \tau - 1$, so that we fall back on the case we started with, *now shewn to be the only one possible.*

Secondly, it is easily seen that two quartic scrolls of chords of the same space cubic, meeting respectively two lines l, l' (Type 7 of Cremona, IV of Sturm), have as residual intersection four generators, and the same is therefore true of two general quartic scrolls with the same nodal cubic (Types 1, III). Consider generally the scrolls of chords of a curve c_μ^h meeting two lines

(respectively), and of order $h + \binom{\mu}{2}$*, $(= N)$.

Then the conditions require

$$N(N - 1) = \mu(\mu - 1)^2,$$

or

$$h^2 + h(\mu^2 - \mu - 1) + \mu(\mu - 1)(\tfrac{1}{4}[\mu^2 - 5\mu] + \tfrac{1}{2}) = 0 ;$$

so that if $h = 1$, in which case the lines are simple directrices†, the only solution is $\mu = 3$. More generally, a first condition is that $4\mu(\mu - 1)^2 + 1$ shall be a perfect square. Solutions are $\mu = 0, 1, 2, 3$ and using these in the way indicated above it is easily shewn that there are no other solutions, for which any one of μ, $\mu - 1$, $\mu - 2$, $\mu - 3$, $\mu^2 + 1$, $\mu^2 + \mu + 4$ is prime. There may be other solutions of the numerical problem, though this appears unlikely‡; but it is easily shewn that there are none corresponding to permissible values of h §.

Of course in most of the cases mentioned the intersection will consist *partly* of generators.

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* Salmon, *op. cit.* vol. II. § 470.

† h is, in fact, the multiplicity of the lines on the scrolls.

‡ There are certainly none $\leqslant 100$.

§ Noether, *loc. cit.* in § 2. $h \geqslant \left(\dfrac{n - 1}{2}\right) - \left[\dfrac{n - 1}{2}\right]\left(n - 2 - \left[\dfrac{n - 1}{2}\right]\right)$; where $[a]$ denotes as usual the greatest integer in a.

Passing back to the case $M = 4$, the quartic of the second species does not occur after $n = 1$, $m = 1$, in our list; and the process of enumeration would lead one to suspect that it does not occur again.

Taking next the case of *a common conic*; we find up to $n = 1$, $m = 1$ inclusive, the following matrices giving $M = 4$

$$\left\| \begin{matrix} 0 & 2_1 & 3_1 \\ 0 & 2_1 & 3_1 \end{matrix} \right\|, \quad \left\| \begin{matrix} 0 & 1_1 & 2_1 \\ 1_0 & 2_1 & 3_1 \end{matrix} \right\|, \quad \left\| \begin{matrix} 1_0 & 1_0 & 2_0 \\ 1_1 & 1_1 & 2_1 \end{matrix} \right\|;$$

$$\left\| \begin{matrix} 0 & 2_1 & 4_2 \\ 0 & 2_1 & 4_2 \end{matrix} \right\|, \quad \left\| \begin{matrix} 0 & 2_1 & 5_3 \\ 0 & 2_1 & 5_3 \end{matrix} \right\|, \text{ etc.,}$$

of which all in the second row, after the first, are to be rejected, since (for $\mu > 2$) a surface of order $\mu + 2$ cannot have a μ-fold conic. These quartics are all of the first species, and clearly *the rational quartic cannot occur in this group.*

§ 4. We pass on to consider methods for determining a second characteristic (say, the number of apparent nodes H) for the curves represented by this type of matrix. We consider in fact the more general problem of the number H for *the intersection of two surfaces with a common system* Θ *of multiple curves, having prescribed mutual intersections but otherwise in general position.* With this we associate *the problem of the intersection of three surfaces under such conditions.* From these the passage to the special problems under consideration is immediate.

The method we employ is to allow the surfaces to degenerate in some suitable way. In the former case we obtain a group of partial curves with a number of mutual intersections, which, as is well known, can serve as an image of the real curve we seek. The number of apparent nodes is equal to the sum of the same characteristic for the partial curves, together with the number of apparent intersections of the curves by pairs. Amongst the apparent intersections however must be reckoned, in general, some of the real intersections on the common multiple curves. In fact, if two parts of the curve intersect in a simple point on the system of common curves Θ, then, with the restrictions involved, they will be images of branches of the curve lying in different sheets of the secant surfaces. Their intersection will clearly represent an *apparent* self-intersection of the real curve. In the case of a double point of the system Θ, it may happen that the actual curve is forced to have there an actual* multiple point. If

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* The term *proper* having acquired, in connection with multiple points in general, a special significance, we prefer to use the term *actual* for self-intersections of curves, where these questions are not taken into consideration.

In the case of an actual node on one of the component curves, our formulae remain unchanged, if expressed entirely in terms of h, instead of R.

r, s are the multiplicities of the parts of Θ, there intersecting, on one surface, and r_1, s_1 on the other, this occurs in general when the inequality between r and s, and that between r_1 and s_1, are opposite (neither being equalities), but not, in general, in the other cases. Thus it does not occur for an actual node on one of the curves of Θ. We exclude the case of triple points or triple intersections of components of Θ. *Neither does it occur in the specialized problem, when the redundant curves have no common intersections.* It appears to be most convenient to leave the number of these actual nodes indeterminate in our formulae, and to determine them in the special cases which occur, which is a quite simple process.

Intersections of the partial curves outside Θ are of two types. Firstly those which arise from a contact of the secant surfaces, or from a nodal curve on one of them. These do not affect H, and are easily determined in application (if we wish to pass from H to the genus p). Secondly we have intersections which do not correspond to actual nodes on the real curve of section, and lie in a simple region of each secant surfaces. They clearly cannot be images of apparent intersections, and hence we have the result

$$H + \mu = \Sigma n_i n_j + \Sigma h_i - \nu,$$

wherein $(n_i h_i)$ *are the characteristics of a partial curve**, ν *is the number of intersections outside* Θ, *and* μ *is the equivalence of the nodes imposed on the curve by multiple points of* Θ. It is necessary to ensure that only simple intersections occur in μ. The method of ensuring this will appear in the calculations. This formulae enables us theoretically to solve all cases which occur. We develop the calculations sufficiently far to cover all cases in which Θ contains at most three components. It appears convenient to consider simpler cases first. This envolves in theory some redundancy of calculation, but not in practice.

The second problem, in which we have three surfaces with the system of multiple curves Θ in common, is much more simple, and actually we treat this case first. Before proceeding to details, however, we require a few lemmas on the intersection of surfaces having a common system of simple curves.

§ 5. LEMMA 1. *Two surfaces have three curves in common; to find the relations between their mutual intersections*, the relative positions being general.

The surfaces are of order α_1, α_2, the curves† of order μ_1, μ_2, μ_3

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* A curve, unless stated to belong to Θ, always means in this connection a partial curve of the image of the real section.

† In the subsequent work a curve (μ) means a curve of order μ.

with σ_{23}, σ_{13}, σ_{12} mutual intersections. Then, from reasoning analogous to that in Salmon's Treatise*, we have

$$\mu_i(\alpha_1-1)(\alpha_2-1)=2h_i+\mu_i(\mu_j+\mu_k)-\sigma_{ij}-\sigma_{ik};\ \ i=1,2,3;\ \ j\neq k\neq i,$$

from which the required results can be obtained. If each c_{μ_i} be compounded of a group of curves with Δ_i intersections, then as far as all the intersections are concerned

$$2\Delta_i+\sigma_{ij}+\sigma_{ik}\sim 0,$$

the symbol implying that we omit all terms independent of the σ. This is the form in which we shall use the result.

LEMMA II. *Two surfaces R^a pass through curves μ, μ'* (simple, or compounded), *and R^β passes through μ' only. To find the number of intersections outside μ and μ' as functions of the intersections.* Considering the intersections of the residual intersection c_ν of the R^a with the curve c_μ, we have

N° of Intersections Sought $\sim$ points on c_ν and $c_{\mu'}$

$$\sim 2\Delta_{\mu'}+\sigma,$$

by Lemma I, wherein σ is the number of intersections of c_μ and $c_{\mu'}$.

LEMMA III. *On the other hand if we have two $R^\beta\equiv c_{\mu'}$ and an $R^a\equiv c_\mu c_{\mu'}$, the number of intersections*

$$f(\alpha,\beta,\beta)\sim 2\Delta_{\mu'}.$$

In fact generally *if three surfaces pass through three groups of curves with common members we may remove from consideration* all those which occur *once* only, and group those which occur *thrice* into a single compound curve.

LEMMA IV. If $R^a\equiv c_{\mu_1}c_{\mu_2}c_{\mu_3}$, $R^\beta\equiv c_{\mu_2}c_{\mu_3}$, $R^\gamma\equiv c_{\mu_3}$, *to find the number of intersections.* This is essentially the same as Lemma II, and the number sought

$$f(\alpha,\beta,\gamma)\sim 2\Delta_{\mu_3}+\sigma_{23}.$$

LEMMA V. *To find the number of intersections of surfaces* $R^a\equiv c_{\mu_1}c_{\mu_2}c_{\mu_3}$, $R^\beta\equiv c_{\mu_2}c_{\mu_3}$, $R^\gamma\equiv c_{\mu_1}c_{\mu_3}$, *outside these curves.* The number of points sought is clearly the number of intersections with R^γ of the residual curve c_ν, intersection of R^a and R^β, outside c_{μ_3}, c_{μ_1}; and the number of these redundant intersections is clearly that of the points of intersection of R^β and c_{μ_1} outside c_{μ_1}, c_{μ_3}. Hence, using Lemma I,

$$f(\alpha,\beta,\gamma)\sim 2\Delta_{\mu_3}+\sigma_{12}+\sigma_{13}+\sigma_{23}.$$

Lastly, LEMMA VI. *To find the number of intersections of surfaces* $R^a\equiv c_{\mu_1}c_{\mu_2}c_{\mu_4}$, $R^\beta\equiv c_{\mu_1}c_{\mu_3}c_{\mu_4}$, $R^\gamma\equiv c_{\mu_2}c_{\mu_3}c_{\mu_4}$ *outside these curves.* This number is equal to $\gamma(\alpha\beta-\mu_1-\mu_4)$ less the number of

intersections of the residual curve $(R^a R^3)$ with c_{μ_2}, c_{μ_3}, and c_{μ_4}. The first of these equals the number of intersections of c_{μ_2} and R^β less its intersections with c_{μ_3}, c_{μ_4}, and c_{μ_1}: the second the same with the suffixes 2 and 3; and α and β; interchanged, but in this we shall have counted the points σ_{23} twice. Using Lemma I for the third, we have finally

$$f(\alpha, \beta, \gamma) \sim 2\Delta_4 + \overset{4}{\underset{1}{\Sigma}} \sigma_{ij}.$$

These are sufficient for the problem as far as we develop it. It is important to notice that *any of the curves may be compound*.

§ 6. *Intersection of Three Surfaces residual to a common group of multiple curves.*

§ 6·1. *Starting with a single curve* c_μ^R of order μ and rank R, and surfaces

$$R^{m_1} \equiv (c_\mu)^r, \; R^{m_2} \equiv (c_\mu)^s, \; R^{m_3} \equiv (c_\mu)^t,$$

in a general manner.

We break R^{m_1} into r surfaces q of suitable order* α passing each once through c_μ, and $m_1 - r\alpha$ planes ϖ in general position; R^{m_2} similarly into q' and ϖ', R^{m_3} into q'' and ϖ''. The points in question are given by $qq'q''$, $qq'\varpi'$, $q\varpi'\varpi''$, $\varpi\varpi'\varpi''$, etc. Using for the first group the formula for the intersection of three surfaces residual to a curve†, we get finally for the number of intersections‡

$$m_1 m_2 m_3 - \mu (\Sigma m_3 rs - 2rst) + R\,rst. \qquad Formula\ I$$

§ 6·2. *We next take a group of curves with no mutual intersections§ and*

$$R^{m_1} \equiv (c_{\mu_1})^{r_1} (c_{\mu_2})^{r_2} (c_{\mu_3})^{r_3}\ldots\ldots,$$
$$R^{m_2} \equiv (c_{\mu_1})^{s_1} \text{ etc.}, \quad R^{m_3} \equiv (c_{\mu_1})^{t_1}, \text{ etc.}$$

We break R^{m_1} into groups of r_i surfaces of suitable order α_i passing simply through c_{μ_i} and $m_1 - \Sigma r_i \alpha_i$ planes in general position, and the remaining two similarly. It is clearly merely sufficient to select those terms which involve at least two distinct curves of the group, and which also do not involve the arbitrary α_i,¶

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* The surfaces (a) must be ∞^1 at least.

† Salmon, *op. cit.* p. 370. Our result agrees with this, of course, and with the formula for $r=2$, $s=t=1$.

‡ In getting this and the subsequent formulae the process adopted is to *reject* all terms involving the a_i. In the earlier cases it is easy to verify that these do disappear, and in fact it must be so always, though special cases may logically require independent examination.

§ It is clearly permissible to exclude $r_i = r_j$ with $s_i = s_j$ for such a pair of curves could be taken together. The same is true throughout.

¶ In special cases there may be no choice for the a_i. In such a case to complete the proof we must theoretically verify that these terms do disappear, or from the fact that it is clearly permissible to assume that the results do not vary *in form* for possible values of the terms involved.

and in fact there are clearly no such terms. Hence the number sought is

$$\pi \equiv m_1 m_2 m_3 - \sum_i \mu_i \{m_3 r_i s_i - 2 r_i s_i t_i\} + \sum_i R_i r_i s_i t_i. \quad \textit{Formula II}$$

§ 6·3. We pass on to consider the modification of this formula when the various curves have mutual intersections, and σ_{ij} will always denote the number of intersections of c_{μ_i} and c_{μ_j}, which are assumed all distinct, and that there are no contacts. *It is necessary to ensure that in the components of the degenerate curve that none of the extra double points fall into their points σ_{ij}.* This we do by the following process. Let R^{m_j} pass r_i times through c_{μ_i}, and let, for the moment,

$$r_i \leqslant r_{i+1} \qquad\qquad\text{(all } i\text{)}.$$

We break R^{m_j} into r_1 surfaces of order α_1 passing through $c_{\mu_1} \dots c_{\mu_\nu}$; $r_2 - r_1$ of order α_2 passing through $c_{\mu_2} \dots c_{\mu_\nu}$; $r_3 - r_2$ of order α_3 through $c_{\mu_3} \dots c_{\mu_\nu}$, etc., etc. If this is done for each surface, the result sought is obtained. It will now be quite clear that a solution is always possible, in which the result sought is a function of the μ_i, R_i, m_i, r_i, etc., etc., and the σ_{ij}. This being so it is sufficient to consider the formulae merely in as far as they concern σ. The results unfortunately are not invariant in form, but depend on whether the sequence of the curves, arranged in order of multiplicity, is the same for each surface, or not. It is tacitly assumed that *at a point of intersection of two components, each surface passes through the curves with the least possible number of sheets.* If this condition is infringed the case must be examined independently.

We believe it useful to examine first *the case of two curves.*

§ 6·4. Case I. $R^{m_1} \equiv (c_{\mu_1})^{r_1} (c_{\mu_2})^{r_2}$; $R^{m_2} \equiv (c_{\mu_1})^{s_1} (c_{\mu_2})^{s_2}$;
$R^{m_3} \equiv (c_{\mu_1})^{t_1} (c_{\mu_2})^{t_2}$;

with $r_1 \leqslant r_2, \;\; s_1 \leqslant s_2, \;\; t_1 \leqslant t_2$.

Breaking R^{m_1} into r_1 α-tics through c_{μ_1} and c_{μ_2}, $r_2 - r_1$ β-tics through c_{μ_2} only, and $m_1 - r_1 \alpha - \beta (r_2 - r_1)$ general planes; etc., then, by Lemmas II and III the coefficient of σ is equal to the number of combinations of type $\alpha \alpha' \beta''$, where the accented symbols denote components of R^{m_2}, R^{m_3} respectively, together with twice the number of combinations $\alpha \alpha' \alpha''$;

$$= \sigma \{ \sum_{r,s,t} r_2 s_1 t_1 - r_1 s_1 t_1 \}, \qquad\qquad \textit{Formula III}$$

which added to Formula II gives the number sought.

§ 6·5. Case II. $r_1 \leqslant r_2, \, s_1 \leqslant s_2, \, t_1 > t_2$. In this case R^{m_3} is to be broken into t_2 and α-tic surfaces α'', $t_1 - t_2$ γ-tic surfaces through c_{μ_1} only, and $m_3 - t_2 \alpha - (t_1 - t_2) \gamma$ planes. Using Lemma V for the intersection of an α-tic, a β-tic, and a γ-tic (which $\sim + \sigma$), we deduce as the modification precisely the expression in Formula III.

§ 6·6. *The General Cases.* We propose to consider the cases which occur for N curves, in as far as is sufficient to complete the problem in all cases when three curves only are involved. We propose to say that the multiplicities $\eta_1, \eta_2, \ldots \eta_N$ satisfy the inequalities.

A : when $\eta_1 \leqslant \eta_2 \leqslant \eta_3 \leqslant \eta_4 \ldots \leqslant \eta_N$,

B : ,, $\eta_1 \leqslant \eta_3 \leqslant \eta_2 \leqslant \eta_4 \ldots \leqslant \eta_N$,

C : ,, $\eta_2 \leqslant \eta_1 \leqslant \eta_3 \leqslant \eta_4 \ldots \leqslant \eta_N$,

D : ,, $\eta_2 \leqslant \eta_3 \leqslant \eta_1 \leqslant \eta_4 \ldots \leqslant \eta_N$,

E : ,, $\eta_3 \leqslant \eta_1 \leqslant \eta_2 \leqslant \eta_4 \ldots \leqslant \eta_N$,

F : ,, $\eta_3 \leqslant \eta_2 \leqslant \eta_1 \leqslant \eta_4 \ldots \leqslant \eta_N$.

We may clearly, with the usual notation, take the r as satisfying (A) so that the cases will be denoted by a two-letter symbol such as AA, DE, the former referring to the s, the latter to the t. We propose to call the sub-group of curves $c_{\mu_i}, c_{\mu_{i+1}} \ldots c_{\mu_N}$ the group Θ_i, or simply (i), while the following groups will also have names attached:—

$(\varpi) : c_{\mu_2}, c_{\mu_4}, c_{\mu_5} \ldots c_{\mu_N}$,　　　$(\psi) : c_{\mu_1}, c_{\mu_4}, c_{\mu_5} \ldots c_{\mu_N}$,

$(\phi) : c_{\mu_1}, c_{\mu_3}, c_{\mu_4} \ldots c_{\mu_N}$,　　　$(\chi) : c_{\mu_1}, c_{\mu_2}, c_{\mu_4} \ldots c_{\mu_N}$.

A symbol such as Δ_i or Δ_ϕ will denote the sum of the number of intersections of all pairs of curves in the corresponding group, and $\overset{y}{\Sigma}_\phi$ etc. will denote a sum over all curves in the ϕ group up to c_{μ_y}, inclusive.

These notations being explained, suppose we have an $R^m \equiv (c_{\mu_i})^{\eta_i}$, $i = 1 \ldots N$. We tabulate the degenerations employed in the various cases.

(A) $\eta^{(1)}$ α_1-tics $\equiv \Theta_1$, $\eta^{(2)}$ α_2-tics $\equiv \Theta_2$, $\eta^{(3)}$ α_3-tics $\equiv \Theta_3$, etc., $\overline{m}$ planes ϖ, wherein $\eta^{(1)} = \eta_1$, $\eta^{(i)} = \eta_i - \eta_{i-1}$, for $i \neq 1$; and $\overline{m} \sim m$.

(B) As (A), together with $(\eta_2 - \eta_3)$ γ-tic surfaces $\equiv \Theta_{\varpi}$, where now $\eta^{(1)} = \eta_1$, $\eta^{(2)} = \eta_3 - \eta_1$, $\eta^{(3)} = 0$, $\eta^{(4)} = \eta_4 - \eta_2$, $\eta^{(i)} = \eta_i - \eta_{i-1}$, when $i \geqslant 5$.

(C) As (A); with $(\eta_1 - \eta_2)$ $\bar{\gamma}$-tics $\equiv \Theta_\phi$, wherein

$\eta^{(1)} = \eta_2$, $\eta^{(2)} = 0$, $\eta^{(3)} = \eta_3 - \eta_1$, $\eta^{(i)} = \eta_i - \eta_{i-1}$, for $i > 3$.

(D) As (A); with $(\eta_3 - \eta_2)$ $\bar{\gamma}$-tics $\equiv \Theta_\phi$, and $(\eta_1 - \eta_3)$ δ-tics $\equiv \Theta_\psi$, wherein

$\eta^{(1)} = \eta_2$, $\eta^{(2)} = \eta^{(3)} = 0$, $\eta^{(4)} = \eta_4 - \eta_1$, $\eta^{(i)} = \eta_i - \eta_{i-1}$, for $i > 4$.

(E) As (A); with $(\eta_2 - \eta_1)$ γ-tics $\equiv \Theta_{\varpi}$, and $(\eta_1 - \eta_3)$ ϵ-tics $\equiv \Theta_\chi$, wherein

$\eta^{(1)} = \eta_3$, $\eta^{(2)} = \eta^{(3)} = 0$, $\eta^{(4)} = \eta_4 - \eta_2$, $\eta^{(i)} = \eta_i - \eta_{i-1}$, for $i > 4$.

(F)   As (A); with $(\eta_2 - \eta_3)$ ϵ-tics $\equiv \Theta_\chi$, and $\eta_1 - \eta_2$ δ-tics $\equiv \Theta_\psi$, wherein

$$\eta^{(1)} = \eta_3,\ \ \eta^{(2)} = \eta^{(3)} = 0,\ \ \eta^{(4)} = \eta_4 - \eta_1,\ \ \eta^{(i)} = \eta_i - \eta_{i-1},\ \text{for } i > 4.$$

Herein η stands for r, s, or t, and *in all the formulae which follow the symbols with upper suffixes must be interpreted in terms of this notation.* The advantage derived is that we are able to state the results for the cases (AB) ... (FF) by means of terms added to that for (AA). We shall put

$$2\Delta_a + \sum_{\beta}^{k-1} \sum_{k}^{N} \sigma_{ij} \equiv \{\alpha, \beta, k\},$$

where α and β may be a numerical symbol or $\phi \dots \psi$, but k is always one of the numerical symbols, and

$$r^{(x)} s^{(y)} + r^{(y)} s^{(x)} = \rho_{xy}, \text{ for } x \neq y,$$
$$= 2\rho_x, \text{ for } x = y;$$
$$\Delta_x + \Delta_y = \Delta(x, y),$$
$$s_i - s_j = s_{ij}, \text{ etc.,}$$

the last definition being independent of the upper suffix notation.

§ 6·7. Taking then the case

$$\text{(AA)} \qquad \left.\begin{array}{l} R^{m_1} \equiv (c_{\mu_i})^{r_i};\quad r_i \leqslant r_{i+1}, \\ R^{m_2} \equiv (c_{\mu_i})^{s_i};\quad s_i \leqslant s_{i+1}, \\ R^{m_3} \equiv (c_{\mu_i})^{t_i};\quad t_i \leqslant t_{i+1}, \end{array}\right\} i = 1 \ \dots \ N,$$

and breaking the surfaces as directed, and using the following results from the lemmas,

$$\begin{array}{ll} f\,((i),(i),(i)) \sim 2\Delta_i, & \text{(Salmon, I. § 355)}, \\ f\,((i),(i),(j)) \sim \{j, i, j\}, & (\textit{Lemma II}), \\ f\,((i),(j),(j)) \sim 2\Delta_j, & (\textit{Lemma III}), \\ f\,((i),(j),(k)) \sim \{k, j, k\}, & (\textit{Lemma IV}), \end{array}$$

(wherein the left-hand sides denote the number of intersections of surfaces respectively through the groups stated, and $i < j, < k$), we plainly get as the number of intersections $\Pi + \Lambda$, where Π is given by Formula II and Λ is equal to

$$2\sum_i r^{(i)} s^{(i)} t^{(i)}\, \Delta_i + \sum_{i<j}\sum_{r,s,t} \left[\sum r^{(i)} s^{(i)} t^{(j)}\right] \{j, i, j\}$$
$$+ 2\sum_{i<j}\sum_{r,s,t} \left[\sum r^{(j)} s^{(j)} t^{(i)}\right] \Delta_j + \sum_{i<j<k}\sum_{r,s,t} \left[\sum r^{(i)} s^{(j)} t^{(k)}\right]\{k, j, k\}.\ \textit{Formula IV}$$

The method being now clear we can pass on to state the results obtained. To each $\Pi + \Lambda$ is supposed added.

$$\begin{array}{ll} (\text{AB}): t_{22}\, L\,(\varpi; r, s), & (\text{AC}): t_{12}\, L\,(\phi; r, s), \\ (\text{AD}): t_{32}\, L\,(\phi; r, s) + t_{13}\, L\,(\psi; r, s), \\ (\text{AE}): t_{21}\, L\,(\varpi; r, s) + t_{13}\, L\,(\chi; r, s), \\ (\text{AF}): t_{12}\, L\,(\psi, r, s) + t_{23}\, L\,(\chi, r, s), & \textit{Formulae V} \end{array}$$

where

$$L\left(\varpi\,;r,s\right)=\rho_{1}\left(\Delta\left(1,\varpi\right)-\sigma_{13}\right)+\left(\rho_{2}+\rho_{12}\right)\Delta\left(2,\varpi\right)+\rho_{3}\,\Delta\left(3,4\right)$$
$$+\left(\rho_{13}+\rho_{23}\right)\Delta\left(2,4\right)+T_{4}\left(1,2\,;\varpi\right)+T_{4}\left(3\,;\varpi\right)+S_{4},$$

$$L\left(\phi\,;r,s\right)=\rho_{1}\,\Delta\left(1,\phi\right)+\rho_{2}\,\Delta\left(2,3\right)+\rho_{12}\,\Delta\left(1,3\right)+T_{3}\left(1\,;\phi\right)$$
$$+\,T_{3}\left(2\,;3\right)+S_{3},$$

$$L\left(\psi\,;r,s\right)=\rho_{1}\left(\Delta\left(1,\psi\right)-\sigma_{23}\right)+\rho_{2}\left\{4,2,4\right\}+\rho_{3}\,\Delta\left(3,4\right)+\rho_{12}\,\Delta\left(1,4\right)$$
$$+\,\rho_{13}\,\Delta\left(4,\phi\right)+\rho_{23}\,\Delta\left(3,4\right)+T_{4}\left(1\,;\psi\right)+T\left(2,3\,;4\right)+S_{4},$$

$$L\left(\chi\,;r,s\right)=\rho_{1}\,\Delta\left(1,\chi\right)+\rho_{2}\,\Delta\left(2,\varpi\right)+\rho_{3}\,\Delta\left(3,4\right)+\rho_{12}\,\Delta\left(1,\varpi\right)$$
$$+\,\rho_{13}\,\Delta\left(1,4\right)+\rho_{23}\,\Delta\left(2,4\right)+T_{4}\left(1\,;\chi\right)+T_{4}\left(2\,;\varpi\right)+T_{4}\left(3\,;4\right)+S_{4};$$

in which we have written

$$S_{h}=2\sum_{h}^{N}\rho_{x}\,\Delta_{x}+\sum_{h\leqslant x<y}^{x=N-1,\,y=N}\rho_{xy}\left\{y,x,y\right\},$$

$$T_{h}\left(\alpha,\beta\,;\gamma\right)=\sum_{h}^{N}\left(\rho_{\alpha x}+\rho_{\beta x}\right)\left\{x,\gamma,x\right\},$$

and $T_{h}\left(\alpha\,;\gamma\right)$ is the same with $\rho_{\beta x}$ dropped. For the remaining formulae we introduce a new abbreviation

$$\sum_{h}^{N}r^{(x)}\left\{x,\beta,x\right\}=g_{h}\left(\beta\right),$$

and we shall use (AB), etc., for the expressions in (V),

(BB): $a+\left(\mathrm{AB}\right)+s_{23}\,t_{23}\,M\left(\varpi\right),$

(BC): $a+\left(\mathrm{AC}\right)+s_{23}\,t_{12}\,M\left(\varpi,\phi\right),$

(BD): $a+\left(\mathrm{AD}\right)+s_{23}\,t_{32}\,M\left(\varpi,\phi\right)+s_{23}\,t_{13}\,M\left(\varpi,\psi\right),$

(BE): $a+\left(\mathrm{AE}\right)+s_{23}\,t_{21}\,M\left(\varpi\right)+s_{23}\,t_{13}\,M\left(\varpi,\chi\right),$

(BF): $a+\left(\mathrm{AF}\right)+s_{23}\,t_{12}\,M\left(\varpi,\psi\right)+s_{23}\,t_{23}\,M\left(\varpi,\chi\right),$ *Formulae VI*

where $a=s_{23}\,L\left(\varpi\,;r,t\right),$

$M\left(\varpi\right)\quad=2\left(r^{(1)}+r^{(2)}\right)\Delta_{\varpi}+r^{(3)}\,\Delta\left(4,\varpi\right)+g_{4}\left(\varpi\right),$

$M\left(\varpi,\phi\right)=r^{(1)}\left(\Delta\left(1,4\right)-\sigma_{12}\right)+r^{(2)}\,\Delta\left(2,4\right)+r^{(3)}\,\Delta\left(3,4\right)+g_{4}\left(4\right),$

$M\left(\varpi,\psi\right)=r^{(1)}\,\Delta\left(1,4\right)+r^{(2)}\,\Delta\left(4,\varpi\right)+g_{3}\left(4\right),$

$M\left(\varpi,\chi\right)=r^{(1)}\,\Delta\left(\varpi,\chi\right)+2r^{(2)}\,\Delta\left(\varpi\right)+r^{(3)}\,\Delta\left(4,\varpi\right)+g_{4}\left(\varpi\right),$

(CC): $b+\left(\mathrm{AC}\right)+s_{12}\,t_{12}\,M\left(\phi\right),$

(CD): $b+\left(\mathrm{AD}\right)+s_{12}\,t_{32}\,M\left(\phi\right)+s_{12}\,t_{13}\,M\left(\phi,\psi\right),$

(CE): $b+\left(\mathrm{AE}\right)+s_{12}\,t_{21}\,M\left(\varpi,\phi\right)+t_{13}\,s_{13}\,M\left(\phi,\chi\right),$

(CF): $b+\left(\mathrm{AF}\right)+s_{12}\,t_{12}\,M\left(\phi,\psi\right)+s_{12}\,t_{23}\,M\left(\phi,\chi\right),$ *Formulae VII*

where $b=s_{12}\,L\left(\phi\,;r,t\right),$

$M\left(\phi\right)\quad=2r^{(1)}\,\Delta_{\phi}+r^{(2)}\,\Delta\left(3,\phi\right)+g_{3}\left(\phi\right),$

$M\left(\phi,\psi\right)=r^{(1)}\,\Delta\left(\psi,\phi\right)+\left(r^{(2)}+r^{(3)}\right)\Delta\left(4,\phi\right)+g_{4}\left(\psi\right),$

$M\left(\phi,\chi\right)=r^{(1)}\,\Delta\left(1,\psi\right)+r^{(2)}\,\Delta\left(1,4\right)+r^{(3)}\,\Delta\left(4,\phi\right)+g_{4}\left(\psi\right).$

Finally,

(DD): $c + (\text{AD}) + s_{32}\, t_{32}\, M(\phi) + (s_{32} t_{13} + s_{13} t_{32})\, M(\phi, \psi) + s_{13} t_{13}\, M(\psi),$

(DE): $c + (\text{AE}) + s_{32}\, t_{21}\, M(\varpi, \phi) + s_{32}\, t_{13}\, M(\phi, \chi) + s_{13}\, t_{21}\, M(\varpi, \psi)$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad + s_{13}\, t_{13}\, M(\psi, \chi).$$

(DF): $c + (\text{AF}) + s_{32}\, t_{12}\, M(\phi, \psi) + s_{32}\, t_{23}\, M(\phi, \chi) + s_{13}\, t_{12}\, M(\psi)$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad + s_{13}\, t_{23}\, M(\psi, \chi),$$

(EE): $d + (\text{AE}) + s_{21}\, t_{21}\, M(\varpi) + (s_{21} t_{13} + s_{13} t_{21})\, M(\varpi, \chi) + s_{13} t_{13}\, M(\chi),$

(EF): $d + (\text{AF}) + s_{21}\, t_{12}\, M(\varpi, \psi) + s_{21}\, t_{23}\, M(\varpi, \chi) + s_{13}\, t_{12}\, M(\psi, \chi)$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad + s_{13}\, t_{13}\, M(\chi),$$

(FF): $s_{12}\, L(\psi, r, t) + s_{23}\, L(\chi, r, t) + (\text{AF}) + s_{12}\, t_{12}\, M(\psi)$
$$\qquad + (s_{12} t_{23} + s_{23} t_{12})\, M(\psi, \chi) + s_{23}\, t_{23}\, M(\chi), \quad \textit{Formulae VIII}$$

where

$$c = s_{32}\, L(\phi, r, t) + s_{13}\, L(\psi, r, t),$$
$$d = s_{21}\, L(\varpi, r, t) + s_{13}\, L(\chi, r, t),$$
$$M(\psi) \quad = 2r^{(1)}\, \Delta_{\psi} + (r^{(2)} + r^{(3)})\, \Delta(4, \psi) + g_4(\psi),$$
$$M(\psi, \chi) = r^{(1)}\, \Delta(\psi, \chi) + r^{(2)}\, \Delta(4, \chi) + r^{(3)}\, \Delta(4, \psi) + g_4(\psi),$$
$$M(\chi, \chi) =: 2r^{(1)}\, \Delta_{\chi} + r^{(2)}\, \Delta(\varpi, \chi) + r^{(3)}(\Delta(4, \chi) - \sigma_{12}) + g_4(\chi).$$

This completes the formulae to the stage indicated, enabling us to solve all cases with three curves or less. It is to be suspected that some of these expressions are identical*, but this question is one of great complexity. They cannot all be identical.

§ 7. We pass on to calculate *the genus of the intersection of two surfaces with a common group of multiple curves*†. We have already explained the method (§ 4), and now pass on to develop the calculation in successive stages.

§ 7·1. *Taking first* $R^{m_1} \equiv (c_{\mu}^{h})^{r}, \quad R^{m_2} \equiv (c_{\mu}^{h})^{s}.$

Breaking R^{m_1} into r surfaces (α) passing simply through c_{μ}, and $(m_1 - r\alpha)$ planes ϖ in arbitrary position, and R^{m_2} similarly into surfaces (α') and ϖ' (the order of α' being also α), we have as components of the degenerate residual curve

$$\alpha_i \alpha_j' \text{ besides } \quad c_{\mu}, \quad \alpha_i \varpi_j', \quad \alpha_i' \varpi_j, \quad \varpi_i \varpi_j.$$

We shall have, in general, one intersection of the components for each intersection, outside c_{μ} of surfaces,

$$\alpha_i \alpha_j \alpha_k', \quad \alpha_i \alpha_j' \alpha_k', \quad \alpha_i \alpha_j' \varpi_k', \quad \alpha_i \alpha_j' \varpi_k;$$
$$\alpha_i \alpha_j \varpi_k', \quad \alpha_i \varpi_j' \varpi_k', \quad \alpha_i' \varpi_j \varpi_k'; \quad \alpha_i \varpi_j \varpi_k, \quad \alpha_i' \varpi_j \varpi_k'.$$

Remembering that for the residual curves $\alpha_i\,\alpha_j'$

$$h' = h + \tfrac{1}{2}\,(\alpha^2 - 2\mu)\,(\alpha - 1)^2,$$

we easily find*, applying § 4,

$$H = \binom{m_1\,m_2}{2} - \left\{ m_1\binom{m_2}{2} + m_2\binom{m_1}{2} \right\} + \mu^2\binom{rs}{2} - \mu rs\,(m_1 m_2 - m_1 - m_2 + 1)$$
$$- l\left\{ r\binom{s}{2} + s\binom{r}{2} \right\} + \mu\left\{ m_1\binom{s}{2} + m_2\binom{r}{2} \right\},$$

$$\textit{Formula IX}$$

l being, as before, $2\mu + R = \mu^2 + \mu - 2h$ (in the case of no actual multiple points). The terms which depend on c_μ we shall call $F(\mu)$. *The case of a common multiple line* is more quickly dealt with directly, giving*

$$H = \binom{m_1 m_2 - rs}{2} - m_1\binom{m_2 - s}{2} - m_2\binom{m_1 - r}{2} - rs\,(m_1 - r)\,(m_2 - s).$$

$$\textit{Formula IX a}$$

§ 7·2. We shall now take *a group of N curves c_{μ_i} with no mutual intersections*. The degenerations will be taken as in § 6·2. It is clear that the result will be derivable from Formula IX, allowing for terms which involve the characteristics of two curves. There is only one group of such terms namely $\Sigma\mu_i\mu_j r_i s_i r_j s_j$, and hence

$$H_0 = \binom{m_1 m_2}{2} - \left\{ m_1\binom{m_2}{2} + m_2\binom{m_1}{2} \right\}$$
$$+ \sum_1^N F(\mu_i) + \sum_1^N \mu_i\mu_j r_i s_i r_j s_j. \qquad \textit{Formula X}$$

§ 7·3. Let us now suppose the curves have mutual intersections (σ_{ij}) of the same nature as explained before. It is convenient to consider first the case of *two curves* c_{μ_1}, c_{μ_2}, as illustrative of the general method. Let then

$$R^{m_1} \equiv (c_{\mu_1})^{r_1}\,(c_{\mu_2})^{r_2}; \quad R^{m_2} \equiv (c_{\mu_1})^{s_1}\,(c_{\mu_2})^{s_2}.$$

Case I. $r_1 \leqslant r_2,\ s_1 \leqslant s_2$. Taking degenerations as in § 6·4, the only partial curves we need consider are $\alpha\alpha'$, $\alpha\beta'$, $\alpha'\beta$, $\beta\beta'$; where $\alpha,\ \beta$ are components of R^{m_1}, and $\alpha',\ \beta'$ of R^{m_2}; the α passing through c_{μ_1} and c_{μ_2} and the β through c_{μ_2} alone. The intersections which arise are $\alpha_i\alpha_j\alpha_k'$, $\alpha_i'\alpha_j'\alpha_k$; $\alpha_i\alpha_j'\beta_k$, $\alpha_i\alpha_j'\beta_k'$, $\alpha_i\alpha_j\beta_k'$, $\alpha_i'\alpha_j'\beta_k$; $\alpha_i\beta_j'\beta_k'$, $\alpha_i'\beta_j\beta_k$, $\alpha_i\beta_j\beta_k'$, $\alpha_i'\beta_j\beta_k'$.

* For these formulae, cf. Severi, "Sulle intersezioni della varietà algebriche....," *Mem. Torino*, ser. 2, t. LII. 1903, p. 61, Chap. III. § 8. The proof here given for Formula IX is quite invalid when c_μ is a complete intersection of two $\bar{a}$-ties, such that it is impossible to have $a \neq \bar{a}$, but an analogous process gives a result equivalent to IX.

By the lemmas these intersections may be counted, and applying § 4:

$$H = H_0 - \sigma \left\{ \frac{r_1^2}{2}(s_2 - s_1) + \frac{s_1^2}{2}(r_2 - r_1) - \frac{1}{2}(r_1 s_2 + r_2 s_1) + r_1 s_1 (r_2 + s_2) \right\}.$$

$$Formula\ XI$$

Case II. $r_1 \leqslant r_2$; $s_1 \geqslant s_2$. As in § 6·5 we break R^{m_1} into r_1 surfaces (α) through c_{μ_1}, c_{μ_2}, $(r_2 - r_1)$ surfaces (β) through c_{μ_2}, and $m - r_1\alpha - \beta(r_2 - r_1)$ planes; and R^{m_2} into s_2 surfaces (α'), $(s_1 - s_2)$ surfaces (γ) through c_{μ_1} and $n - s_2\alpha - \gamma(s_1 - s_2)$ planes. Analysing as before

$$H + \mu = H_0 - \sigma \left\{ \frac{s_2^2}{2}(r_1 - r_2) + \frac{r_1^2}{2}(s_2 - s_1) - \frac{1}{2}(r_1 + r_2)s_1 + s_1 r_2(s_2 + r_1) \right\}$$

$$Formula\ XII$$

where μ is the equivalent multiplicity of the actual multiple points imposed at the points σ.

Passing then to *the general case*, we have surfaces of order m_1, m_2 containing multiply the N curves. We may suppose the multiplicities r_i for (m_1) satisfy the inequalities (A) of § 6·6, so that we have six cases corresponding to those for (m_2). Remembering that for the residual intersection of two surfaces with a common compound curve, $(-h)$ is essentially equal to the number of intersections of the components, and employing the same degenerations as before, we may confine ourselves to a statement of the results obtained. We shall employ the following abbreviations, in addition to those of § 6·6:

$$\binom{r^{(i)}}{2} = R^{(i)} \text{ etc.}; \quad \overset{N}{\underset{i}{\Sigma}} r^{(x)} \{x, \alpha, x\} = k(i, \alpha);$$

$$2 \overset{N}{\underset{a}{\Sigma}} R^{(x)} \Delta_x + \overset{x=N-1, y=N}{\underset{a \leqslant x < y}{\Sigma\Sigma}} r^{(x)} s^{(y)} \{y, x, y\} = \Sigma_a, \quad r^{(i)} + s^{(i)} = \tau_i.$$

Case (AA). $H = H_0 - H'$ where

$$H' = \overset{N}{\underset{1}{\Sigma}} \rho_i (\tau_i - 1) \Delta_i + \underset{i<j}{\Sigma\Sigma} \{\rho_{ij} \Delta_j + (s^{(j)} R^{(i)} + s^{(i)} R^{(j)} + \rho_i \tau_j)\{j, i, j\}$$
$$+ 2(r^{(i)} S^{(j)} + s^{(i)} R^{(j)} + \rho_j \tau_i) \Delta_j\}$$
$$+ \tfrac{1}{2} \underset{i<j<k}{\Sigma\Sigma\Sigma} (\rho_{ij} \tau_k + \rho_{ik} \tau_j + \rho_{jk} \tau_i) \{k, j, k\}.$$

For the remaining cases, to give $H + \mu$ we *subtract* the following quantities from $H_0 - H'$:

(AB): $K(s_{23}, \varpi)$,   (AC): $K(s_{12}, \phi)$,

(AD): $K(s_{32}, \phi) + K(s_{13}, \psi)$,

(AE): $K(s_{21}, \varpi) + K(s_{13}, \chi)$,

(AF): $K(s_{12}, \psi) + K(s_{13}, \chi)$,   *Formulae XIII*

where $K(\theta, \alpha) = \theta(N_a + L_a + L_a') + \binom{\theta}{2}M_a$; in which L_a has been previously defined, and the N_a arise from the apparent nodes of the extra partial terms, and are

$$N(\varpi) = r^{(1)}\Delta_{\varpi} + r^{(2)}\Delta_{\varpi} + r^{(3)}\Delta_4 + \kappa_4,$$
$$N(\phi) = r^{(1)}\Delta_{\phi} + r^{(2)}\Delta_3 + \kappa_3,$$
$$N(\psi) = r^{(1)}\Delta_{\psi} + (r^{(2)} + r^{(3)})\Delta_4 + \kappa_4,$$
$$N(\chi) = r^{(1)}\Delta_{\chi} + r^{(2)}\Delta_{\varpi} + r^{(3)}\Delta_4 + \kappa_4,$$

in which $\overset{N}{\underset{i}{\Sigma}} r^{(x)}\Delta_x \equiv \kappa_i$.

Finally:

$$L'(\varpi) = R^{(1)}(\Delta(1, \varpi) - \sigma_{13}) + R^{(2)}\Delta(2, \varpi) + R^{(3)}\Delta(3, 4)$$
$$+ \{r^{(1)}r^{(2)}\Delta(2, \varpi) + (r^{(1)} + r^{(2)})[r^{(3)}\Delta(2, 4) + k(4, \varpi)]$$
$$+ r^{(3)}k(4, 4)\} + \Sigma_4,$$

$$L'(\phi) = R^{(1)}\Delta(1, \phi) + R^{(2)}\Delta(2, 3)$$
$$+ \{r^{(1)}r^{(2)}\Delta(1, 3) + r^{(1)}k(3, \phi) + r^{(2)}k(3, 3)\} + \Sigma_3,$$

$$L'(\psi) = R^{(1)}\Delta(1, \psi) - R^{(1)}\sigma_{23} + R^{(2)}(\Delta(2, 4) - \sigma_{23}) + R^{(3)}\Delta(3, 4)$$
$$+ \{r^{(1)}r^{(2)}\Delta(1, 4) + r^{(1)}r^{(3)}\Delta(4, \phi) + r^{(1)}k(4, \psi)$$
$$+ r^{(2)}r^{(3)}\Delta(3, 4) + (r^{(2)} + r^{(3)})k(4, 4)\} + \Sigma_4,$$

$$L'(\chi) = R^{(1)}\Delta(1, \chi) + R^{(2)}\Delta(2, \varpi) + R^{(3)}\Delta(3, 4)$$
$$+ \{r^{(1)}r^{(2)}\Delta(1, \varpi) + r^{(1)}r^{(3)}\Delta(1, 4) + r^{(1)}k(4, \chi)$$
$$+ r^{(3)}k(4, 4) + r^{(2)}r^{(3)}\Delta(2, 4) + r^{(2)}k(4, \varpi)\} + \Sigma_4.$$

§ 8. Parallel to the formulae of the preceding section we have a group of formulae, giving the number of intersections of the curve, whose characteristics were sought, with the given base curves. The general method of obtaining this will be to select as simple a surface as possible passing through the curve in question and applying the formulae of § 6, though the comportment at the intersections of two base curves must be examined carefully. We therefore omit a detailed discussion. We can also read off the results from the degeneration method employed, and will state the results in a few of the simpler cases.

N common multiple curves c_{μ_i} in arbitrary position. The curve meets

c_{μ_i} in $\mu_i(r_i m_2 + s_i m_1) - r_i s_i(2\mu_i + R_i)$ points.   *Formula XIV*

Taking secondly two common curves with σ common intersections.
Case I (§ 7·3); the curve meets c_{μ_1} and c_{μ_2} in the number of points given by XIV less $\{\sigma(r_1 s_2 + r_2 s_1 - r_1 s_1)\}$ for c_{μ_1}, and less $\sigma r_1 s_1$ for c_{μ_2}.

Case II. The curve meets c_{μ_1} and c_{μ_2} (outside the σ points of intersection) in the number of points given by XIV less $\sigma s_2 r_2$ for c_{μ_1} and $\sigma r_1 s_1$ for c_{μ_2}; and in addition passes $(r_2 - r_1)(s_1 - s_2)$ times through each point σ.

§ 9. We pass on now to a brief consideration of *the matrix representation of varieties in* S_4. The fundamental results required in this case have been given by Severi[*], and we proceed to consider respectively the cases of surfaces[†] and curves[‡]. Three-dimensionalities (in S_4) we shall call *forms*, in accordance with Severi's nomenclature.

A surface in S_4 is given by precisely the same types of matrix as for curves in S_3, but interpreted as involving five coordinates (homogeneously). Thus the characteristics of a space section are obtainable by methods already known, both in the general case, and in the case when redundant *surfaces* have to be introduced (§ 8). Further the theory of the normality of these matrices proceeds as before. In the first few orders this will suffice to determine the surface, but it has been shewn (I, n. 4) that, in general, the following characteristics are required to determine a surface of S_4 of " general " character.

(1) The order, n;

(2) The rank of a space section, a;

(3) The class, n', or the number to tangent spaces in a pencil;

(4) The number of apparent triple points, t.

From these b, the order of the apparent nodal curve; d, the number of *improper* nodes[§]; j the number of tangent lines through a point[¶], and other less important characteristics are determinable. Of these we know (1) and (2) in all cases. In the case of a matrix whose elements are general the remainder can be calculated by the repeated use of the methods of S, §§ 5, 9, which are too long to be given in detail. $d = 0$ for all such surfaces, and by S, § 9, n. 20, Eqn. (2) j is double the h of a space section.

In the case of surfaces not represented by a general matrix, and for which therefore we have to introduce redundant simpler surfaces, our degeneration method involves points of considerable logical difficulty, though in simple cases it may be successfully employed.

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* In particular in "Intorno ai punti doppi de una superficie...," *Rend. Palermo*, t. xv. 1901, p. 33; "Sulle intersezioni della varietà algebriche...," *Mem. Torino*, ser. 2, t. LII. 1903, p. 61. When all the elements are linear, cf. F. P. White, "The projective generation...," *Proc. Cam. Phil. Soc.* vol. XXI. 1922, Pt. 3, and the references there given. Severi's papers are cited as I and S, respectively.

† Two-dimensionalities, denoted by v_2^n, etc.   ‡ Denoted as in S_3 by c_n^p, etc.

§ Biplanar points, such that every line through one such is a chord of the surface. The genus of a space section through such a point is unaffected (I, n. 1, 5 *et seq.*). *Proper* nodes, biplanar or otherwise, are unessential singularities.

¶ ·Or the first *ceto* in the terminology of S, § 1.

§ 10. To pass on to the case of *curves* in S_4, these are given by a matrix of l rows and $(l+2)$ columns (the curve being thereby given as the intersection of three forms residual to a surface); and if none of the elements of the matrix are of zero order we shall call the matrix normal. Taking first the case when all the elements are general, the order is given by calculating the number of intersections of three surfaces in S_3, residual to a given curve (Salmon, *loc. cit.* in § 6·1). If the matrix be normal it is easily shewn that the order is not less than $\frac{1}{6}l(l+1)(l+2)$. Thus the curves which occur are even more restricted in type than in S_3. They include, of course, all plane curves. Generally *the curves represented by a normal matrix of l rows are complete intersections of forms residual to a surface represented by a normal matrix of $(l-1)$ rows.* The converse is true for $l \leqslant 2$ only. We may calculate the rank* of the curve by the methods of S, §§ 5, 12. The result is extremely complicated in the general case, but for the matrix of two rows

$$\left\| \begin{array}{cccc} (m_1) & (m_1+h_1) & (m_1+h_2) & (m_1+h_3) \\ (m_2) & (m_2+h_1) & (m_2+h_2) & (m_2+h_3) \end{array} \right\| = 0,$$

the result is :

$$R = N\left[3(m_1+m_2) + \sum_{1}^{3} h_i - 3 \right] - 2\,m_1 m_2 \left[m_1^2 + m_1 m_2 + m_2^2 \right.$$
$$\left. + (m_1+m_2)\left(\sum_{1}^{3} h_i \right) + \sum_{1}^{3} h_1 h_2 \right]. \qquad Formula \ XV$$

Some Examples.

Conics: $\left\| \begin{array}{cccc} (0) & (1) & (1) & (2) \\ (0) & (1) & (1) & (2) \end{array} \right\|$; $\quad \left\| \begin{array}{cccc} (1) & (1) & (1) & (2) \\ (0) & (0) & (0) & (1) \end{array} \right\|$.

Cubics: None.

Quartics: $\left\| \begin{array}{cccc} (0) & (1) & (1) & (4) \\ (0) & (1) & (1) & (4) \end{array} \right\|$; $\quad \left\| \begin{array}{cccc} (1) & (1) & (1) & (4) \\ (0) & (0) & (0) & (3) \end{array} \right\|$ (Plane).

$\left\| \begin{array}{cccc} (1) & (1) & (2) & (2) \\ (0) & (0) & (1) & (1) \end{array} \right\|$; $\quad \left\| \begin{array}{cccc} (0) & (1) & (2) & (2) \\ (0) & (1) & (2) & (2) \end{array} \right\|$, $(c_4^1 \text{ of } S_3)^+_1$

$\left\| \begin{array}{cccc} (1) & (1) & (1) & (1) \\ (1) & (1) & (1) & (1) \end{array} \right\|$. (Normal in S_4.)

For all higher orders there are two analogous matrices for the plane curve, which we therefore omit in future. Similarly any complete intersection necessarily occurs, and we omit those which belong to S_3, for which we have two corresponding matrices.

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* Number of tangent lines meeting a plane.

† The upper suffix in this paragraph denotes the genus.

Quintics, Sextics: No others.

Septimics: $\left\|\begin{array}{cccc} (1) & (1) & (1) & (2) \\ (1) & (1) & (1) & (2) \end{array}\right\|$ (c_7^3 of S_4)*.

Octavics: $\left\|\begin{array}{cccc} (1) & (2) & (2) & (2) \\ (0) & (1) & (1) & (1) \end{array}\right\|$, $\left\|\begin{array}{cccc} (2) & (2) & (2) & (2) \\ (0) & (0) & (0) & (0) \end{array}\right\|$

(c_8^5 of S_4 intersection of 3 quadric forms).

Curves of order 9. No others.

This exhausts all curves of order $\leqslant 9$, which can be represented by a general matrix. For $n = 10$ we start matrices of three rows, begining with the c_{10}^6, for which all the elements are linear.

§ 11. Let us next consider briefly the introduction of redundant curves. Let each element equated to zero represent a form in which a c_μ is multiple to the order indicated by the corresponding suffix

$$\left\| \begin{array}{cccc} m_a & (m+h)_{a+\rho} & (m+k)_{a+\sigma} & (m+l)_{a+\tau} \\ n_\beta & (n+h)_{\beta+\rho} & (n+k)_{\beta+\sigma} & (n+l)_{\beta+\tau} \end{array} \right\| = 0.$$

For brevity we omit the inequalities which these integers must satisfy. *The order* of the curve is, *in the general case,* given by
$$M = (m_1^3 + m_1^2 m_2 + m_1 m_2^2 + m_2^3) + (m_1^2 + m_1 m_2 + m_2^2)(h + k + l)$$
$$+ (m_1 + m_2)(hk + hl + kl) - \mu \{(\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3)$$
$$+ (\alpha^2 + \alpha\beta + \beta^2)(\rho + \sigma + \tau) + (\alpha + \beta)(\rho\sigma + \sigma\tau + \rho\tau) + \rho\sigma\tau\},$$
$$\textit{Formula XVI}$$

as may be seen by considering a space section.

A second characteristic may in all cases be determined by applying the formulae to be given in § 12. As an example let us quote the simplest matrix for the space cubic, namely,

$$\left\| \begin{array}{cccc} 1_0 & 1_1 & 1_1 & 1_1 \\ 1_0 & 1_1 & 1_1 & 1_1 \end{array} \right\| = 0,$$

the redundant curve being a line.

A second method of representing those curves, which cannot be represented by a general matrix, is to employ the introduction of one or more redundant *surfaces.* We may represent this by precisely the same matrix as in § 11. The order may in all cases be calculated by §§ 6–8. We may indicate the process for the most important case of a common plane $\mu = 1$.

Then taking the section of the figure by an arbitrary space S_3, the surfaces represented by the elements of the first column equated to zero cut again in curve c_N^R, for which, by Formula IX a, the order $N = mn - \alpha\beta$,

the rank $R = 2m \binom{n-\beta}{2} + 2n \binom{m-\alpha}{2} + 2(\alpha+\beta)(m-\alpha)(n-\beta)$;

* The upper suffix in this paragraph denotes the genus.

and which cuts the common line l, section of the redundant in σ points, where, by Formula XIV,

$$\sigma = m\beta + n\alpha - 2\alpha\beta.$$

Finally, using Formulae II and III with

$$r_1 = s_1 = t_1 = 1;$$
$$r_2 = \alpha + \beta + \rho, \quad s_2 = \alpha + \beta + \sigma, \quad t_2 = \alpha + \beta + \tau;$$
$$m_1 = m + n + h, \quad m_2 = m + n + k, \quad m_3 = m + n + l;$$

we obtain for the order sought

$$(M+h)(M+k)(M+l) - \{\Sigma(M+h)(\omega+\sigma)(\omega+\tau)$$
$$+ 2(\omega+\rho)(\omega+\sigma)(\omega+\tau)\} - (nm-\alpha\beta)(3M+h+k+l-2)$$
$$+ R + (m\beta+n\alpha-2\alpha\beta)(3\omega+\rho+\sigma+\tau-1), \quad Formula\ XVII$$

where $M = m + n$, $\omega = \alpha + \beta$. A second characteristic may, in special cases*, be determined by the formulae of the next section, or in special cases, directly. As an example let us quote a second matrix for the space cubic

$$\left\|\begin{array}{cccc} 0 & 1_0 & 2_1 & 2_1 \\ 0 & 1_0 & 2_1 & 2_1 \end{array}\right\| = 0,$$

the redundant surface being a plane.

§ 12. We pass on to determine formulae for *the curve of intersection of three forms in S_4 residual to a system of curves and surfaces*. We shall use the same terminology for the characteristics, using for a surface

$$2n + a = l.$$

We shall use $i, j, \ldots$ as suffixes for the curves, and $x, y, \ldots$ for the surfaces, so that in the formulae we may have one of the first group equal to one of the second, but never two of the same group.

We now take

$$F^{m_1} \equiv (c_{\mu_i})^{r_i} (V^{n_x})^{r_x}, \qquad i = 1 \ldots N, x = 1 \ldots M,$$

and two others of order m_2, m_3 with s and t in place of r. We shall use the following abbreviations:

$$r_a s_a t_a = \rho_a; \quad \Sigma r_a s_a t_\beta = \rho_{a\beta}; \quad \Sigma \binom{r_a}{2} s_a t_a = \kappa_a,$$

$$\Sigma m_1 s_a t_a = \varpi_a; \quad r_a + s_a + t_a = S_a.$$

It is important to notice that certain symbols are re-employed in a new sense. None of the old abbreviations are employed as such.

The methods being quite parallel to those used for S_3, it will be usually sufficient to state the results.

---

* In particular when a pair such as α and β are both zero; or if one element be zero, the curve being now a complete intersection residual to one multiple surface.

The curves and surfaces are in arbitrary position, namely the surfaces have point intersections only, the curves none with each other, or with the surfaces.

$$H = \binom{m_1 m_2 m_3}{2} - \Sigma m_1 m_2 \binom{m_3}{2}$$
$$+ \sum_{i,x} \left[\{\mu_i\} + \{n_x\} + \mu_i \mu_j \rho_i \rho_j + n_x n_y \left\{ \varpi_x \varpi_y + \sum_{r,s,t} \binom{r_x}{2} s_y t_y \right\} \right.$$
$$\left. + l_x l_y \rho_x \rho_y - n_x l_y \varpi_x \rho_y + \mu_i n_x \rho_i \varpi_x - \mu_i l_x \rho_i \rho_x \right],$$

where the summation is for all i and x, in the manner above explained, and

$$2\{\mu\} = \rho^2(\mu^2 - R) + \mu(A - 3\rho S),$$
$$2\{n\} = (l\rho - n\varpi)^2 + l(12\rho + 6\kappa - \varpi - A) - n \left[2\varpi (m_1 m_2 m_3 - \Sigma m + 1) \right.$$
$$\left. + \Sigma m_3^2 rs + 6\rho + 6\kappa - 2\Sigma m_1 m_2 \binom{t}{2} \right] + 2n'(\rho + \kappa)^*,$$

Formula XVIII

where the suffixes are to be inserted where required, and

$$A \equiv \rho \left[(m_1 + m_2 + m_3) - 2m_1 m_2 m_3 + 4 \right] + \varpi (S - 2),$$

with the same omission of suffixes i or x.

The intersections with $c_{(i)}$ and $V^{(x)}$ are respectively in number

$$\mu_i \varpi_i - \rho_i (3\mu_i + R_i),$$
$$n_x (\Sigma m_1 m_2 t_x - \Sigma m_1 s_x t_x + 3\rho_x) - a_x (\varpi_x - \rho_x) + n_x' \rho_x^*,$$

which are independent, as they clearly should be, of the remaining curves and surfaces.

§ 13·1. We may solve the case of a group of curves with mutual intersections exactly as for S_3 (§ 7·2). For brevity we state the results for the case of *two curves with* σ *intersections. All the expressions of this paragraph* § 13·1 *are to be multiplied by* σ *and substracted from the formulae for* σ *equal to zero.*

Case I. $r_1 \geqslant r_2, \quad s_1 \geqslant s_2, \quad t_1 \geqslant t_2.$

$2H:$ $\rho_2 (S_2 + 8) + (S_2 - 2) \Sigma r_2 s_2 t_1,$

and the intersections with the first and second curves are respectively modified by ρ_2 and $(\Sigma r_2 s_2 t_1 - \rho_2)$.

Case II. $r_1 \geqslant r_2, \quad s_1 \geqslant s_2, \quad t_2 \geqslant t_1.$

Introducing the new terms

$$r_1 s_2 + r_2 s_1 = t, \text{ etc.}, \quad s_1 - s_2 = s_{12}, \text{ etc.},$$

---

* The summations in this equation are, of course, over m_1, m_2, m_3.

the modification for $H + \mu$ is

$$r_2 s_2 \dot{t}_1 + \rho_2 (r_{12} + s_{12}) + r \binom{s_2}{2} + s \binom{r_2}{2} + t_1 t_{21} (t - r_2 s_2)$$
$$+ (t + 2r_2 s_2) \binom{t_1}{2} + s_{12} r_{12} \binom{t_{21}}{2},$$

and those for the intersections, respectively,

$$t_{21} (t - r_2 s_2) + r_2 s_2 t_1; \ \ t_2 (t - 2r_2 s_2) + r_2 s_2 t_1,$$

while the curve has $t_{21} (r_1 s_1 - r_2 s_2)$ branches through each of the σ points.

§ 13·2. The more important case of *a curve c_μ and surface V^n with σ points Q in common* requires a little more attention, and we will give the method in outline. We shall denote the respective multiplicities by r etc., r' etc. If α, β, γ denote, respectively, forms through both constructs, through the curve alone, and through the surface alone, we start by observing that the surfaces $(\alpha\alpha)$, $(\alpha\beta)$, $(\beta\beta)$ pass through c, $\alpha\gamma$ meets it outside Q, $\beta\gamma$ both at Q and outside Q, and $\gamma\gamma$ not at all. Further $\alpha\alpha$, $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, $\gamma\gamma$ have intersections with V along curves; but $\beta\beta$ only in a finite number of points which include the Q. This enables us to insert the unbracketed numbers in Table I, giving the intersections with c and V of the various curves, as functions of σ (cf. § 5). Thence we pass to Table II, giving the number of intersections of the groups of four forms involved, and this in turn gives the bracketed figures of the Table I.

Table I

		Curve	$\alpha\alpha\alpha$	$\alpha\alpha\beta$	$\alpha\beta\beta$	$\alpha\alpha\gamma$	$\alpha\beta\gamma$	$\beta\beta\gamma$
Intersections with	V		$[-\sigma]$	$-\sigma$	$-\sigma$	$[0]$	0	$-\sigma$
,,	,,	c	$[-2\sigma]$	$[-\sigma]$	$[0]$	$-\sigma$	$-\sigma$	$-\sigma$

		Curve	$\beta\beta\beta$	$\alpha\gamma\gamma$	$\gamma\gamma\gamma$	$\beta\gamma\gamma$
Intersections with	V		0	$[0]$	0	0
,,	,,	c	0	0	0	0

In addition only $\beta\beta\gamma$ will pass through the points Q. This and the similar results for the remainder are given directly.

Table II

Group	$\alpha\alpha\alpha\alpha$	$\alpha\alpha\alpha\beta$	$\alpha\alpha\beta\gamma$	$\alpha\beta\beta\gamma$	$\alpha\alpha\alpha\gamma$	$\alpha\alpha\beta\beta$
Number	3σ	2σ	σ	σ	σ	σ

Remainder, 0.

It would be tedious to carry this through in detail.

We will confine ourselves to giving an example of each stage. *Firstly* the curve $\alpha\beta\gamma$ may be considered as the intersection of a

surface $\beta\gamma$ and form α, and therefore meets c in the points other than Q where c meets $\beta\gamma$. Since the total intersection $\beta\gamma \cdot \alpha$ contains a curve (in V) through Q, $\alpha\beta\gamma$ itself will not pass through Q. As intersection of $\alpha\gamma$ and a form β it meets V in the points where $\alpha\gamma \cdot V$ meets β, which on reduction are zero in number. *Secondly* the points $\alpha\alpha\beta\gamma$ being the intersections outside V and c of $\alpha\beta\gamma$ and α are σ in number; and *thirdly* since the points $\alpha\alpha\alpha\gamma$ are the intersections of $\alpha\alpha\alpha$ and γ and are σ in number, the curve $\alpha\alpha\alpha$ must meet V in $(-\sigma)$ points.

The h of the partial curves other than $\alpha\alpha\alpha$ are all zero. We proceed to find this outstanding one by a method, which also affords a useful check on the above calculation. I and J denote its intersections with V and c respectively, r its rank*. Then taking the Jacobian of the three α and two arbitrary spaces (as in S, § 6, n. 15), and considering its intersections with $\alpha\alpha\alpha$ and c,

$$r + I + 2J \sim I + 2\sigma \sim 0.$$

Regarding the two curves in question as a compound curve, intersection residual to V of the three α, we have, from the formula (S, § 13, n. 37) for the rank, $r + 2I \sim -2(\sigma + J)$; and from that for the intersections with V, $J + \sigma \sim 0$. We thus verify the values of I and J, and obtain for our h the value (-2σ).

We shall denote by r etc., r' etc. the multiplicities of c and V respectively. If Θ denote one such symbol, we take as components of the corresponding forms

$$\Theta' \text{ forms } \alpha, \quad \Theta - \Theta' \text{ forms } \beta, \quad \text{if } \Theta \geqslant \Theta',$$

and $\qquad \Theta$ forms α, $\quad \Theta' - \Theta$ forms γ, $\quad$ if $\Theta < \Theta'$,

together with a suitable number of spaces. To cover all cases let us say that we take as components of (m_i), ξ_i forms α, η_i forms β, ζ_i forms γ. $(i = 1, 2, 3.)$ The four cases are taken as

I. $r \geqslant r'$, $s \geqslant s'$, $t \geqslant t'$; $\qquad$ II. $r < r'$, $s < s'$, $t < t'$;

III. $r < r'$, $s < s'$, $t \geqslant t'$; $\qquad$ IV. $r \geqslant r'$, $s \geqslant s'$, $t < t'$;

and we put

$\xi_i \eta_j + \xi_j \eta_i = \zeta_k'$, cyclically, $i \neq j \neq k$; $\quad \xi_1 + \xi_2 + \xi_3 = \xi$, etc.

Then the modification for $H + \mu$ is

$$- \sigma \left\{ \xi_1 \xi_2 \xi_3 (\tfrac{3}{2}\xi + 2\eta + \zeta - \tfrac{3}{2}) + \Sigma \binom{\xi_1}{2} (\xi_1' + \eta_1' + \zeta_1' + \eta_2\eta_3) \right.$$
$$\left. + \Sigma \binom{\eta_1}{2} (\eta_1' + \xi_2\xi_3) + \Sigma \xi_1\zeta_1' (\eta_1 + \zeta_1) + \eta_1\eta_2\zeta_3 (\xi_1 + \xi_3) \right\},$$

---

* This is used in a different sense already, but confusion is impossible.

and those for the intersections with V and c are respectively

$$I_1 = -\sigma\left[\xi_1\xi_2\xi_3 + \Sigma(\xi_1\xi_2\eta_3 + \xi_1\eta_2\eta_3) + \eta_1\eta_2\zeta_3\right],$$

$$I_2 = -\sigma\left[2\,\xi_1\xi_2\xi_3 + \Sigma\,\xi_1\xi_2(\eta_3 + \zeta_3) + \Sigma\,\xi_1\xi_1' + \eta_1\eta_2\zeta_3\right],$$

the summations being over the suffixes 1, 2, 3. *The curve will pass* $\eta_1\eta_2\zeta_3$ *times through each point* Q. We give the results explicitly for the important cases I and II, the results being multiplied by σ and subtracted from those for $\sigma = 0$. ρ, ρ' and S, S' are defined as at the beginning of § 12, and

$$\Sigma\,rst' = \lambda, \quad \Sigma\,r's't = \lambda'.$$

Case I. $\qquad\qquad r \geqslant r', \quad s \geqslant s', \quad t \geqslant t'.$

$$2H:\ \rho'(7 - S + S') + \lambda'(S - S' - 3) + \lambda(S - 1);$$

$$I_1:\ \lambda - \lambda' + \rho'; \quad I_2:\ \lambda' - \rho'.$$

Case II. $\qquad\qquad r < r', \quad s < s', \quad t < t'.$

$$2H:\ \rho(1 + S' - S) + \lambda(S - 2);$$

$$I_1:\ \rho; \quad I_2:\ \lambda - \rho.$$

§ 14. For the sake of completeness we conclude by giving formulae *for the intersections of four forms in* S_4 *residual to a group of multiple curves and surfaces*. It will be sufficient to state the results, the notation being as in § 13, the fourth form being of order m_4 with multiplicities u_i, u_x. In the case of N curves, and M surfaces, we have, if $\tau_a \equiv r_a s_a t_a u_a$:

$$
\begin{aligned}
I = m_1 m_2 m_3 m_4 &- \sum_i\left[\mu_i(\Sigma\,m_1 s_i t_i u_i - 3\tau_i) - R_i\tau_i\right] \\
&- \sum_x\left[\Sigma\,m_1 m_2 t_x u_x - l_x\,\Sigma\,m_1 s_x t_x u_x + (3n_x + n_x' + 3a_x)\,\tau_x\right] \\
&- \underset{x,y}{\Sigma\Sigma}\,n_x n_y\,\Sigma\,r_x s_x t_y u_y, \qquad\qquad \textit{Formula XIX}
\end{aligned}
$$

unmodified Σ denoting summation over the suffixes $1, \ldots 4$. In the case of *two curves with* σ *intersections* we have three cases:

$$\text{I.}\quad r_1 \geqslant r_2, \quad s_1 \geqslant s_2, \quad t_1 \geqslant t_2, \quad u_1 \geqslant u_2.$$

$$\text{II.}\quad r_1 \geqslant r_2, \quad s_1 \geqslant s_2, \quad t_1 \geqslant t_2, \quad u_1 < u_2.$$

$$\text{III.}\quad r_1 \geqslant r_2, \quad s_1 \geqslant s_2, \quad t_1 < t_2, \quad u_1 < u_2.$$

In the two first cases we have to add

$$\sigma\left[\Sigma\,r_2 s_2 t_2 u_1 - 2\tau_2\right],$$

in the last case

$$\sigma\{t_2 u_2(r_2 s_1 + r_1 s_2) + r_1 s_1(t_1 u_2 + t_2 u_1) + u_1 t_1 r_2 s_2 - u_2 t_2 r_1 s_1 - \tau_1 - \tau_2\}.$$

Lastly, in the case of *a curve and surface with* σ *intersections*, we have to add

$$\sigma\{3\,\xi_1\xi_2\xi_3\xi_4 + \Sigma\left[\xi_1\xi_2\xi_3(2\eta_4 + \zeta_4) + \xi_1\eta_3\zeta_4(\xi_2 + \eta_2) + \xi_1\xi_2\eta_3\eta_4\right]\},$$

defining the ξ, η, ζ as before.

§ 15. *Conclusion.* There would be no difficulty in the extension of these formulae to any case other than those in which one or more pairs of surfaces have curve intersection. In this case the *order* of the curve is itself modified (by an amount determinable from § 6), but the determination of a second characteristic requires more fundamental analysis than that which we have been using. We could, by our method, complete the formulae as soon as formulae are found for the intersection of forms passing simply through two surfaces with a curve intersection. This in turn introduces the question as to whether we can replace a *surface* in problems of intersection, etc., by a system of planes with line intersections (such that the lines of section have themselves point intersection), and, if so, how. These questions will also occur in the theory of space of higher dimension.

Hankel Transforms. By E. C. TITCHMARSH, Balliol College, Oxford. [*Communicated by* Mr G. H. HARDY.]

1. The notion of Fourier transforms arises from Fourier's integral formula,

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos xu \, du \int_0^\infty \cos xt f(t) \, dt, \quad \ldots\ldots\ldots(1)$$

which gives the reciprocal relations

$$\left. \begin{aligned} f(x) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu \, F(u) \, du, \\ F(x) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xu f(u) \, du, \end{aligned} \right\} \quad \ldots\ldots\ldots\ldots(2)$$

connecting two functions $f(x)$ and $F(x)$. Each of the two functions so related is said to be the Fourier transform of the other. The formula (1) is ordinarily proved under the hypothesis that $\int_0^\infty |f(t)| \, dt$ exists, and that $f(t)$ is of bounded variation in the neighbourhood of the point $t = x$. When, however, we come to study the relations (2) directly, we find that the ordinary theory of the repeated integral is not enough ; we cannot deduce from it any theorem of the form 'if $f(x)$ satisfies certain conditions, so does $F(x)$, and the reciprocity holds.' The only case in which a satisfactory theory has been established is that in which $\int_0^\infty \{f(t)\}^2 \, dt$ exists ; even in this case the integrals in (2) do not generally exist, and we have to express the reciprocal relations in the form

$$\left. \begin{aligned} f(x) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} \, F(u) \, du, \\ F(x) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dx} \int_0^\infty \frac{\sin xu}{u} f(u) \, du, \end{aligned} \right\} \quad \ldots\ldots\ldots(3)$$

which reduce to (2) when differentiation under the sign of integration is permissible. There is then a theorem of the desired character, viz. that *if* $\int_0^\infty \{f(x)\}^2 \, dx$ *exists, then* $\int_0^\infty \{F(x)\}^2 \, dx$ *exists, and the reciprocity holds.*

Plancherel* has proved this by showing that the reciprocal

* M. Plancherel, *Rendiconti di Palermo*, xxx (1910), 289–335.

relation belongs to a general type of functional transformations, by which one function of integrable* square is transformed into another in the following way. We have two sequences of ortho-gonal functions

$$\phi_1(x), \ \phi_2(x), \ \dots, \ \phi_n(x), \ \dots,$$
$$\psi_1(x), \ \psi_2(x), \ \dots, \ \psi_n(x), \ \dots.$$

We form the 'Fourier series,' with respect to the first sequence, of a function of integrable square $f(x)$,

$$f(x) \sim a_1\phi_1(x) + a_2\phi_2(x) + \dots;$$

then the transform of $f(x)$ in this system is the function $F(x)$ which has the same Fourier coefficients with respect to the second sequence,

$$F(x) \sim a_1\psi_1(x) + a_3\psi_2(x) + \dots.$$

Plancherel's theory is of a very general character, but its applica-tion to the ordinary Fourier transforms is by no means immediate. It depends on the expression of $\cos xy$ in the form†

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos xy = \sum_{n=0}^{\infty} \phi_n(x) \, \psi_n(y),$$

where the ϕ's and ψ's form systems of orthogonal functions in $(0, \infty)$; or at any rate, it is necessary that the result obtained by integrating term by term with respect to x and y should be true. The functions found are

$$\phi_n(x) = \frac{2^{\frac{1}{2}} e^x}{n!} \frac{d^n}{dx^n} (e^{-2x} x^n),$$
$$\psi_n(y) = \frac{2}{\pi^{\frac{1}{2}}} \frac{2}{(1+y^2)^{n+1}} \sum_{k=0}^{n} (-1)^{n+k} \binom{2n+1}{2k} y^{2k}.$$

The possibility of being expressed in this way is, from Plancherel's point of view, the fundamental property of $\cos xy$ which gives rise to Fourier transforms.

2. Reciprocal relations of a more general character than Fourier's can be derived from Hankel's integral formula,

$$f(x) = \int_0^\infty J_\nu(ux) \, u \, du \int_0^\infty J_\nu(ut) \, t f(t) \, dt.$$

We have, in fact, writing $x^{-\frac{1}{2}} f(x)$ for $f(x)$,

$$f(x) = \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux) \, F(u) \, du,$$
$$F(x) = \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux) f(u) \, du.$$

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* I use integrable as meaning 'integrable in the sense of Lebesgue,' unless Riemann integration is referred to explicitly.

† *Loc. cit.* § 20.

Two functions which are connected by relations of this nature we shall call *Hankel transforms* each of the other. No direct theory of this reciprocity appears to exist. To derive such a theory from Plancherel's work, it would be necessary to discover for $(xy)^{\frac{1}{2}} J_\nu(xy)$ an expression like that given above for $\cos xy$. On the other hand, we may notice a certain similarity between the expressions for an arbitrary function by Hankel's repeated integral and by a Fourier-Bessel series; and this suggests that a theory of Hankel transforms might be founded on quite a different property of Bessel functions, namely the orthogonal property which gives rise to the Fourier-Bessel series. It is the object of the present paper to develop this idea. Here the notion of a transform arises in the first place from the correspondence between a function of integrable square, and its Fourier coefficients with respect to a given orthogonal sequence

$$\phi(x),\ \phi(2x),\ \ldots,\ \phi(nx),\ \ldots.$$

This is developed by means of the theory of Riemann integration into a correspondence between two functions of integrable square. At this point expansions in *series* disappear from our theory altogether, and the transformation is expressed in terms of integrals alone. The theory does not aim at the generality of Plancherel's, but its application to Fourier and Hankel transforms appears to be more immediate and natural.

3. We are to consider a number of integrals of the form

$$\int_b^a \phi(x) f(x)\, dx,$$

where $f(x)$ is merely an integrable function, but $\phi(x)$ is bounded and integrable in Riemann's sense, and indeed continuous in the applications. We can turn to account our knowledge of $\phi(x)$ while preserving the generality of $f(x)$ by using the following lemma.

LEMMA. *Let $f(x)$ be an integrable function in the interval (a, b), and $\phi(x)$ a bounded function, integrable in Riemann's sense, in the same interval. The interval is divided into n parts by the points*

$$a = x_0,\, x_1,\, x_2,\, \ldots,\, x_{n-1},\, x_n = b.$$

When the number of these points is so increased that the greatest partial interval tends to zero,

$$\sum_{i=0}^{n-1} \phi(x_i) \int_{x_i}^{x_{i+1}} f(x)\, dx \to \int_a^b \phi(x) f(x)\, dx.$$

Since an integrable function is the difference between two positive integrable functions, we may suppose without loss of generality that $f(x)$ is positive.

Let M_i, m_i, be the upper and lower bounds of $\phi(x)$ in (x_i, x_{i+1}). Then

$$\sum_{i=0}^{n-1} m_i \int_{x_i}^{x_{i+1}} f(x)\, dx \leqq \sum_{i=0}^{n-1} \phi(x_i) \int_{x_i}^{x_{i+1}} f(x)\, dx \leqq \sum_{i=0}^{n-1} M_i \int_{x_i}^{x_{i+1}} f(x)\, dx.$$

But also

$$\sum_{i=0}^{n-1} m_i \int_{x_i}^{x_{i+1}} f(x)\, dx \leqq \int_a^b \phi(x) f(x)\, dx \leqq \sum_{i=0}^{n-1} M_i \int_{x_i}^{x_{i+1}} f(x)\, dx.$$

Hence it is sufficient to prove that

$$\sum_{i=0}^{n-1} (M_i - m_i) \int_{x_i}^{x_{i+1}} f(x)\, dx \to 0.$$

We define the function $\psi(x)$ by the equalities

$$\psi(x) = M_i - m_i;\quad x_i \leqq x < x_{i+1} \quad (i = 0, 1, \ldots, n-1).$$

If σ denotes the set of intervals (and also the measure of the set) in which

$$\psi(x) \geqq \delta,$$

then

$$\int_a^b \psi(x)\, dx \geqq \sigma \delta.$$

On the other hand, since $\phi(x)$ is integrable in Riemann's sense we can make the greatest partial interval so small that

$$\int_a^b \psi(x)\, dx < \omega,$$

ω being any positive number; and then

$$\sigma < \omega/\delta.$$

Hence $\quad \displaystyle\int_a^b \psi(x) f(x)\, dx \leqq \delta \int_a^b f(x)\, dx + \int_\sigma \psi(x) f(x)\, dx$

$$\leqq \delta \int_a^b f(x)\, dx + 2M \int_\sigma f(x)\, dx,$$

if $\phi(x) \leqq M$, so that $\psi(x) \leqq 2M$. By choosing first δ and then ω, we can make the right-hand side as small as we please, which proves the theorem.

COROLLARY. *Suppose that, in the lemma, $\phi(x)$ is replaced by $\phi(\theta x)$. Then*

$$\sum_{i=0}^{n-1} \phi(\theta x_i) \int_{x_i}^{x_{i+1}} f(x)\, dx \to \int_a^b \phi(\theta x) f(x)\, dx,$$

uniformly for $0 < \theta_0 \leqq \theta \leqq \theta_1$.

It is evident, from the previous proof, that we have merely to prove that we can take a division of the interval (a, b),

$$a = x_0, \ x_1, \ \ldots, \ x_{n-1}, \ x_n = b,$$

such that
$$\int_a^b \psi(x, \theta) \, dx < \omega$$

for all prescribed values of θ, $\psi(x, \theta)$ being the function corresponding to $\phi(\theta x)$ in the way that $\psi(x)$ corresponds to $\phi(x)$.

We can choose a number η such that

$$\int_{\theta_0 a}^{\theta_1 b} \psi(\xi) \, d\xi < \omega \theta_0,$$

if the interval $(\theta_0 a, \ \theta_1 b)$ is divided up into intervals each less than η. *A fortiori*

$$\int_{\theta a}^{\theta b} \psi(\xi) \, d\xi < \omega \theta_0$$

for every θ, with the same choice of η.

If we divide the interval $(\theta a, \theta b)$ by the points

$$\theta a = \theta x_0, \ \theta x_1, \ \ldots, \ \theta x_n = \theta b,$$

we have
$$\psi(x, \theta) = \psi(\theta x),$$

for in $(\theta x_i, \ \theta x_{i+1})$, $\phi(x)$ takes the same values as $\phi(\theta x)$ does in (x_i, x_{i+1}). Hence

$$\int_a^b \psi(x, \theta) \, dx = \int_a^b \psi(\theta x) \, dx = \frac{1}{\theta} \int_{\theta a}^{\theta b} \psi(\xi) \, d\xi < \omega,$$

provided that $\text{Max} \, \{\theta(x_{i+1} - x_i)\} < \eta$ for every θ, which is true if $\text{Max}(x_{i+1} - x_i) < \eta / \theta_1$.

4. We may recall the following theorems in the theory of mean convergence[*]. We state them, as they will be used, with an infinite interval of integration.

4·1. If $F_a(x)$ is of integrable square in $(0, \infty)$ for all values of the parameter a, and

$$\lim \int_0^\infty \{F_a(x) - F_b(x)\}^2 \, dx = 0,$$

when $a \to \infty$, $b \to \infty$, in any manner, then there exists a function $F(x)$ of integrable square, defined uniquely almost everywhere, such that

$$\lim_{a \to \infty} \int_0^\infty \{F(x) - F_a(x)\}^2 \, dx = 0.$$

$F_a(x)$ is said to converge in mean (*en moyenne, im Mittel*) to $F(x)$.

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[*] For this theory see E. Fischer, *Comptes Rendus*, CXLIV, 1022–1024; H. Weyl, *Math. Annalen*, LXVII (1909), 225–245; F. Riesz, *Math. Annalen*, LXIX (1910), 449–497; W. H. and G. C. Young, *Quarterly Journal*, XLIV (1913), 49–88; and Plancherel's paper referred to above.

4·2. If $g(x)$ is also of integrable square in $(0, \infty)$, then

$$\lim_{a \to \infty} \int_0^\infty g(x) \{F(x) - F_a(x)\} \, dx = 0.$$

This is easily proved by means of Schwarz's inequality. In particular

$$\lim_{a \to \infty} \int_0^X \{F(x) - F_a(x)\} \, dx = 0.$$

4·3. We have also

$$\lim_{a \to \infty} \int_0^\infty \{F_a(x)\}^2 \, dx = \int_0^\infty \{F(x)\}^2 \, dx,$$

For, by 4·2, $\lim\limits_{a \to \infty} \int_0^\infty F(x) \{F(x) - F_a(x)\} \, dx = 0,$

and the result stated is equivalent to

$$\lim_{a \to \infty} \int_0^\infty \left[\{F(x) - F_a(x)\}^2 - 2F(x) \{F(x) - F_a(x)\} \right] dx = 0.$$

5. *The existence of transforms.* Let $\phi(x)$ be a bounded function integrable in Riemann's sense, in any finite interval, and let

$$\int_0^1 \phi(mx) \phi(nx) \, dx = \begin{array}{ll} 0 & (m \neq n), \\ 1 & (m = n). \end{array}$$

Then $\int_0^\lambda \phi\left(\frac{mx}{\lambda}\right) \phi\left(\frac{nx}{\lambda}\right) dx = \begin{array}{ll} 0 & (m \neq n), \\ \lambda & (m = n). \end{array}$

Let $f(x)$ be a function of integrable square in $(0, \infty)$, and let

$$h_n = \int_{\frac{n}{\lambda}}^{\frac{n+1}{\lambda}} f(x) \, dx, \quad (n = 1, 2, \ldots).$$

Then by Schwarz's inequality,

$$h_n^2 \leqq \int_{\frac{n}{\lambda}}^{\frac{n+1}{\lambda}} \{f(x)\}^2 \, dx \times \int_{\frac{n}{\lambda}}^{\frac{n+1}{\lambda}} 1^2 dx = \frac{1}{\lambda} \int_{\frac{n}{\lambda}}^{\frac{n+1}{\lambda}} \{f(x)\}^2 \, dx.$$

Let $\Phi_n = \sum\limits_{m=1}^n h_m \phi\left(\frac{mx}{\lambda}\right), \quad (n = 1, 2, \ldots).$

We define n and p as functions of λ by the inequalities

$$\frac{n}{\lambda} < a \leqq \frac{n+1}{\lambda}, \quad \frac{n+p+1}{\lambda} \leqq b < \frac{n+p+2}{\lambda},$$

where a and b are any positive numbers, and $a < b$.

Then it follows from § 3 that

$$\lim_{\lambda \to \infty} (\Phi_{n+p} - \Phi_n) = \lim_{\lambda \to \infty} \sum_{i=1}^{p} \phi \left\{ \frac{(n+1)x}{\lambda} \right\} \int_{\frac{n+1}{\lambda}}^{\frac{n+i+1}{\lambda}} f(u)\, du = \int_a^b \phi(ux) f(u)\, du,$$

uniformly for $0 < x_1 \leqq x \leqq x_2$. Again

$$\int_0^\lambda (\Phi_{n+p} - \Phi_n)^2\, dx = \lambda \sum_{i=1}^{p} h^2_{n+i} \leqq \int_{\frac{n+1}{\lambda}}^{\frac{n+p+1}{\lambda}} \{f(x)\}^2\, dx,$$

so that, *a fortiori*,

$$\int_{x_1}^{x_2} (\Phi_{n+p} - \Phi_n)^2\, dx \leqq \int_a^b \{f(x)\}^2\, dx,$$

if $0 < x_1 < x_2 < \lambda$. Making $\lambda \to \infty$, we have

$$\int_{x_1}^{x_2} \left\{ \int_a^b \phi(ux) f(u)\, du \right\}^2 dx \leqq \int_a^b \{f(x)\}^2\, dx.$$

Making $x_1 \to 0$, $x_2 \to \infty$, we have

$$\int_0^\infty \left\{ \int_a^b \phi(ux) f(u)\, du \right\}^2 dx \leqq \int_a^b \{f(x)\}^2\, dx.$$

But $\int_a^b \{f(x)\}^2\, dx \to 0$ when $a \to \infty$, $b \to \infty$; hence, by the theorem referred to in § 4·1, $\int_0^a \phi(ux) f(u)\, du$ converges in mean when $a \to \infty$, to a function $F(x)$ of integrable square in $(0, \infty)$.

$F(x)$ is said to be the *transform* of $f(x)$.

5·1. To include the case of Bessel functions we have to make some slight modifications in the above proof. Instead of $\phi(nx)$ we have $c_n \phi(j_n x)$, where $c_n \to c$, $j_n = An + B + o(1)$, when $n \to \infty$. We now take

$$h_n = \int_{j_n/\lambda}^{j_{n+1}/\lambda} f(x)\, dx,$$

and we find

$$h_n^2 \leqq \frac{K}{\lambda} \int_{j_n/\lambda}^{j_{n+1}/\lambda} \{f(x)\}^2\, dx,$$

where K is an absolute constant. We also take

$$\Phi_n = \sum_{m=1}^{n} h_m c_m \phi \left(\frac{j_m x}{\lambda} \right),$$

$$\frac{j_n}{\lambda} < a \leqq \frac{j_{n+1}}{\lambda}, \quad \frac{j_{n+p+1}}{\lambda} < b < \frac{j_{n+p+2}}{\lambda},$$

and we have

$$\lim_{\lambda \to \infty} (\Phi_{n+p} - \Phi_n) = \lim_{\lambda \to \infty} \sum_{i=1}^{p} c_{n+1} \phi\left(\frac{j_{n+i}x}{\lambda}\right) \int_{j_{n+i}/\lambda}^{j_{n+i+1}/\lambda} f(u)\, du$$

$$= c \int_a^b \phi(ux) f(u)\, du,$$

uniformly in x, when $\lambda \to \infty$.

This would follow at once from the Lemma if $c_n = c$; but it is also true if $c_n \to c$, since

$$\left| \sum_{i=1}^{p} (c_{n+1} - c) \phi\left(\frac{j_{n+i}x}{\lambda}\right) \int_{j_{n+i}/\lambda}^{j_{n+i+1}/\lambda} f(u)\, du \right| \leqq \max_{n+1 \leqq m \leqq n+p} |c_m - c| M \int_a^b |f(u)|\, du.$$

The proof that $\int_0^a \phi(ux) f(u)\, du$ converges in mean to a function of integrable square can now be completed as before.

5·2. If we put

$$\phi(x) = x^{\frac{1}{2}} J_\nu(x), \quad c_n = (\tfrac{1}{2} j_n)^{\frac{1}{2}} J_{\nu+1}(j_n),$$

where $\nu \geqq -\frac{1}{2}$, and j_n is the nth positive zero of $J_\nu(x)$, then the conditions of the previous section are satisfied. Hence when $a \to \infty$,

$$F_a(x) \equiv \int_0^a (ux)^{\frac{1}{2}} J_\nu(ux) f(u)\, du$$

converges in mean to a function $F(x)$ of integrable square.

$F(x)$ is said to be the *Hankel transform* of $f(x)$.

Hankel transforms reduce to Fourier cosine transforms in the case $\nu = -\frac{1}{2}$, and to Fourier sine transforms for $\nu = \frac{1}{2}$.

6. *An explicit formula for $F(x)$.*

We return to the general case of §5. Let $\lambda(x)$ be a function whose square is integrable over any finite interval, and let

$$\mu(x, u) = \int_0^x \lambda(t) \phi(ut)\, dt.$$

Then it follows from §4·2 that

$$\lim_{a \to \infty} \int_0^x \lambda(t) F_a(t)\, dt = \int_0^x \lambda(t) F(t)\, dt.$$

But $\quad\displaystyle\int_0^x \lambda(t) F_a(t)\, dt = \int_0^x \lambda(t)\, dt \int_0^a \phi(ut) f(u)\, du$

$$= \int_0^a \mu(x, u) f(u)\, du.$$

Hence $\quad\displaystyle\int_0^\infty \mu(x, u) f(u)\, du = \int_0^x \lambda(t) F(t)\, dt.$

Hence we have almost everywhere

$$F(x) = \frac{1}{\lambda(x)} \frac{d}{dx} \int_0 \mu(x, u) f(u) \, du.$$

In the case $\phi(x) = x^{\frac{1}{2}} J_\nu(x)$, let $\lambda(x) = x^{\nu + \frac{1}{2}}$. Then

$$\mu(u, x) = \int_0^x t^{\nu+1} J_\nu(ut) \, dt = x^{\nu+1} J_\nu(ux)/u.$$

Thus $\qquad F(x) = x^{-\nu - \frac{1}{2}} \frac{d}{dx} \left\{ x^{\nu+1} \int_0^\infty u^{-\frac{1}{2}} J_{\nu+1}(ux) f(u) \, du \right\}.$

7. We have now to prove a crude form of Hankel's integral theorem, viz. that *if $f(x)$ is integrable in $(0, a)$, and $x < a$, then*

$$\int_0^x t^{\nu + \frac{1}{2}} f(t) \, dt = x^{\nu+1} \int_0^\infty J_{\nu+1}(ux) \, du \int_0^a J_\nu(ut) \, t^{\frac{1}{2}} f(t) \, dt.$$

The theorem is true if we can invert the order of integration on the right; for

$$\int_0^\infty J_{\nu+1}(ux) J_\nu(ut) \, du = \begin{matrix} t^\nu/x^{\nu+1}, & (t < x), \\ 0 & , & (t > x). \end{matrix}$$

To justify the inversion, it is sufficient to prove that

$$\lim_{N \to \infty} \int_0^A t^{\frac{1}{2}} f(t) \, dt \int_N^\infty J_{\nu+1}(ux) J_\nu(ut) \, du = 0.$$

Since $\qquad \lim_{N \to \infty} \int_N^\infty J_{\nu+1}(ux) J_\nu(ut) \, du = 0$

for all values of t, it is sufficient to prove that the modulus of

$$t^{\frac{1}{2}} \int_N^\infty J_{\nu+1}(ux) J_\nu(ut) \, du$$

is less than an integrable function of t independent of N^*. Since $t^{\frac{1}{2}} \int_0^x$ satisfies this condition, we can consider instead

$$t^{\frac{1}{2}} \int_0^N J_{\nu+1}(ux) J_\nu(ut) \, du = t^{-\frac{1}{2}} \int_0^{tN} J_{\nu+1}\left(\frac{xz}{t}\right) J_\nu(z) \, dz$$

$$= t^{-\frac{1}{2}} \int_0^k + t^{-\frac{1}{2}} \int_k^{tN}, \quad (tN > k)$$

$$= t^{-\frac{1}{2}} \int_0^{tN}, \qquad (tN < k),$$

---

* C. de la Vallée-Poussin, *Cours d'Analyse*, vol. I (ed. 3), p. 264, Theorem 2.

k being an absolute constant. Since $J_{\nu+1}\left(\dfrac{xz}{t}\right)$ and $z^{-\nu}J_\nu(z)$ are bounded for $z < k$ and all values of t, $\displaystyle\int_0^k$ is bounded in the first case, and $\displaystyle\int_0^{tN}$ in the second. We have still to consider $\displaystyle\int_k^{tN}$; here we can substitute the asymptotic expansions for the Bessel functions, and we get

$$t^{-\frac{1}{2}}\int_k^{tN} J_{\nu+1}\left(\frac{xz}{t}\right) J_\nu(z)\,dz = O\int_k^{tN} \sin\left(\frac{xz}{t} - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right)\cos\left(z - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right)\frac{dz}{z}$$

$$= O\int_k^{tN}\left[\sin z\left(\frac{x}{t} - 1\right) - \cos\left\{z\left(\frac{x}{t} + 1\right) - \nu\pi\right\}\right]\frac{dz}{z}$$

$$= O(1),$$

for all values of N and t.

8. *The reciprocity.* The formula of the preceding section may be written

$$\int_0^x t^{\nu+\frac{1}{2}} f(t)\,dt = x^{\nu+1}\int_0^\infty u^{-\frac{1}{2}} J_{\nu+1}(ux)\, F_a(u)\,du, \quad (x < a).$$

Let $\mathscr{F}(x)$ be the transform of $F(x)$. Then, by § 6,

$$\int_0^x t^{\nu+\frac{1}{2}}\,\mathscr{F}(t)\,dt = x^{\nu+1}\int_0^\infty u^{-\frac{1}{2}} J_{\nu+1}(ux)\, F(u)\,du.$$

Subtracting, we have, if $a > x$,

$$\int_0^x t^{\nu+\frac{1}{2}}\{\mathscr{F}(t) - f(t)\}\,dt = x^{\nu+1}\int_0^\infty u^{-\frac{1}{2}} J_{\nu+1}(ux)\{F(u) - F_a(u)\}\,du.$$

But, by § 4·2,

$$\lim_{a\to\infty}\int_0^\infty u^{-\frac{1}{2}} J_{\nu+1}(ux)\{F(u) - F_a(u)\}\,du = 0.$$

Hence

$$\int_0^x t^{\nu+\frac{1}{2}}\{\mathscr{F}(t) - f(t)\}\,dt = 0$$

for all values of x, so that we have almost everywhere

$$\mathscr{F}(x) = f(x).$$

9. *The relation between integrals of squares.*

Let
$$f_1(x) = f(x), \quad x < a,$$
$$= 0, \qquad x > a.$$

Then $F_a(x)$ is the transform of $f_1(x)$, so that conversely $f_1(x)$ is the transform of $F_a(x)$, and we have

$$\int_0^b (ux)^{\frac{1}{2}} J_\nu(ux) F_a(u)\, du \to f_1(x)$$

in mean, when $b \to \infty$. Hence, by § 4·2,

$$\lim_{b \to \infty} \int_0^a f(x) \left\{ f_1(x) - \int_0^b (ux)^{\frac{1}{2}} J_\nu(ux) F_a(u)\, du \right\} dx = 0.$$

But
$$\int_0^a f(x) f_1(x)\, dx = \int_0^a \{f(x)\}^2\, dx,$$

and

$$\int_0^a f(x)\, dx \int_0^b (ux)^{\frac{1}{2}} J_\nu(ux) F_a(u)\, du = \int_0^b F_a(u)\, du \int_0^a (ux)^{\frac{1}{2}} J_\nu(ux) f(x)\, dx$$
$$= \int_0^b \{F_a(u)\}^2\, du.$$

Hence
$$\int_0^a \{f(x)\}^2\, dx = \int_0^\infty \{F_a(u)\}^2\, du,$$

and making $a \to \infty$ and using § 4·3, we have

$$\int_0^\infty \{f(x)\}^2\, dx = \int_0^\infty \{F(u)\}^2\, du. \quad \ldots\ldots\ldots\ldots(1)$$

If also $G(x)$ is the transform of $g(x)$, then $F(x) + G(x)$ is the transform of $f(x) + g(x)$, and we have as an immediate corollary of (1)

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F(x)\, G(x)\, dx. \quad \ldots\ldots\ldots\ldots(2)$$

On the Fifth Book of Euclid's Elements (Addendum* to Fifth Paper). By M. J. M. HILL, Sc.D., LL.D., F.R.S., Astor Professor of Mathematics in the University of London.

Euc. V, 24, Corollary 1.

If $A > B$, if $(A : C) = (X : Z)$, and if $(B : C) = (Y : Z)$, then $(A - B : C) = (X - Y : Z)$.

Compare $(A - B : C)$ with *any* rational fraction *whatever*, r/s.

Then (see Art. 5, p. 89, of the Third Paper of this series in Volume XXII of the *Transactions*) it is sufficient to consider only the following alternatives.

(i) If $(A - B : C) > r/s$, then $s(A - B) > rC$. Choose n so that $n[s(A-B)-rC] > 2C$. Choose uC the *greatest* multiple of C which is *less* than nsA.

$$\therefore uC < nsA \gtreqless (u + 1)C.$$

Choose vC the *least* multiple of C which is *greater* than nsB†.

$$\therefore vC > nsB \geqq (v - 1)C.$$

(ii) If $(A-B : C) < r/s$, then $s(A-B) < rC$. Choose n so that $n[rC-s(A-B)] > 2C$. Choose uC the *least* multiple of C which is *greater* than nsA.

$$\therefore uC > nsA \geqq (u - 1)C.$$

Choose vC the *greatest* multiple of C which is *less* than nsB‡.

$$vC < nsB \gtreqless (v + 1)C.$$

(Then it is shown in the footnotes below that, of the integers u, v determined as above, u is in each case greater than v.)

Now $0 < nsA - uC \gtreqless C$,	Now $0 < uC - nsA \gtreqless C$,
and $0 < vC - nsB \gtreqless C$.	and $0 < nsB - vC \gtreqless C$.
$\therefore 0 < ns(A-B)-(u-v)C \gtreqless 2C$,	$\therefore 0 < (u-v)C - ns(A-B) \gtreqless 2C$,
but $ns(A - B) - nrC > 2C$.	but $nrC - ns(A-B) > 2C$.
$\therefore nrC < (u - v)C$.	$\therefore nrC > (u - v)C$.
$\therefore v + nr < u$.	$\therefore v + nr > u$.
$\therefore vZ + nrZ < uZ$.	$\therefore vZ + nrZ > uZ$.
Now $nsA > uC$,	Now $nsA < uC$,
and $(A : C) = (X : Z)$.	and $(A : C) = (X : Z)$.
$\therefore nsX > uZ$.	$\therefore nsX < uZ$.

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* This is intended to be read after p. 461 of Volume XXII of the *Transactions*. It contains proofs of the two corollaries to Euc. V, 24, on the same lines as the proof of the main proposition there given.

† In this case	‡ In this case
$$nsA - nsB - nrC > 2C.$$	$uC > nsA > nsB > vC.$
$$\therefore nsA - C > nsB + C,$$	
but $nsA - C \gtreqless uC$	
and $nsB + C \geqq vC$.	
$$\therefore uC > vC.$$	$$\therefore uC > vC.$$
$$\therefore u > r.$$	$$\therefore u > v.$$

Also $nsB < vC$,   Also $nsB > vC$,

and $(B:C) = (Y:Z)$.   and $(B:C) = (Y:Z)$.

$\therefore nsY < vZ$. $\therefore nsY > vZ$.

$\therefore nsX > uZ > vZ + nrZ$. $\therefore nsX < uZ < vZ + nrZ$.

$\therefore nsX > nsY + nrZ$. $\therefore nsX < nsY + nrZ$.

$\therefore s(X - Y) > rZ$. $\therefore s(X - Y) < rZ$.

$\therefore (X - Y : Z) > r/s$. $\therefore (X - Y : Z) < r/s$.

Hence r/s cannot lie between $(A - B : C)$ and $(X - Y : Z)$ but r/s represents *any* rational fraction *whatever*. Therefore *no* rational fraction *whatever* can lie between

$$(A - B : C) \text{ and } (X - Y : Z).$$
$$\therefore (A - B : C) = (X - Y : Z).$$

Euc. V, 24, Corollary 2.

If $(A_1 : B) = (X_1 : Y)$ ....(1),

if $(A_2 : B) = (X_2 : Y)$ ....(2),

....

if $(A_m : B) = (X_m : Y)$ ....(m),

then $(A_1 + A_2 + ... + A_m : B) = (X_1 + X_2 + ... + X_m : Y)$.

The easiest way to do this is by successive applications of the main proposition. The following proof (including that of the main proposition) is on the same lines as the proof of the main proposition on pp. 460–1 of Vol. XXII.

Compare $(A_1 + A_2 + ... + A_m : B)$ with *any* rational fraction *whatever*, r/s. It is necessary to consider only the two alternatives.

(i) If $(A_1 + A_2 + ... + A_m : B) > r/s$,   (ii) If $(A_1 + A_2 + ... + A_m : B) < r/$

then $s(A_1 + A_2 + ... + A_m) > rB$.   then $s(A_1 + A_2 + ... + A_m) < rB$.

Choose n so large that   Choose n so large that

$n[s(A_1 + A_2 + ... + A_m) - rB] > mB$. $n[rB - s(A_1 + A_2 + ... + A_m)] > m.$

Let $u_1 B, u_2 B, ..., u_m B$ be the *greatest* multiples of B which are *less* than $nsA_1, nsA_2, ..., nsA_m$ respectively.   Let $u_1 B, u_2 B, ..., u_m B$ be the *le* multiples of B which are *greater* th $nsA_1, nsA_2, ..., nsA_m$ respectively.

$\therefore 0 < nsA_1 - u_1 B \gtreqless B$, $0 < u_1 B - nsA_1 \gtreqless B$,

$0 < nsA_2 - u_2 B \gtreqless B$, $0 < u_2 B - nsA_2 \gtreqless B$,

....

$0 < nsA_m - u_m B \gtreqless B$. $0 < u_m B - nsA_m \gtreqless B$.

$0 < ns(A_1 + A_2 + ... + A_m) - (u_1 + u_2 + ... + u_m)B \gtreqless mB$, $\therefore 0 < (u_1 + u_2 + ... + u_m)B - ns(A_1 + A_2 + ... + A_m)$

but $ns(A_1 + A_2 + ... + A_m) - nrB > mB$.   but $nrB - ns(A_1 + A_2 + ... + A_m) > m.$

$\therefore (u_1 + u_2 + ... + u_m)B > nrB$. $\therefore (u_1 + u_2 + ... + u_m)B < nrB$.

$\therefore (u_1 + u_2 + ... + u_m) > nr$. $\therefore (u_1 + u_2 + ... + u_m) < nr$.

Now $nsA_1 > u_1 B,$ Now $nsA_1 < u_1 B,$

but $(A_1 : B) = (X_1 : Y).$ but $(A_1 : B) = (X_1 : Y).$

$\therefore\ nsX_1 > u_1 Y.$ $\therefore\ nsX_1 < u_1 Y.$

Similarly $nsX_2 > u_2 Y,$ Similarly $nsX_2 < u_2 Y,$

....

$nsX_m > u_m Y,$ $nsX_m < u_m Y.$

$\therefore ns(X_1 + X_2 + ... + X_m) > (u_1 + u_2 + ... + u_m)\,Y.$ $\therefore ns(X_1 + X_2 + ... + X_m) < (u_1 + u_2 + ... + u_m)\,Y.$

$\therefore\ ns(X_1 + X_2 + ... + X_m) > nrY.$ $\therefore\ ns(X_1 + X_2 + ... + X_m) < nrY.$

$\therefore\ (X_1 + X_2 + ... + X_m : Y) > r/s.$ $\therefore\ (X_1 + X_2 + ... + X_m : Y) < r/s.$

Therefore r/s cannot lie between

$(A_1 + A_2 + ... + A_m : B)$ and $(X_1 + X_2 + ... + X_m : Y).$

But r/s represents *any* rational fraction *whatever*.

Therefore *no* rational fraction *whatever* can lie between

$(A_1 + A_2 + ... + A_m : B)$ and $(X_1 + X_2 + ... + X_m : Y).$

$\therefore\ (A_1 + A_2 + ... + A_m : B) = (X_1 + X_2 + ... + X_m : Y).$

The Magnetic Field of a Helix. By Dr H. LAMB.

The coils used in electromagnetic experiments are as a rule so closely wound that they may be regarded for most purposes as cylindrical conducting shells round which a uniform current flows. For instance the field in the interior of a long solenoid is sensibly uniform except near the windings. For this reason the following investigation can claim little more than mathematical interest; but the form of the results is remarkable, and lends itself if need be to numerical computation, by the methods of the theory of Bessel Functions.

1. It is required to find the magnetic field due to a unit current flowing in the helix whose equations are

$$\xi = a \cos \phi, \quad \eta = a \sin \phi, \quad z = k\phi \ \dots\dots\dots\dots(1),$$
where
$$k = a \tan \alpha \dots\dots\dots\dots\dots\dots\dots(2),$$

if α be the slope.

Let us first calculate the magnetic force at a point on the axis. Since it is indifferent what point is taken, we choose the origin. If we write

$$R = \sqrt{(\xi^2 + \eta^2 + \zeta^2)} \ \dots\dots\dots\dots\dots\dots(3),$$

the longitudinal component is

$$\gamma_0 = \int \frac{\xi d\eta - \eta d\xi}{R^3} = \int_{-\infty}^{\infty} \frac{a^2 d\phi}{(a^2 + k^2\phi^2)^{\frac{3}{2}}} = \frac{2}{k} \ \dots\dots\dots(4).$$

Since the current per unit length of the axis is $1/2\pi k$, this agrees with the known result for a closely wound spiral.

The transverse components are

$$\alpha_0 = \int \frac{\eta d\zeta - \zeta d\eta}{R^3} = ka \int_{-\infty}^{\infty} \frac{\sin \phi - \phi \cos \phi}{(a^2 + k^2\phi^2)^{\frac{3}{2}}} d\phi = 0 \dots\dots(5),$$

$$\beta_0 = \int \frac{\zeta d\xi - \xi d\zeta}{R^3} = - ka \int_{-\infty}^{\infty} \frac{\phi \sin \phi + \cos \phi}{(a^2 + k^2\phi^2)^{\frac{3}{2}}} d\phi \ \dots\dots(6).$$

By a partial integration we find

$$\int_0^{\infty} \frac{\phi \sin \phi d\phi}{(a^2 + k^2\phi^2)^{\frac{3}{2}}} = \frac{1}{k^2} \int_0^{\infty} \frac{\cos \phi d\phi}{(a^2 + k^2\phi^2)^{\frac{1}{2}}} = \frac{1}{k^3} K_0 \left(\frac{a}{k} \right) \dots\dots(7),$$

where K_0 is the Bessel Function of zero order, of pure imaginary argument, of the second kind*. Moreover

$$\int_0^{\infty} \frac{\cos \phi d\phi}{(a^2 + k^2\phi^2)^{\frac{3}{2}}} = \frac{1}{k^2 a} K_1 \left(\frac{a}{k} \right) \ \dots\dots\dots\dots(8).$$

---

* The notation of Watson, *Theory of Bessel Functions*, pp. 172, 185, is followed in this paper.

Since
$$K_0(z) + \frac{1}{z} K_1(z) = -K_1'(z) \quad \ldots\ldots\ldots\ldots(9),$$

by a standard formula of reduction*, we have finally

$$\beta_0 = \frac{2a}{k^2} K_1'\left(\frac{a}{k}\right) \quad \ldots\ldots\ldots\ldots\ldots(10).$$

The magnetic force at any point O on the axis of the helix consists therefore of a longitudinal component (γ_0) and a transverse component (β_0) at right angles to the line drawn from O to the nearest point of the helix. The ratio of the two components is

$$\beta_0/\gamma_0 = zK_1'(z)\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where $z = a/k = \cot\alpha$. The magnitude of this ratio gives an indication of the degree to which the field, in the neighbourhood of the axis, deviates from uniformity. A few numerical values are appended:

$a/k = 5,$	10,	15,	25,
$\alpha = 11° 19',$	$5° 43',$	$3° 49',$	$2° 17',$
$\pi k/a = ·628,$	·314,	·209,	·126,
$-\beta_0/\gamma_0 = 2·25 \times 10^{-2},$	$1·96 \times 10^{-4},$	$1·57 \times 10^{-6},$	$8·84 \times 10^{-11}.$

The quantity $\pi k/a$ measures the ratio of the "step" of the helix to its diameter.

2. We proceed to calculate the field at any point P not on the axis. The coordinates of P being

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \quad \ldots\ldots\ldots(12),$$

we write
$$R = \sqrt{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}}$$
$$= \sqrt{\{a^2 - 2ar\cos(\phi-\theta) + r^2 + (z-k\phi)^2\}} \quad \ldots\ldots(13).$$

The longitudinal force at P is then

$$\gamma = \int \frac{(y-\eta)\,d\xi - (x-\xi)\,d\eta}{R^3} = \int_{-\infty}^{\infty} \frac{a^2 - ar\cos(\phi-\theta)}{R^3}\,d\phi\ldots(14).$$

It is obvious from geometrical considerations that γ must be a periodic function of $\theta - z/k$, of period 2π. We may therefore assume

$$\gamma = R_0 + \sum_{1}^{\infty} \left\{R_n \cos n\left(\theta - \frac{z}{k}\right) + S_n \sin n\left(\theta - \frac{z}{k}\right)\right\} \ldots(15),$$

where the coefficients are functions of r, to be determined.

Moreover, since γ must satisfy the equation

$$\frac{\partial^2\gamma}{\partial r^2} + \frac{1}{r}\frac{\partial\gamma}{\partial r} + \frac{1}{r^2}\frac{\partial^2\gamma}{\partial\theta^2} + \frac{\partial^2\gamma}{\partial z^2} = 0 \quad \ldots\ldots\ldots\ldots(16),$$

we must have
$$\frac{\partial^2 R_0}{\partial r^2} + \frac{1}{r}\frac{\partial R_0}{\partial r} = 0 \quad \ldots\ldots\ldots\ldots\ldots(17),$$

$$\frac{\partial^2 R_n}{\partial r^2} + \frac{1}{r}\frac{\partial R_n}{\partial r} - n^2\left(\frac{1}{r^2} + \frac{1}{k^2}\right) R_n = 0 \quad \ldots\ldots\ldots\ldots(18),$$

* Watson, p. 79.

with a similar equation for S_n. Since γ must be finite for $r = 0$, we have, for the internal field,

$$\gamma = A_0 + \overset{\infty}{\underset{1}{\Sigma}} \left\{ A_n \cos n\left(\theta - \frac{z}{k}\right) + B_n \sin n\left(\theta - \frac{z}{k}\right) \right\} I_n\left(\frac{nr}{k}\right) \ldots(19),$$

a series of "Kapteyn type."

In determining the coefficients A_0, A_n, B_n, by comparison with (14), we may without loss of generality put $z = 0$. We have, then,

$$2\pi A_0 = \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{a^2 - ar\cos(\phi - \theta)}{\{a^2 - 2ar\cos(\phi - \theta) + r^2 + k^2\phi^2\}^{\frac{3}{2}}} \, d\phi \, d\theta \ldots(20).$$

The order of integration may be inverted. If we integrate first with respect to θ, writing $\theta = \omega + \phi$ we get

$$\int_{-\phi}^{2\pi - \phi} \frac{a(a - r\cos\omega)\,d\omega}{(a^2 - 2ar\cos\omega + r^2 + k^2\phi^2)^{\frac{3}{2}}} \ldots \ldots \ldots(21),$$

where the limits of integration may be replaced by 0 and 2π, since it is immaterial over what complete cycle of ω the integral is taken. If we now go back to the original order, the integration with respect to ϕ gives

$$\int_{-\infty}^{\infty} \frac{d\phi}{(\rho^2 + k^2\phi^2)^{\frac{3}{2}}} = \frac{2}{k\rho^2} \ldots \ldots \ldots \ldots(22),$$

where

$$\rho^2 = a^2 - 2ar\cos\omega + r^2 \ldots \ldots \ldots \ldots(23).$$

Thus (20) is equivalent to

$$2\pi A_0 = \frac{2a}{k} \int_0^{2\pi} \frac{(a - r\cos\omega)\,d\omega}{a^2 - 2ar\cos\omega + r^2} \ldots \ldots \ldots(24).$$

The definite integral is equal to $2\pi/a$ or 0, according as $r \lessgtr a$. Hence on the present hypothesis that $r < a$ we have

$$A_0 = 2/k \ldots \ldots \ldots \ldots \ldots \ldots(25)$$

in agreement with (4).

Again,

$$\pi A_n I_n\left(\frac{nr}{k}\right) = \int_0^{2\pi} \cos n\theta \int_{-\infty}^{\infty} \frac{a^2 - ar\cos(\phi - \theta)}{\{a^2 - 2ar\cos(\phi - \theta) + r^2 + k^2\phi^2\}^{\frac{3}{2}}} \, d\phi \, d\theta \ldots(26).$$

Inverting the order and putting $\theta = \omega + \phi$, as before, we obtain

$$\int_{-\infty}^{\infty} \cos n\phi \int_0^{2\pi} \frac{a(a - r\cos\omega)\cos n\omega}{(\rho^2 + k^2\phi^2)^{\frac{3}{2}}} \, d\omega \, d\phi \ldots\ldots(27).$$

Reversing again, and integrating first with respect to ϕ, we note that

$$(a - r\cos\omega) \int_{-\infty}^{\infty} \frac{\cos n\phi\,d\phi}{(\rho^2 + k^2\phi^2)^{\frac{3}{2}}} = -\frac{\partial}{\partial a} \int_{-\infty}^{\infty} \frac{\cos n\phi\,d\phi}{(\rho^2 + k^2\phi^2)^{\frac{1}{2}}}$$

$$= -\frac{2}{k}\frac{\partial}{\partial a} K_0\left(\frac{n\rho}{k}\right) \ldots\ldots(28).$$

Now when $r < a$ we have *

$$K_0\left(\frac{n\rho}{k}\right) = K_0\left(\frac{na}{k}\right) I_0\left(\frac{nr}{k}\right) + 2\sum_1^\infty K_s\left(\frac{na}{k}\right) I_s\left(\frac{nr}{k}\right) \cos s\omega \ldots(29).$$

Hence $\pi A_n I_n\left(\frac{nr}{k}\right) = -\frac{2a}{k}\frac{\partial}{\partial a}\int_0^{2\pi} K_0\left(\frac{n\rho}{k}\right)\cos n\omega\, d\omega$

$$= -\frac{4\pi na}{k^2} K_n'\left(\frac{na}{k}\right) I_n\left(\frac{nr}{k}\right),$$

or $\qquad A_n = -\frac{4na}{k^2} K_n'\left(\frac{na}{k}\right)$ ....(30).

It is not necessary to go through the corresponding process to find B_n. It is evident from geometrical considerations that (when $z = 0$) γ must be an even function of θ, so that $B_n = 0$.

The general value of γ for internal points is therefore

$$\gamma = \gamma_0\left\{1 - \frac{2a}{k}\sum_1^\infty n K_n'\left(\frac{na}{k}\right) I_n\left(\frac{nr}{k}\right)\cos n\left(\theta - \frac{z}{k}\right)\right\}\ldots(31).$$

The corresponding expression for the magnetic potential is

$$\Omega = -\int \gamma dz = -\gamma_0\left\{z + 2a\sum_1^\infty K_n'\left(\frac{na}{k}\right) I_n\left(\frac{nr}{k}\right)\sin n\left(\theta - \frac{z}{k}\right)\right\}\ldots(32).$$

Hence at the origin we have

$$\alpha_0 = -\left(\frac{\partial \Omega}{\partial r}\right)_{\theta=0} = 0, \quad \beta_0 = -\left(\frac{\partial \Omega}{\partial r}\right)_{\theta=\frac{1}{2}\pi} = \frac{2a}{k^2} K_1'\left(\frac{a}{k}\right)\ldots(33),$$

as already found in § 1.

The investigation for the external field would follow a similar course. We should find

$$\gamma = \gamma_0\left\{0 - \frac{2a}{k}\sum_1^\infty I_n'\left(\frac{na}{k}\right) K_n\left(\frac{nr}{k}\right)\cos n\left(\theta - \frac{z}{k}\right)\right\}\ \ldots(34),$$

where the cipher represents the value of the integral in (24) when $r > a$.

The series in (31) and (34) are the Fourier expansions of a function of θ which, together with its derivatives, is evidently continuous whatever the value of r other than a. They are therefore necessarily convergent. The known properties of I_n and K_n indicate (as we should expect) that the convergence will be more rapid the greater the value of a/k, i.e. the smaller the angle α of the spiral; also that as regards internal points the convergence will be slower the greater the distance from the axis. These features are illustrated by the following table, which relates to the case of a rather open spiral, where $a/k = 5$, or $\pi k/a = \cdot628$, so that the "step" is about five-eighths of the diameter. The corresponding value of α is $11°\ 19'$. The results are given for $z = 0$. For other values of z the angle θ must be replaced by $(\theta - z/k)$.

* Watson, p. 361.

r/a	γ/γ_0
·0	1
·2	$1 + ·0254 \cos \theta + ·0003 \cos 2\theta + \ldots$
·4	$1 + ·0716 \cos \theta + ·0030 \cos 2\theta + ·0001 \cos 3\theta + \ldots$
·6	$1 + ·1779 \cos \theta + ·0215 \cos 2\theta + ·0027 \cos 3\theta + ·0003 \cos 4\theta + \ldots$
·8	$1 + ·4392 \cos \theta + ·1504 \cos 2\theta + ·0531 \cos 3\theta + ·0180 \cos 4\theta + \ldots$
1·2	$- ·3007 \cos \theta - ·1019 \cos 2\theta - ·0414 \cos 3\theta - \ldots$
1·4	$- ·1016 \cos \theta - ·0125 \cos 2\theta - ·0019 \cos 3\theta - \ldots$
1·6	$- ·0348 \cos \theta - ·0016 \cos 2\theta - \ldots$
1·8	$- ·0120 \cos \theta - ·0002 \cos 2\theta - \ldots$
2·0	$- ·0042 \cos \theta - \ldots$

By way of contrast I have examined the case of $a/k = 25$, where $\alpha = 2° 17'$, and the step is about one-eighth of the diameter. I find for $r/a = ·8$

$$\gamma/\gamma_0 = 1 + ·0075 \cos \theta + \ldots \quad \ldots\ldots\ldots\ldots\ldots(35),$$

the remaining terms being negligible, to the order of accuracy indicated by the number of decimal places. For $r/a = ·6$, and for smaller values of r, the ratio γ/γ_0 is unity, to the same order. The field is sensibly uniform over about two-thirds of the diameter.

For larger values of a/k, i.e. smaller values of α, the earlier terms of the series are obtained with considerable accuracy by means of the asymptotic formulae *

$$K_n(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}, \quad I_n(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}} \quad \ldots\ldots\ldots(36).$$

These give, for the coefficient of the general term in the series (31) for γ/γ_0,

$$- \frac{2a}{k} n K_n'\left(\frac{na}{k}\right) I_n\left(\frac{nr}{k}\right) = \left(\frac{a}{r}\right)^{\frac{1}{2}} e^{-n(a-r)/k} \ldots\ldots\ldots(37),$$

whilst in the case of (34) we have

$$- \frac{2a}{k} n I_n'\left(\frac{na}{k}\right) K_n\left(\frac{nr}{k}\right) = -\left(\frac{a}{r}\right)^{\frac{1}{2}} e^{-n(r-a)/k} \ldots\ldots\ldots(38).$$

The approximation requires that r/k, as well as a/k should be large.

* Watson, p. 202.

On Errors of Observation. By Dr W. BURNSIDE.

An observer or experimenter who proposes to obtain the value of an astronomical or physical constant in general makes several separate determinations and obtains a number of discordant values.

Denoting these values by $x_1, x_2, \ldots, x_n$, and writing

$$\sum_{1}^{n} x_i = s_1, \quad \sum_{1}^{n} x_i^2 = s_2,$$

the result is generally stated as follows :—

The value of the physical constant is $\dfrac{1}{n} s_1$, and the probable error ϵ of this value is

$$\epsilon = \cdot 6745 \sqrt{\dfrac{s_2 - \dfrac{1}{n} s_1^2}{n\,(n-1)}} \; ;$$

the latter statement implying that the true value of the constant is equally likely to lie within or without the range from

$$\frac{1}{n} s_1 - \epsilon \;\; \text{to} \;\; \frac{1}{n} s_1 + \epsilon.$$

This statement is made on the assumption that the n determinations are made by the same method, or by methods judged to be equally reliable. This assumption will be adhered to here.

It is clear that, apart from the assumption of a law of error of the determinations, no definite statement could possibly be made about the probable error of the result. The above statement is, in fact, deduced from the assumption that the errors follow what is known as Gauss's Law. This is that there is a constant h (precision-constant) such that when δy is small enough the probability of an error in one determination, lying between y and $y + \delta y$, is

$$\sqrt{\frac{h}{\pi}}\, e^{-hy^2}\, \delta y.$$

On this assumption the probability that n determinations give results lying between x_1 and $x_1 + \delta x_1$, x_2 and $x_2 + \delta x_2$, $\ldots$, x_n and $x_n + \delta x_n$ is

$$\left(\frac{h}{\pi}\right)^{\frac{n}{2}} e^{-h\sum_{1}^{n}(x_i - x_0)^2}\, \delta x_1 \,\delta x_2 \ldots \delta x_n,$$

where h is the unknown precision-constant and x_0 the unknown true value.

The particular values of h and x_0 which make this probability as great as possible are

$$x_0 = \frac{1}{n}\, s_1, \quad h = \frac{n}{2\left(s_2 - \dfrac{1}{n}\, s_1^2\right)}.$$

It is apparently assumed that since these values of h and x_0 make the probability of what has been observed to happen as great as possible, they are the best values to take. The probable error of a single determination is given in terms of the precision-constant by the formula $\dfrac{\cdot 6745}{\sqrt{2h}}$. The probable error of $\dfrac{1}{n}\, s_1$, which is the mean of n separate determinations is $\dfrac{1}{\sqrt{n-1}}$ of the probable error of a single determination and this leads to the expression for ϵ given above.

It would be difficult to justify the assumption that, because a particular value of the precision-constant makes the probability of the observed event as great as possible, the precision-constant necessarily has that value. Moreover a further examination will shew that there is no need to make this particular assumption.

Denote by $y_1, y_2, \ldots, y_n$ the errors in n separate determinations, and suppose that $y_1 - y_i$ is found to lie between α_{i-1} and $\alpha_{i-1} + \delta_{i-1}$, where δ_{i-1} is extremely small compared to α_{i-1} $(i = 2, 3, \ldots, n)$. The probability that this may be the case is the integral of

$$\left(\frac{h}{\pi}\right)^{\frac{n}{2}} e^{-h\sum_{1}^{n} y_i^2}\, dy_1 dy_2 \ldots dy_n$$

over the range given by

$$\alpha_i \leqslant y_1 - y_{i+1} \leqslant \alpha_i + \delta_i \ (i = 1, 2, \ldots, n-1).$$

Taking new variables

$$Y_0 = y_1, \ Y_i = y_1 - y_{i+1} \ (i = 1, 2, \ldots, n-1),$$

then

$$dy_1 dy_2 \ldots dy_n = d Y_0 d Y_1 \ldots d Y_{n-1},$$

$$\sum_{1}^{n} y_i^2 = n \left(Y_0 + \frac{1}{n}\sum_{1}^{n-1} Y_i \right)^2 + \sum_{1}^{n-1} Y_i^2 - \frac{1}{n}\left(\sum_{1}^{n-1} Y_i \right)^2$$

and the range for the new variables is

$$-\infty < Y_0 < \infty, \quad \alpha_i \leqslant Y_i \leqslant \alpha_i + \delta_i.$$

Hence the probability of the observed event is

$$\frac{1}{\sqrt{n}}\left(\frac{h}{\pi}\right)^{\frac{n-1}{2}} e^{-h\left(\sum_{1}^{n-1} \alpha_i^2 - \frac{1}{n}\left(\sum_{1}^{n-1} \alpha_i \right)^2 \right)} \delta_1 \delta_2 \ldots \delta_{n-1}.$$

If
$$\alpha_i = x_1 - x_{i+1}, \quad (i = 1, 2, \ldots, n-1),$$

this is
$$\frac{1}{\sqrt{n}} \left(\frac{h}{\pi}\right)^{\frac{n-1}{2}} e^{-h\left(s_2 - \frac{1}{n}s_1^2\right)} \delta_1 \delta_2 \ldots \delta_{n-1}.$$

If $p\delta h$ were the *à priori* probability, *i.e.* the probability before the set of determinations had been made, that the precision-constant lay between h and $h + \delta h$, then the *à posteriori* probability that this may be the case is proportional to

$$p h^{\frac{n-1}{2}} e^{-h\left(s_2 - \frac{1}{n}s_1^2\right)} \delta h.$$

It will be equal to k times this expression if

$$\frac{1}{k} = \int_0^\infty p h^{\frac{n-1}{2}} e^{-h\left(s_2 - \frac{1}{n}s_1^2\right)} dh.$$

When h is the precision-constant, the probability that $\frac{1}{n}\sum_1^n y_i$ lies between $-\gamma$ and γ is the integral of

$$\left(\frac{h}{\pi}\right)^{\frac{n}{2}} e^{-h\sum_1^n y_i^2} dy_1 dy_2 \ldots dy_n$$

over the range
$$-\gamma \leqslant \frac{1}{n}\sum_1^n y_i \leqslant \gamma.$$

Hence the *à posteriori* probability, *i.e.* the probability after the n determinations have been made, that $\frac{1}{n}\sum_1^n y_i$ should lie between $-\gamma$ and γ is the integral of

$$\frac{k p h^{n-\frac{1}{2}}}{\pi^{\frac{n}{2}}} e^{-h\left(\sum_1^n y_i^2 + s_2 - \frac{1}{n}s_1^2\right)} dh \, dy_1 dy_2 \ldots dy_n,$$

over the range
$$0 \leqslant h < \infty,$$
$$-\gamma \leqslant \frac{1}{n}\sum_1^n y_i \leqslant \gamma.$$

It is clear that this can only be evaluated if p is assumed to be a known function of h. Some assumption must be made and the simplest is that p is independent of h, *i.e.* that *à priori* all values of the precision-constant are equally likely. With this assumption

$$\frac{1}{kp} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sigma^{\frac{n+1}{2}}}, \qquad \sigma = s_2 - \frac{1}{n}s_1^2,$$

and the *à posteriori* probability is the integral of

$$\frac{\sigma^{\frac{n+1}{2}}}{\pi^{\frac{n}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}\,h^{n-\frac{1}{2}}\,e^{-h\left(\sigma+\overset{n}{\underset{1}{\Sigma}}y_i^2\right)}\,dh\,dy_1 dy_2\ldots dy_n,$$

over the range stated above. Carrying out the integration with respect to h, the result is the integral of

$$\frac{\sigma^{\frac{n+1}{2}}}{\pi^{\frac{n}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}\,\frac{\Gamma(n+\frac{1}{2})}{\left(\sigma+\overset{n}{\underset{1}{\Sigma}}y_i^2\right)^{n+\frac{1}{2}}}\,dy_1 dy_2\ldots dy_n,$$

over the range $\qquad -\gamma \leqslant \dfrac{1}{n}\overset{n}{\underset{1}{\Sigma}}y_i \leqslant \gamma.$

If an orthogonal transformation is made from y's to z's, where

$$z_1 = \frac{\overset{n}{\underset{1}{\Sigma}}y_i}{\sqrt{n}},$$

the result is the integral of

$$\frac{\sigma^{\frac{n+1}{2}}\,\Gamma(n+\frac{1}{2})}{\pi^{\frac{n}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}\,\frac{dz_1 dz_2\ldots dz_n}{\left(\sigma+\overset{n}{\underset{1}{\Sigma}}z_i^2\right)^{n+\frac{1}{2}}}$$

where the limits with respect to $z_2, z_3, \ldots, z_n$ are $\pm \infty$, and with respect to z_1 are $\pm \gamma\sqrt{n}$. The integrations with respect to $z_2, z_3, \ldots, z_n$ present no difficulty, but the results for n even or odd must be stated separately. It is found that the *à posteriori* probability q that $\dfrac{1}{n}\overset{n}{\underset{1}{\Sigma}}y_i$ should lie between $-\gamma$ and γ is given by

$$q = \frac{1\,.\,3\ldots n}{2\,.\,4\ldots(n-1)}\int_0^{\gamma\sqrt{\frac{n}{\sigma}}}\frac{dt}{(1+t^2)^{\frac{n}{2}+1}}, \qquad n \text{ odd,}$$

$$q = \frac{2}{\pi}\,\frac{2\,.\,4\ldots n}{1\,.\,3\ldots(n-1)}\int_0^{\gamma\sqrt{\frac{n}{\sigma}}}\frac{dt}{(1+t^2)^{\frac{n}{2}+1}}, \qquad n \text{ even.}$$

When γ is infinite q is unity, in either case, as it should be; so that the formula may be written

$$q = \frac{\displaystyle\int_0^{\gamma\sqrt{\frac{n}{\sigma}}} \frac{dt}{(1+t^2)^{\frac{n}{2}+1}}}{\displaystyle\int_0^\infty \frac{dt}{(1+t^2)^{\frac{n}{2}+1}}}.$$

From this it follows that $q = \frac{1}{2}$ when $\gamma = \theta_n \sqrt{\dfrac{\sigma}{n}}$, if

$$\int_0^{\theta_n} \frac{dt}{(1+t^2)^{\frac{n}{2}+1}} = \int_{\theta_n}^\infty \frac{dt}{(1+t^2)^{\frac{n}{2}}}.$$

Mr David Gibb, M.A., of Edinburgh, has had the great kindness to calculate θ_n from this equation, for the lower integral values of n. His results are given in the following table:

n	θ_n
2	·4416087
3	·3703468
4	·3249200
5	·2929416
6	·2686977
7	·2497452
8	·2342386
9	·2212980
10	·2103585

The final conclusion is that if x_0 is the true value of the constant to be determined, then $\dfrac{1}{n} s_1 - x_0$ is equally likely to lie within or without the interval from $-\theta_n \sqrt{\dfrac{s_2 - \dfrac{1}{n} s_1^2}{n}}$ to $+\theta_n \sqrt{\dfrac{s_2 - \dfrac{1}{n} s_1^2}{n}}$.
The ratio of the breadth of this interval to the breadth of the interval given by the standard formula is $\dfrac{\theta_n \sqrt{n-1}}{\cdot6745}$. This number is less than unity for all values of n up to 10, and for the smaller values of n is materially less than unity.

The standard formula for the probable error is arrived at by assuming (i) that the errors follow Gauss's law, (ii) that the

precision-constant has that particular value which makes the probability of the observed set of values as great as possible. The alternative formula here put forward, which gives a smaller value for the probable error, is arrived at by making the same first assumption and replacing the second by the assumption that *à priori* all values of the precision-constant are equally likely.

It is a matter of individual judgment which form of second assumption is the more reasonable.

The Solution of a certain Partial Difference Equation. By Dr
W. BURNSIDE.

1. Denote by G a symbol operating on the first and second
suffixes of $u_{p,q}$, such that

$$Gu_{p,q} \equiv 4u_{p,q} - u_{p-1,q} - u_{p+1,q} - u_{p,q-1} - u_{p,q+1}.$$

It is proposed to determine $u_{p,q}$, subject to the conditions

(i) $Gu_{p,q} = 0$ for all positive and negative integral
values (including zero values) of p and
q, except that

$$Gu_{p_1,q_1} = 1,$$
$$Gu_{p_2,q_2} = -1:$$

(ii) Limit $u_{p,q} = 0$ as $|p|$ increases to infinity, and as $|q|$
increases to infinity.

2. Let a, b, r be positive integers, and put

$$v_{a,b,r} = \frac{1}{4^{a+b+2r}} \frac{(a+b+2r)!\,(a+b+2r)!}{r!\,(a+r)!\,(b+r)!\,(a+b+r)!}.$$

It will be found, by using the known approximation for $n!$, that,
when r is large compared to $a+b$,

$$v_{a,b,r} = \frac{1}{\pi r} + \text{terms in } \frac{1}{r^2}.$$

From this it follows at once that

$$\sum_0^\infty (v_{a,b,r} - v_{a',b',r})$$

is a uniformly convergent series.

Now, the symbol G affecting only the first two suffixes of $v_{a,b,r}$,

$$Gv_{a,b,r} = v_{a,b,r}\left[4 - \frac{(a+b+2r+1)^2}{4(a+r+1)(a+b+r+1)} - \frac{(a+b+2r+1)^2}{4(b+r+1)(a+b+r+1)} \right.$$
$$\left. - \frac{4(a+r)(a+b+r)}{(a+b+2r)^2} - \frac{4(b+r)(a+b+r)}{(a+b+2r)^2} \right]$$

$$= v_{a,b,r}\left[\frac{4r}{a+b+2r} - \frac{(a+b+2r+1)^2(a+b+2r+2)}{4(a+r+1)(b+r+1)(a+b+r+1)} \right]$$

$$= w_{a,b,r} - w_{a,b,r+1},$$

where

$$w_{a,b,r} = \frac{1}{4^{a+b+2r-1}} \frac{(a+b+2r-1)!\,(a+b+2r-1)!\,(a+b+2r)}{(r-1)!\,(a+r)!\,(b+r)!\,(a+b+r)!},$$

and $w_{a,b,0}$ is zero. It is to be noticed that if either a or b is zero the expression $Gu_{a,b,0}$ has no meaning: and that $w_{a,b,r}$ approaches zero as r increases.

Now
$$G \sum_0^x (v_{a,b,r} - v_{a',b',r}) = \sum_0^x (w_{a,b,r} - w_{a,b,r+1} - w_{a',b',r} + w_{a',b',r+1}).$$

Hence if $a \geqslant 1,\ b \geqslant 1,\ a' \geqslant 1,\ b' \geqslant 1$

$$G \sum_0^x (u_{a,b,r} - u_{a',b',r}) = 0.$$

3. Put
$$4u_{p,q} = \sum_0^x (v_{|p-p_1|,\,|q-q_1|,\,r} - v_{|p-p_2|,\,|q-q_2|,\,r}).$$

Then, if no one of the four quantities $p - p_1,\ q - q_1,\ p - p_2,\ q - q_2$ is zero, it follows from the last paragraph that

$$Gu_{p,q} = 0.$$

Suppose that $p = p_1$, while neither $q - q_1$ nor $q - q_2$ is zero. Then

$$4Gu_{p_1,q} = \sum_0^x \begin{pmatrix} 4v_{0,q',r} - 2v_{1,q',r} - v_{0,q'-1,r} - v_{0,q'+1,r} \\ -4v_{p'',q'',r} + v_{p''-1,q'',r} + v_{p''+1,q'',r} + v_{p'',q''-1,r} + v_{p'',q''-1,r} \end{pmatrix}$$

where
$$q' = |q - q_1|,\quad p'' = |p_1 - p_2|,\quad q'' = |q - q_2|.$$

It has been seen in the last paragraph that the part of this sum arising from the second line is zero.

Now the first line under the sign of summation is

$$\frac{1}{4^{q'+2r}} \frac{(q'+2r)!\,(q'+2r)!}{r!\,r!\,(q'+r)!\,(q'+r)!} \left[4 - \frac{1}{2}\frac{(q'+2r+1)^2}{(r+1)\,(q'+r+1)} - 4\frac{(q'+r)^2}{(q'+2r)^2} - \frac{1}{4}\frac{(q'+2r+1)^2}{(q'+r+1)^2} \right]$$

$$= \frac{1}{4^{q'+2r}} \frac{(q'+2r)!\,(q'+2r)!}{r!\,r!\,(q'+r)!\,(q'+r)!} \left[\frac{4r\,(2q'+3r)}{(q'+2r)^2} - \frac{(q'+2r+1)^2\,(2q'+3r+3)}{4\,(r+1)\,(q'+r+1)^2} \right]$$

$$= \frac{1}{4^{q'-2r-1}} \frac{(q'+2r-1)!\,(q'+2r-1)!\,(2q'+3r)}{(r-1)!\,r!\,(q'+r)!\,(q'+r)!} - \frac{1}{4^{q'+2r+1}} \frac{(q'+2r+1)!\,(q'+2r-1)!\,(2q'-3r+3)}{r!\,(r+1)!\,(q'+r+1)!\,(q'+r+1)!}.$$

Hence the whole sum is zero, so that

$$Gu_{p_1,q} = 0 \qquad \text{if } q \neq q_1.$$

In the same way it is shewn that

$$\begin{aligned} Gu_{p,q_1} &= 0 & \text{if } p \neq p_1, \\ Gu_{p,q_2} &= 0 & \text{if } p \neq p_2, \\ Gu_{p_2,q} &= 0 & \text{if } q \neq q_2. \end{aligned}$$

If $p = p_1$, $q = q_1$, then
$$4Gu_{p_1, q_1}$$
$$= \sum_0^\infty \left(\begin{array}{lll} 4v_{0, 0, r} & - 2v_{1, 0, r} & - 2v_{0, 1, r} \\ - 4v_{p', q', r} + v_{p'-1, q', r} + v_{p'+1, q', r} + v_{p', q'-1, r} + v_{p', q'+1, r} \end{array} \right),$$
where $p' = |p_1 - p_2|$, $q' = |q_1 - q_2|$,
$$= 4 \sum_0^\infty (v_{0, 0, r} - v_{0, 1, r})$$
$$= 4 \sum_0^\infty \left(\frac{1}{4^{2r}} \frac{2r! \, 2r!}{r! \, r! \, r! \, r!} - \frac{1}{4^{2r+1}} \frac{(2r+1)! \, (2r+1)!}{r! \, r! \, (r+1)! \, (r+1)!} \right)$$
$$= 4 \sum_0^\infty \left(\frac{1}{4^{2r}} \frac{2r! \, 2r!}{r! \, r! \, r! \, r!} - \frac{1}{4^{2r+2}} \frac{(2r+2)! \, (2r+2)!}{(r+1)! \, (r+1)! \, (r+1)! \, (r+1)!} \right)$$
$$= 4.$$

In a similar way it is shewn that Gu_{p_2, q_2} is -1.

4. When p is large enough, the leading term in $v_{p, q, r}$, for all values of r is
$$\frac{1}{p + q + 2r} \, e^{-\frac{p^2 + q^2}{p + q + 2r}}.$$

The sum of the first $[p^{2-a}]$ terms of this series, where α is an arbitrary small (< 1) positive quantity, is obviously less than
$$p^{1-a} e^{-\frac{1}{2} p a},$$
and this approaches zero as p increases.

When r is large compared to $p + q$, the leading term
$$v_{p, q, r} - v_{p+a, q+\beta, r} \quad \text{is} \quad \frac{Ap}{r^2},$$
where A is a finite constant; and the sum of terms of this form from $r = [p^{2-a}]$ to infinity is a finite multiple of $\frac{1}{p^{1-a}}$, which also approaches zero as limit when p increases. It follows that
$$\underset{p = \infty}{\mathrm{Lt}} \sum_0^\infty (v_{p, q, r} - v_{p+a, q+\beta, r}) = 0.$$

Hence the solution
$$u_{p, q} = \tfrac{1}{4} \sum_0^\infty (v_{|p-p_1|, |q-q_1|, r} - v_{|p-p_2|, |q-q_2|, r})$$
satisfies both conditions (i) and (ii).

5. If this solution were not unique, and $u'_{p, q}$ were a second solution, then $v_{p, q} = u_{p, q} - u'_{p, q}$ is such that
$$Gv_{p, q} = 0,$$
for all values of p and q without exception; while $v_{p, q}$ approaches zero as limit both when $|p|$ increases and when $|q|$ increases.

Since $Gv_{p,q} = 0$ without exception, one of the four quantities $v_{p-1,q}$, $v_{p+1,q}$, $v_{p,q-1}$, $v_{p,q+1}$ must be greater than $v_{p,q}$. If it is $v_{p+1,q}$, say, the same reasoning applies to this and it is possible to pass from one pair of values (p, q) to an adjacent one continually with increasing values of $v_{p,q}$. This is clearly impossible in consideration of the second condition that $v_{p,q}$ satisfies.

It follows that the above solution $u_{p,q}$ is unique.

6. If neither q_1 nor q_2 is negative, and

$$U_{p,q} = \tfrac{1}{4} \sum_0^\infty (v_{|p-p_1|, |q-q_1|, r} - v_{|p-p_2|, |q-q_2|, r})$$

$$+ \tfrac{1}{4} \sum_0^\infty v_{|p-p_1|, |q+q_1+1|, r} - v_{|p-p_2|, |q+q_2+1|, r}),$$

then $U_{p,0} = U_{p,-1}$ for all values of p.

Hence $U_{p,q}$ is the unique solution, when the conditions are

(i) $GU_{p,q} = 0$ for all values of p and all non-negative values of q, except

$$G U_{p_1,q_1} = 1, \quad G U_{p_2,q_2} = -1;$$

(ii) $U_{p,0} = U_{p,-1}$ for all values of p;

(iii) Limit $U_{p,q} = 0$, when $|p|$ tends to infinity, and when
$$q \quad\quad ,, \quad\quad\quad ,,$$

Similarly, if p_1, q_1, p_2, q_2 are not negative, and

$$W_{p,q} = \tfrac{1}{4} \sum_0^\infty (v_{|p-p_1|, |q-q_1|, r} - v_{|p-p_2|, |q-q_2|, r})$$

$$+ \tfrac{1}{4} \sum_0^\infty (v_{|p+p_1+1|, |q-q_1|, r} - v_{|p+p_2+1|, |q-q_2|, r})$$

$$+ \tfrac{1}{4} \sum_0^\infty (v_{|p-p_1|, |q+q_1+1|, r} - v_{|p-p_2|, |q+q_2+1|, r})$$

$$+ \tfrac{1}{4} \sum_0^\infty (v_{|p+p_1+1|, |q+q_1+1|, r} - v_{|p+p_2+1|, |q+q_2+1|, r})$$

is the unique solution if the conditions are

(i) $G W_{p,q} = 0$, for all non-negative values of p and q, except
$$G W_{p_1,q_1} = 1, \quad\quad G W_{p_2,q_2} = -1.$$

(ii) $W_{p,0} = W_{p,-1}$ for all values of p.
$W_{0,q} = W_{-1,q}$ for all values of q.

(iii) Limit $W_{p,q} = 0$, where p tends to infinity, and when
$$q \quad ,, \quad\quad ,,$$

A chapter from Ramanujan's note-book. By Mr G. H. HARDY.

[*Read* 5 February, 1923.]

1. I have in my possession a manuscript note-book which Ramanujan left with me when he returned to India in 1919. It contains some 300 pages of closely written formulae. The first 132 pages, which are written in a peculiar green ink, and seem to contain the results of his early Indian work, are more systematic than the rest, the formulae being arranged in 'chapters', each of which is composed of a numbered sequence of theorems and 'examples'. It is only occasionally that there is any indication of a proof, and a systematic verification of the results would be a very heavy undertaking.

One chapter which I have studied with considerable care is Chapter XII. This chapter contains 47 main theorems, many of them followed by a separate statement of a number of corollaries and particular cases. Thus formulae 40 is followed by 'examples' 1–18.

The subject of the chapter is the hypergeometric series*

$$_3F_2(a_1, a_2, a_3; \beta_1, \beta_2; 1) = F\begin{pmatrix} a_1, a_2, a_3 \\ \beta_1, \beta_2 \end{pmatrix} = 1 + \frac{a_1 \cdot a_2 \cdot a_3}{1 \cdot \beta_1 \cdot \beta_2} + \frac{a_1 (a_1+1) a_2 (a_2+1) a_3 (a_3+1)}{1 \cdot 2 \cdot \beta_1 (\beta_1+1) \beta_2 (\beta_2+1)} + \ldots, \quad \ldots(1 \cdot 1)$$

and it contains in a condensed form practically everything that is known about the summation and transformation of this series. It was inevitable that most of it should have been anticipated by one writer or another; but the literature of the subject is very scattered and obscure, and the chapter has more than a merely historical interest. I propose to give here a short summary of its contents, with some references to later chapters of the note-book, and comments and additions of my own. I have not tried to preserve Ramanujan's notation; without elaborate explanations, this would make the paper almost unintelligible.

I am indebted to Prof. G. N. Watson and Mr F. J. W. Whipple for a number of references and suggestions; especially for the use of a manuscript prepared by Mr Whipple.

2. I write, for shortness,

$$a^{(n)} = a (a + 1) \ldots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)},$$

$$a_{(n)} = a (a - 1) \ldots (a - n + 1) = \frac{\Gamma(a + 1)}{\Gamma(a - n + 1)}.$$

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* I follow the notation of Barnes. See E. W. Barnes, 'The asymptotic expansion of integral functions defined by generalised hypergeometric series,' *Proc. London Math. Soc.* (2), 5 (1906), 59–116.

Then the fundamental formula is

$$\sum_{n=0}^{\infty} (-1)^n (s+2n) \frac{s^{(n)}}{1^{(n)}} \frac{(x+y+z+u+2s+1)^{(n)}}{(x+y+z+u+s)_{(n)}} \prod_{x,\,y,\,z,\,u} \frac{(x)_{(n)}}{(x+s+1)^{(n)}}$$

$$= \frac{s}{\Gamma(s+1)\,\Gamma(x+y+z+u+s+1)} \prod_{x,\,y,\,z,\,u} \frac{\Gamma(x+s+1)\,\Gamma(y+z+u+s+1)}{\Gamma(z+u+s+1)}\,.$$
$$\ldots\ldots(2{\cdot}1;\ \text{R. XII. 1})^{*}$$

The formula is true if any one of x, y, z, u, or

$$-x-y-z-u-2s-1$$

is a positive integer (so that the series terminates). Ramanujan is not explicit about this, but observes that it is not true without restriction. It may be expressed as a relation between two terminating hypergeometric series of the type

$$_6F_5(\alpha_1, \ldots, \alpha_6;\ \beta_1, \ldots, \beta_5;\ 1).$$

When $u = -\tfrac{1}{2}s$, the series on the left of (2·1) reduces to

$$s\, _5F_4 \begin{pmatrix} s,\,-x,\,-y,\,-z,\,x+y+z+\tfrac{3}{2}s+1 \\ x+s+1,\,y+s+1,\,z+s+1,\,-x-y-z-\tfrac{1}{2}s \end{pmatrix},$$

and we obtain the sum of this series when one of x, y, z, or $-x-y-z-\tfrac{3}{2}s-1$ is a positive integer.

Ramanujan had been anticipated in the discovery of this very remarkable identity by Dr J. Dougall†. There is no proof in the note-book, and it was with much difficulty that I found one; but no proof that I could find is equal, in elegance and simplicity, to Dougall's. Let us suppose that the identity is true for

$$u = 0,\ 1,\ 2,\ \ldots,\ m-1,$$

and endeavour to prove it for $u = m$. In this case the formula asserts (when we clear of fractions) the identity of two polynomials in x, of degree $2m$. It follows from our assumption, and the symmetry of the formula in x and u, that it is true for

$$x = 0,\ 1,\ 2,\ \ldots,\ m-1.$$

Further, the formula is symmetrical in the arguments

$$x,\ -x-y-z-u-2s-1;$$

and it is therefore true for

$$x = -y-z-2s-m-1,\ \ldots,\ -y-z-2s-2m.$$

Thus the polynomials are equal for $2m$ values of x, and it is enough to verify their equality for one other value. We choose the value $x = -y-z-s-1$, which is a pole of the last term only of

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* 'R. XII. 1' and so on refer to actual formulae in the note-book.

† J. Dougall, 'On Vandermonde's Theorem, and some more general expansions,' *Proc. Edinburgh Math. Soc.*, 25 (1907), 114–132. Ramanujan probably discovered the identity about 1911 or 1912.

the series (2·1); the verification in this case is immediate, and the proof is completed.

3. Suppose that u, in (2·1), is a positive integer, and make $u \to \infty$. If we take the limit term by term, we obtain

$$\overset{\infty}{\underset{n=0}{\Sigma}} \, (-1)^n (s+2n) \frac{s^{(n)}}{1^{(n)}} \underset{x,\,y,\,z}{\Pi} \frac{x_{(n)}}{(x+s+1)^{(n)}} = \frac{s\Gamma\,(x+y+z+s+1)}{\Gamma\,(s+1)} \underset{x,\,y,\,z}{\Pi} \frac{\Gamma\,(x+s+1)}{\Gamma\,(y+z+s+1)}.$$
$$\text{......(3·1; R. xii. 5)}$$

This formula holds whenever the series is convergent, that is to say when

$$\mathbf{R}\,(x+y+z+s+1) > 0. \qquad \text{......(3·11)}$$

It is not particularly difficult to justify the passage to the limit directly*. There is another method, of a less elementary type, which deserves notice, since it avoids practically all calculation and is continually useful in problems of this kind.

We use the following theorem of Carlson †:—

Suppose that $f(z)$ is an analytic function of the complex variable z which satisfies the following conditions:

(i) $f(z)$ *is regular throughout an angle* **A** *of magnitude greater than or equal to* π, *with its vertex at a point* $z = \zeta$ *on the real axis, and symmetrical about the axis*;

(ii) $|f(z)| < Ce^{A|z|}$, *where* A *and* C *are constants, and* $A < \pi$, *throughout* **A**;

(iii) $f(z) = 0$ *at each of the points*

$$z = \zeta + n \qquad (\zeta > 0, \, n = 0, \, 1, \, 2, \, \ldots).$$

Then $f(z)$ *is identically zero.*

The series derived from (3·1) by suppressing the factors involving z, viz.

$$s - (s+2)\frac{s}{1}\,\frac{x}{x+s+1}\,\frac{y}{y+s-1} + (s+4)\frac{s\,(s+1)}{1\,.\,2}\,\frac{x\,(x-1)}{(x+s+1)\,(x+s+2)}\,\frac{y\,(y-1)}{(y+s+1)\,(y+s+2)} + \ldots$$

is absolutely and uniformly convergent for real values of x, y, s greater than certain positive values ξ, η, σ. Further, if $z = u + iv$ and $u \geqq \zeta > 0$, then

$$\left| \frac{z-n}{z+s+n+1} \right| = \sqrt{\left\{ \frac{(u-n)^2 + v^2}{(u+s+n+1)^2 + v^2} \right\}} < 1$$

for $n = 0, 1, 2, \ldots$. Hence the left-hand side of (3·1) is absolutely and uniformly convergent throughout the angle **A** whose magnitude is π and vertex ζ, and represents a function regular and bounded throughout **A**. The same is true of the right-hand side

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* See Dougall, *loc. cit.*

† F. Carlson, 'Sur une classe de séries de Taylor,' Dissertation, Uppsala (1914), 58. See also S. Wigert, 'Sur un théorème concernant les fonctions entières,' *Arkiv för Matematik*, 11 (1916), no. 22; G. H. Hardy, 'On two theorems of F. Carlson and S. Wigert', *Acta Math.*, 42 (1920), 327–339.

of the equation. Hence $f(z)$, the difference of the two sides, satisfies conditions (i) and (ii), with $A = 0$. It also satisfies (iii), since (3·1) is known to be true for positive integral z, and therefore it is true identically in z. It follows by analytic continuation that (3·1) is true for all values of the variables which satisfy (3·11).

In (3·1) put $z = -\tfrac{1}{2}s$. We obtain

$$_3F_2\left(\begin{matrix} -x, \ -y, \ s \\ x+s+1, \ y+s+1 \end{matrix}\right) = \frac{\Gamma\left(\tfrac{1}{2}s+1\right)\Gamma\left(x+s+1\right)\Gamma\left(y+s+1\right)\Gamma\left(x+y+\tfrac{1}{2}s+1\right)}{\Gamma\left(s+1\right)\Gamma\left(x+\tfrac{1}{2}s+1\right)\Gamma\left(y+\tfrac{1}{2}s+1\right)\Gamma\left(x+y+s+1\right)}; \quad .. \ (3·2; \ \text{R. xii. } 9)$$

a beautiful formula due to A. C. Dixon*. It sums the series $_3F_2$ when

$$\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \alpha_3 + 1;$$

and includes as a special case the sum of the cubes of the binomial expansion, first evaluated generally by F. Morley†.

Other special cases of (3·1) noted by both Ramanujan and Dougall are‡

$$s + (s+2)\left(\frac{s}{1}\right)^3 + (s+4)\left\{\frac{s(s+1)}{1 \cdot 2}\right\}^3 + \ldots = \frac{\sin s\pi}{\pi} \frac{\Gamma\left\{\tfrac{1}{2}(s+1)\right\}\Gamma\left\{\tfrac{1}{2}(1-3s)\right\}}{\left[\Gamma\left\{\tfrac{1}{2}(1-s)\right\}\right]^3}, \ \ldots(3·31; \ \text{R. xii. } 32)$$

$$s - (s+2)\left(\frac{s}{1}\right)^3 + (s+4)\left\{\frac{s(s+1)}{1 \cdot 2}\right\}^3 + \ldots = \frac{\sin s\pi}{\pi}, \qquad \ldots(3·32; \ \text{R. xii. } 33)$$

$$s + (s+2)\left(\frac{s}{1}\right)^4 + (s+4)\left\{\frac{s(s+1)}{1 \cdot 2}\right\}^4 + \ldots = \frac{\sin^2 s\pi}{2\cos s\pi}\frac{\{\Gamma(s)\}^2}{\Gamma(2s)}. \qquad \ldots(3·33; \ \text{R. xii. } 30)$$

Ramanujan gives a multitude of further curious formulae. I note as typical

$$1 + 3\frac{x-1}{x+1} + 5\frac{(x-1)(x-2)}{(x+1)(x+2)} + \ldots = x, \qquad \ldots\ldots(3·41; \ \text{R. xii. } 40·5)$$

$$1 - 3\frac{x-1}{x+1} + 5\frac{(x-1)(x-2)}{(x+1)(x+2)} - \ldots = 0, \qquad \ldots\ldots(3·42; \ \text{R. xii. } 40·8)$$

$$1 + 3\left(\frac{x-1}{x+1}\right)^2 + 5\left\{\frac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^2 - \ldots = \frac{x^2}{2x-1}, \qquad \ldots\ldots(3·43; \ \text{R. xii. } 40·2)$$

$$1 - 3\left(\frac{x-1}{x+1}\right)^2 + 5\left\{\frac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^2 - \ldots = \frac{\{\Gamma(x+1)\}^2}{\Gamma(2x)}, \qquad \ldots\ldots(3·44; \ \text{R. xii. } 40·4)$$

$$1 - 3\left(\frac{x-1}{x+1}\right)^3 + 5\left\{\frac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^3 - \ldots = \frac{\{\Gamma(x+1)\}^3\Gamma(3x-1)}{\{\Gamma(2x)\}^3}, \ \ldots(3·45; \ \text{R. xii. } 40·1)$$

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* A. C. Dixon, 'Summation of a certain series,' *Proc. London Math. Soc.* (1), 35 (1902), 284–289.

† F. Morley, 'On a series of cubes,' *Proc. London Math. Soc.* (1), 34 (1902), 397–402. See also A. C. Dixon, 'On the sum of the cubes of the coefficients in a certain expansion by the binomial theorem,' *Messenger of Math.*, 20 (1891), 79–80; H. W. Richmond, 'The sum of the cubes of the coefficients in $(1-x)^{2n}$,' *ibid.*, 21 (1892), 77–78; P. A. MacMahon, 'The sums of the powers of the binomial coefficients,' *Quarterly Journal*, 33 (1902), 274–288. In these papers the exponent is a positive integer.

Another proof of Dixon's formula has been given recently by Watson: see G. N. Watson, 'Dixon's Theorem on generalised hypergeometric functions', *Proc. London Math. Soc., Records, etc.* for 26 April, 1923.

‡ I do not repeat the convergence conditions, which a reader can supply.

$$1 + \frac{x-1}{x+1} + \frac{(x-1)\,(x-2)}{(x+1)\,(x+2)} + \ldots = 2^{2x-1}\frac{\{\Gamma\,(x+1)\}^2}{\Gamma\,(2x+1)}, \qquad \ldots\ldots(3\cdot46;\; \text{R. xii. } 40\cdot6)$$

$$1 - \frac{x-1}{x+1} + \frac{(x-1)\,(x-2)}{(x+1)\,(x+2)} + \ldots = \frac{x}{2x-1}, \qquad \ldots\ldots(3\cdot47;\; \text{R. xii. } 40\cdot7)$$

$$1 - \frac{1}{3}\frac{x-1}{x+1} + \frac{1}{5}\frac{(x-1)\,(x-2)}{(x+1)\,(x+2)} - \ldots = 2^{4x}\frac{\{\Gamma\,(x+1)\}^4}{4x\,\{\Gamma\,(2x+1)\}^2}, \qquad \ldots\ldots(3\cdot48;\; \text{R. xii. } 40\cdot11)$$

$$1 + \left(\frac{x-1}{x+1}\right)^2 + \left\{\frac{(x-1)\,(x-2)}{(x+1)\,(x+2)}\right\}^2 + \ldots = \frac{2x}{4x-1}\frac{\{\Gamma\,(x+1)\}^4\Gamma\,(4x+1)}{\{\Gamma\,(2x+1)\}^4}\ldots(3\cdot49,\text{R. xii.} 40\cdot3)$$

But Dougall gives one particularly elegant formula which Ramanujan seems to have overlooked, viz.

$$1 - 2\frac{xyz}{(x+1)\,(y+1)\,(z+1)} + 2\frac{x\,(x-1)\,y\,(y-1)\,z\,(z-1)}{(x+1)\,(x+2)\,(y+1)\,(y+2)\,(z+1)\,(z+2)} - \ldots$$
$$= \frac{\Gamma\,(x+1)\,\Gamma\,(y+1)\,\Gamma\,(z+1)\,\Gamma\,(x+y+z+1)}{\Gamma\,(y+z+1)\,\Gamma\,(z+x+1)\,\Gamma\,(x+y+1)}. \qquad \ldots\ldots(3\cdot5)$$

This is obtained by dividing $(3\cdot1)$ by s and making $s = 0$.

4. When looking for a proof of the formula $(3\cdot1)$, I found that it has an interesting connection with the theory of Bessel's functions.

We start from the formula[*]

$$J_\rho(u) = (\tfrac{1}{2}u)^{\rho-s} \sum_{n=0}^{\infty} (s+2n)\frac{\Gamma\,(s+n)}{\Gamma\,(\rho+n+1)}\frac{\Gamma\,(\rho-s+1)}{n!\,\Gamma\,(\rho-s-n+1)}J_{s+2n}(u), \quad \ldots(4\cdot1)$$

valid so long as ρ, s, and $\rho - s$ are not negative integers. We write $s + x$ for ρ, multiply by

$$J_{y-z}(u)\, u^{-x-y-z-1},$$

and integrate term by term from $u = 0$ to $u = \infty$. Using the formula[†]

$$\int_0^\infty J_\alpha(u) J_\beta(u)\, u^\gamma\, du = 2^\gamma\frac{\Gamma\,\{\tfrac{1}{2}\,(1+\alpha+\beta+\gamma)\}\,\Gamma\,(1-\gamma)}{\Gamma\,\{\tfrac{1}{2}\,(1+\alpha-\beta-\gamma)\}\,\Gamma\,\{\tfrac{1}{2}\,(1-\alpha+\beta-\gamma)\}\,\Gamma\,\{\tfrac{1}{2}\,(1-\alpha-\beta+\gamma)\}}, \quad \ldots(4\cdot2)$$

we find that the integral

$$\int_0^\infty J_{x+s}(u)\, J_{y-z}(u)\, u^{-x-y-z-s-1}\, du,$$

is equal, on the one hand to

$$2^{-x-y-z-s-1}\frac{\Gamma\,(x+y+z+s+1)\,\Gamma\,(-z)}{\Gamma\,(z+x+s+1)\,\Gamma\,(x+y+s+1)\,\Gamma\,(y+1)},$$

and on the other to

$$2^{-x-y-z-s-1}\frac{\Gamma\,(s)\,\Gamma\,(-z)\,\Gamma\,(y+z+s+1)}{\Gamma\,(y+1)\,\Gamma\,(x+s+1)\,\Gamma\,(y+s+1)\,\Gamma\,(z+s+1)}\,S,$$

[*] G. N. Watson, *Theory of Bessel functions*, 139. The formula was proved by Sonine under the restriction that $\rho - s$ is a positive integer (when the series is finite). See N. Nielsen, *Cylinderfunktionen*, 275, formula (3).

[†] Watson, *ibid.*, 403.

where S is the series (3·1). Equating these values we obtain (3·1), which may therefore be deduced directly from (4·1).

The justification of the term by term integration is rather troublesome, and, as my object is merely to show a formal connection, I omit the details of the analysis. It will be found that all the formulae employed have a meaning if

$$\mathbf{R}(x+y+z+s+1) > 0, \ \mathbf{R}(y+z+s+1) > 0, \ \mathbf{R}(-z) > 0.$$

We prove the result first on the assumption that the two last conditions are satisfied and that x is positive and sufficiently large, extending it afterwards, by analytic continuation, to its full region of validity.

There are other elegant formulae, not included in those given by Ramanujan or Dougall, which can be obtained in a similar manner. Thus from

$$(-1)^{n-1} \frac{\Gamma(n-x)\Gamma(n-y)}{\Gamma(x+n+1)\Gamma(y+n+1)} = \frac{2\pi}{\sin x\pi \, \Gamma(x+y+1)} \int_0^\infty J_{2n}(2u) J_{x-y}(2u) u^{-x-y-1} \, du,$$
$$\cos(2u\sin\theta) = J_0(2u) + 2J_2(2u)\cos 2\theta + \ldots,$$

we deduce

$$1 - \frac{2xy}{(x+1)(y+1)}\cos 2\theta + \frac{2x(x-1)y(y-1)}{(x+1)(x+2)(y+1)(y+2)}\cos 4\theta - \ldots = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+1)} F(-x,-y,\tfrac{1}{2},\sin^2\theta).$$
$$\ldots\ldots(4\text{·}3)$$

Similarly

$$\sin\theta - \frac{xy}{(x+2)(y+2)}\sin 3\theta + \frac{x(x-1)y(y-1)}{(x+2)(x+3)(y+2)(y+3)}\sin 5\theta - \ldots = \frac{\Gamma(x+2)\Gamma(y+2)}{\Gamma(x+y+2)}\sin\theta F(-x,-y,\tfrac{3}{2},\sin^2\theta).$$
$$\ldots\ldots(4\text{·}4)$$

If we give θ the special values 0 or $\frac{1}{2}\pi$, the right-hand sides can be evaluated in finite form, and we obtain special cases of (3·1).

5. Returning to the fundamental formula (2·1), we suppose that x, y, or z is a positive integer, divide by s, write $m-s$ for u, and make $s \to \infty$. We obtain the formula

$$_3F_2\left(\begin{matrix} -x, \, -y, \, -z \\ m+1, \, -x-y-z-m \end{matrix}\right) = \frac{\Gamma(m+1)\,\Gamma(y+z+m)\,\Gamma(z+x+m)\,\Gamma(x+y+m)}{\Gamma(x+m+1)\,\Gamma(y+m+1)\,\Gamma(z+m+1)\,\Gamma(x+y+z+m+1)},$$
$$\ldots\ldots(5\text{·}1; \text{ R. xii. 3})$$

due to Saalschütz*. It gives the value of $_3F_2$ when

$$\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 - 1, \qquad \ldots\ldots(5\text{·}11)$$

and one of the α's is a negative integer; and may be proved directly (as was pointed out to me by Prof. Watson), by equating coefficients in the identity

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, x).$$

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* L. Saalschütz, 'Eine Summationsformel,' *Zeitschrift für Math.*, 35 (1890), 186–188, and a later paper 'Über einen Specialfall der hypergeometrischen Reihe dritter Ordnung,' *ibid.*, 36 (1891), 278–295, 321–327. Also, W. F. Sheppard, *Proc. Lond. Math. Soc.* (2), 10 (1911), 469–478.

The formula is not true without the restriction on the α's. Ramanujan does not say this explicitly; but he stated elsewhere a more general formula which includes that of Saalschütz and replaces it when the α's are arbitrary. This formula* is

$$_3F_2\begin{pmatrix} \alpha_1, \ \alpha_2, \ \alpha_3 \\ \beta_1, \ \beta_2 \end{pmatrix} = \frac{\Gamma(\beta_1)\,\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\,\Gamma(\beta_1-\alpha_2)}\,_3F_2\begin{pmatrix} \alpha_1, \ \alpha_2, \ \beta_2-\alpha_3 \\ \alpha_1+\alpha_2-\beta_1+1, \ \beta_2 \end{pmatrix}$$

$$+\frac{\Gamma(\beta_1)\,\Gamma(\beta_2)\,\Gamma(\alpha_1+\alpha_2-\beta_1)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\beta_2-\alpha_3)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}\,_3F_2\begin{pmatrix} \beta_1-\alpha_1, \ \beta_1-\alpha_2, \ \beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3 \\ \beta_1-\alpha_1-\alpha_2+1, \ \beta_1+\beta_2-\alpha_1-\alpha_2 \end{pmatrix},$$

$$\dots\dots(5\cdot2)$$

and is valid if

$$\mathbf{R}\,(\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2)<0, \ \ \mathbf{R}\,(\alpha_3-\beta_1+1)>0. \ \ \dots(5\cdot21)$$

If (5·11) is satisfied, the first series on the right of (5·2) has two equal elements, and may be summed by the well known formula of Gauss. We thus obtain

$$_3F_2\begin{pmatrix} \alpha_1, \ \alpha_2, \ \beta_1+\beta_2-\alpha_1-\alpha_2-1 \\ \beta_1, \ \beta_2 \end{pmatrix} = \frac{\Gamma(\beta_1)\,\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\,\Gamma(\beta_1-\alpha_2)}\,\frac{\Gamma(\beta_2)\,\Gamma(\beta_2-\alpha_1-\alpha_2)}{\Gamma(\beta_2-\alpha_1)\,\Gamma(\beta_2-\alpha_2)}$$

$$+\frac{1}{\alpha_1+\alpha_2-\beta_1}\,\frac{\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}\,_3F_2\begin{pmatrix} \beta_1-\alpha_1, \ \beta_1-\alpha_2, \ 1 \\ \beta_1-\alpha_1-\alpha_2+1, \ \beta_1+\beta_2-\alpha_1-\alpha_2 \end{pmatrix}, \ \dots(5\cdot3)$$

which reduces to the formula of Saalschütz if α_1 or α_2 is a negative integer. The formula (5·3) is given by Saalschütz in his second paper referred to above†. The more general formula (5·2) is one of a family of relations investigated systematically by Thomae‡ and later independently by Whipple.

Prof. Watson has communicated to me the following very simple proof of Ramanujan's formula (5·2). We have, subject to convergence conditions which it is hardly necessary to repeat,

$$_3F_2\begin{pmatrix} \alpha_1, \ \alpha_2, \ \alpha_3 \\ \beta_1, \ \beta_2 \end{pmatrix} = \frac{\Gamma(\beta_2)}{\Gamma(\alpha_3)\,\Gamma(\beta_2-\alpha_3)}\int_0^1 x^{\alpha_3-1}\,(1-x)^{\beta_2-\alpha_3-1}\,_2F_1(\alpha_1,\alpha_2;\beta_1;\,x)\,dx. \ (5\cdot4)$$

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* Quoted on p. 1 of my notice of Ramanujan in *Proc. London Math. Soc.* (2), 19 (1920), formula (10).

† There is naturally a similar formula in which β_1 and β_2 are interchanged. Comparing the two we obtain the relation (A)

$$\frac{1}{\alpha_1+\alpha_2-\beta_1}\,_3F_2\begin{pmatrix} \beta_1-\alpha_1, \ \beta_1-\alpha_2, \ 1 \\ \beta_1-\alpha_1-\alpha_2+1, \ \beta_1+\beta_2-\alpha_1-\alpha_2 \end{pmatrix} = \frac{1}{\alpha_1+\alpha_2-\beta_2}\,_3F_2\begin{pmatrix} \beta_2-\alpha_1, \ \beta_2-\alpha_2, \ 1 \\ \beta_2-\alpha_1-\alpha_2+1, \ \beta_1-\beta_2-\alpha_1-\alpha_2 \end{pmatrix}.$$

This formula was first given by Saalschütz (*loc. cit. supra*), and is a special case of a more general formula due to Thomae and rediscovered by A. C. Dixon (see § 6). Later, M. J. M. Hill and F. J. W. Whipple ('A reciprocal relation between generalised hypergeometric series,' *Quarterly Journal*, 41 (1910), 128–135) rediscovered (A) and proved it algebraically, and E. W. Barnes ('A transformation of generalised hypergeometric series,' *ibid.*, 136–140), proving (A) by complex integration, rediscovered the generalisation of Thomae and Dixon. See also J. H. C. Searle, 'The summation of certain series,' *Messenger of Math.*, 38 (1909), 138–144.

‡ J. Thomae, 'Ueber die Funktionen u. s. w.' *J. für Math.*, 87 (1879), 26–73.

But*
$$_2F_1(\alpha_1, \alpha_2; \beta_1; x) = \frac{\Gamma(\beta_1)\Gamma(\beta_1 - \alpha_1 - \alpha_2)}{\Gamma(\beta_1 - \alpha_1)\Gamma(\beta_1 - \alpha_2)} {}_2F_1(\alpha_1, \alpha_2; \alpha_1 + \alpha_2 - \beta_1 + 1; 1 - x)$$
$$+ \frac{\Gamma(\alpha_1 + \alpha_2 - \beta_1)\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (1-x)^{\beta_1 - \alpha_1 - \alpha_2} {}_2F_1(\beta_1 - \alpha_1, \beta_1 - \alpha_2; \beta_1 - \alpha_1 - \alpha_2 + 1; 1 - x). \quad ...(5\cdot5)$$

Substituting from (5·5) in (5·4), and integrating term by term, we obtain (5·2).

6. The same process, of multiplication by $x^{\alpha_3 - 1}(1-x)^{\beta_2 - \alpha_3 - 1}$ and integration from 0 to 1, applied to the identity†

$$_2F_1(\alpha_1, \alpha_2; \beta_1; x) = (1-x)^{\beta_1 - \alpha_1 - \alpha_2} {}_2F_1(\beta_1 - \alpha_1, \beta_1 - \alpha_2; \beta_1; x),$$

leads to the formula‡

$$_3F_2\begin{pmatrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{pmatrix} = \frac{\Gamma(\beta_2)\Gamma(\beta_1 - \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(\beta_2 - \alpha_3)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)} {}_3F_2\begin{pmatrix} \beta_1 - \alpha_1, \beta_1 - \alpha_2, \alpha_3 \\ \beta_1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 \end{pmatrix}. \quad ...(6\cdot1; \text{ R. xiii. 12})$$

This important theorem, which may be expressed by saying that

$$_3F_2\begin{pmatrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{pmatrix} \Big/ \Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)$$

is a symmetric function of the five arguments

$$\beta_1, \beta_2, \beta_1 + \beta_2 - \alpha_2 - \alpha_3, \beta_1 + \beta_2 - \alpha_3 - \alpha_1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2,$$

is due to Thomae §. It was afterwards found independently by A. C. Dixon‖ and by Barnes¶.

A repetition of the transformation leads to

$$_3F_2\begin{pmatrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{pmatrix} = \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(\alpha_1)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_3)} {}_3F_2\begin{pmatrix} \beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3, \beta_1 - \alpha_1, \beta_2 - \alpha_1 \\ \beta_1 + \beta_2 - \alpha_1 - \alpha_2, \beta_1 + \beta_2 - \alpha_1 - \alpha_3 \end{pmatrix}$$
$$......(6\cdot2)$$

(the formula given explicitly by Dixon); and the theorem is expressed completely by (6·1), (6·2) and the obvious results of permutations of the α's and the β's. Conversely, we can deduce (6·1) from (6·2).

Ramanujan gives an elementary proof of (6·2), which is in essentials the same as that given by Thomae**. We have, by Gauss's formula

$$\frac{\Gamma(x+y+s+q+1)}{\Gamma(x+s+q+1)\Gamma(y+s+q+1)} = \frac{1}{\Gamma(s+q+1)} {}_2F_1\begin{pmatrix} -x, -y \\ s+q+1 \end{pmatrix}$$
$$= \Gamma(x+1)\Gamma(y+1) \sum_{p=0}^{x} \frac{1}{p!\,\Gamma(x-p+1)\Gamma(y-p+1)\Gamma(s+p+q+1)}.$$

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* See, for example, Whittaker and Watson, *Modern Analysis*, ed. 2, p. 285.
† Whittaker and Watson, *loc. cit.*, p. 280.
‡ In all that follows I omit convergence conditions, which can be immediately supplied. The interest lies in the formulae.
§ Thomae, *l.c. supra*, 5·2.
‖ A. C. Dixon, 'On a certain double integral,' *Proc. London Math. Soc.* (2), 2 (1904), 8–15.
¶ Barnes, *l.c. supra*.
** Mr Whipple's manuscript (written with knowledge of Dixon's, but in ignorance of Thomae's work) contains a substantially identical proof.

Hence

$$\frac{\Gamma(x+y+s+1)}{\Gamma(x+s+1)\,\Gamma(y+s+1)}\, {}_3F_2\left(\begin{array}{c} -a,\; -b,\; x+y+s+1 \\ x+s+1,\; y+s+1 \end{array}\right)$$

$$= \Gamma(a+1)\,\Gamma(b+1)\sum_{q=0}^{\infty}\frac{\Gamma(x+y+s+q+1)}{q!\,\Gamma(a-q+1)\,\Gamma(b-q+1)\,\Gamma(x+s+q+1)\,\Gamma(y+s+q+1)}$$

$$= \sum_{p,\,q=0}^{\infty}\frac{\Gamma(x+1)\,\Gamma(y+1)\,\Gamma(a+1)\,\Gamma(b+1)}{p!\,q!\,\Gamma(x-p+1)\,\Gamma(y-p+1)\,\Gamma(a-q+1)\,\Gamma(b-q+1)\,\Gamma(s+p+q+1)}$$

$$= \frac{\Gamma(a+b+s+1)}{\Gamma(a+s+1)\,\Gamma(b+s+1)}\, {}_3F_2\left(\begin{array}{c} -x,\; -y,\; a+b+s+1 \\ a+s+1,\; b+s+1 \end{array}\right),$$

in virtue of the symmetry of the double series. This is (6·2). The argument requires that the double series shall be absolutely convergent; and this is certainly so if the real part of s is sufficiently large[*].

When α_3 is a negative integer, the formula (6·1) reduces to an identity between rational functions. In this case there is a still simpler inductive proof. Suppose $\alpha_3 = -N$, and consider both sides of (6·1) as functions of β_1. The only poles of either side are $\beta_1 = -n$, where $0 \leqq n < N$. Expressing the condition that the residues of the two sides are the same, we are led to an identity of the same type, but in which N is replaced by a smaller integer, and the result is thus established inductively. We have then to remove the restriction on α_3, and this we can do, as in § 3, by using Carlson's theorem, though in this case the details are rather more troublesome.

There is a similar proof of (5·1). Here one of x, y, or z is essentially integral, so that the inductive proof is particularly simple. I have also constructed a proof on these lines of (5·2), but it is more elaborate.

7. Our information about the series (1·1) is now as follows. There are two cases (§ 3 and § 5) in which it can be summed in finite terms, one (Dixon) in which the parameters are connected by *two* relations, and one (Saalschütz) in which they obey one only, but there is a further arithmetic restriction. There is also the symmetry theorem of Thomae and Dixon (involving no restriction on the parameters), and by means of this we can find a number of other summable cases. There is, so far as I know, no other general relation between *two* functions ${}_3F_2$, but there are a multitude of *three-term* relations, of which Ramanujan's formula (5·2) is one.

These relations have, as I stated in § 5, been investigated

[*] Suppose for example, that s is positive and greater than $2n$, where n is an integer. Then

$$\Gamma(s+p+q+1) > \Gamma(n+p+1)\,\Gamma(n+q+1),$$

and the series may be compared with the product of two absolutely convergent simple series.

systematically by Thomae and Whipple. One such relation may be found without difficulty by the calculus of residues, viz.

$$\sum_{a_1,\,a_2,\,a_3} \frac{\Gamma\left(a_2 - a_1\right)\Gamma\left(a_3 - a_1\right)}{\Gamma\left(1 - a_1\right)\Gamma\left(\beta_1 - a_1\right)\Gamma\left(\beta_2 - a_1\right)} \,_3F_2\left(\begin{matrix} a_1,\, 1 + a_1 - \beta_1,\, 1 + a_1 - \beta_2 \\ 1 + a_1 - a_2,\, 1 + a_1 - a_3 \end{matrix}\right) = 0. \quad \ldots(7\cdot1)$$

This is a corollary of the four-term relation

$$\frac{\Gamma(a)\,\Gamma(a_2)\,\Gamma(a_3)}{\Gamma(\beta_1)\,\Gamma(\beta_2)} \,_3F_2\left(\begin{matrix} a_1,\, a_2,\, a_3 \\ \beta_1,\, \beta_2 \end{matrix}\right) = \sum_{a_1,\,a_2,\,a_3} e^{\pm \pi i a_1}\, \frac{\Gamma(a_1)\,\Gamma(a_2 - a_1)\,\Gamma(a_3 - a_1)}{\Gamma(\beta_1 - a_1)\,\Gamma(\beta_2 - a_2)} \,_3F_2\left(\begin{matrix} a_1,\, 1 + a_1 - \beta_1,\, 1 + a_1 - \beta_2 \\ 1 + a_1 - a_2,\, 1 + a_1 - a_3 \end{matrix}\right)$$
$$\ldots\ldots(7\cdot2)$$

(where either sign may be taken throughout), which follows immediately from a consideration of the integral*

$$\frac{1}{2\pi i} \int e^{\pm z\pi i} \frac{\Gamma\left(1 - z\right)\Gamma\left(\alpha_1 + z\right)\Gamma\left(\alpha_2 + z\right)\Gamma\left(\alpha_3 + z\right)}{\Gamma(\beta_1 + z)\,\Gamma(\beta_1 + z)}\, dz.$$

These relations can of course be transformed by the symmetry theorem, and there are many cases in which they simplify materially when one of the parameters is integral.

It is improbable that Ramanujan made any use of complex integration, and there is no evidence that he had investigated the three-term relations systematically. It seems clear, however, that he was familiar with the substance of almost all the results that I have stated.

8. I have not met with any relation connecting these functions which cannot be deduced from one or other of the preceding formula. For example, Dougall proves the relation

$$_3F_2\left(\begin{matrix} -a,\, -b,\, 1 \\ c+1,\, d+1 \end{matrix}\right) + \frac{cd}{(a+1)(b+1)} \,_3F_2\left(\begin{matrix} 1-c,\, 1-d,\, 1 \\ a+2,\, b+2 \end{matrix}\right) = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(d+1)\Gamma(a+b+c+d+1)}{\Gamma(a+c+1)\Gamma(a+d+1)\Gamma(b+c+1)\Gamma(b+d+1)}.$$
$$\ldots\ldots(8\cdot1)$$

This may be deduced from (7·2) by supposing that $\alpha_3 = 1$.

Prof. Watson communicated to me recently the formula

$$_3F_2\left(\begin{matrix} -n,\, \lambda,\, 2\lambda + 2\mu + n - 1 \\ 2\lambda,\, \lambda + \mu \end{matrix}\right) = (-2)^n \frac{n\,!}{(\frac{1}{2}n)\,!} \frac{\Gamma\left(\lambda + \tfrac{1}{2}n\right)\Gamma\left(\mu + \tfrac{1}{2}n\right)\Gamma\left(2\mu\right)\Gamma\left(\lambda + \mu\right)}{\Gamma\left(\lambda + \mu + \tfrac{1}{2}n\right)\Gamma\left(2\lambda + n\right)\Gamma\left(\lambda\right)\Gamma\left(\mu\right)}, \quad \ldots(8\cdot2)$$

in which n is an even integer (if n is an odd integer, the series vanishes). This may be deduced from (5·2) by taking

$$\alpha_1 = -n,\ \alpha_2 = \lambda,\ \alpha_3 = 1 - \lambda - \mu - n,\ \beta_1 = 1 - \lambda - n,\ \beta_2 = \lambda + \mu.$$

In this case the third term of (5·2) disappears, the second is a multiple of Watson's function, and the first may be summed by (3·2).

---

* This is the method used in Mr Whipple's manuscript. Thomae's methods are elementary.

Another interesting formula of Watson's [*] is

$$
{}_3F_2\begin{pmatrix} \nu+\mu+1, \ \nu+\tfrac{1}{2}, \ \nu-\mu+1 \\ 2\nu+2, \ \nu+\tfrac{3}{2} \end{pmatrix} = \frac{\Gamma(2\nu+2)}{2\Gamma(\nu+\mu+1)\,\Gamma(\nu-\mu+1)}
$$

$$
\times \left\{ \psi\left(\frac{\nu+\mu+2}{2}\right) + \psi\left(\frac{\nu-\mu+2}{2}\right) - \psi\left(\frac{\nu+\mu+1}{2}\right) - \psi\left(\frac{\nu-\mu+1}{2}\right) \right\},
$$
$$\dots\dots(8\cdot3)$$

where ψ means Γ'/Γ. The deduction of this is rather more intricate. I begin by proving (as a limiting case of $(3\cdot2)$), that

$$
\frac{a}{b}\frac{1}{a+b} + \frac{a(a+1)}{b(b+1)}\frac{1}{a+b+1} + \dots = \psi(a+b) - \psi(b-a) - \psi\left(\frac{a+b+1}{2}\right) + \psi\left(\frac{b-a+1}{2}\right).
$$
$$\dots\dots(8\cdot4)$$

Now assign to the parameters, in $(5\cdot2)$, the values they have in the function $(8\cdot3)$. The left-hand side is then Watson's function; the first function on the right has two equal parameters $\mu+\tfrac{1}{2}$, and is therefore expressible as a product of Gamma-functions, while the third reduces to a series of the type $(8\cdot4)$. After reduction, we obtain $(8\cdot3)$.

9. I conclude by a few words concerning formulae given by Ramanujan but not directly concerned with the series $(1\cdot1)$. These are contained in the next chapter (Ch. XIII) of the note-book. They include a number of the formulae for the linear transformation of the ordinary hypergeometric function $F(\alpha, \beta, \gamma, x)$; those, roughly, which, like the formula

$$
F(\alpha, \beta, \gamma, x) = \left(\frac{1}{1-x}\right)^a F\left(\alpha, \gamma-\beta, \gamma, -\frac{x}{1-x}\right); \ \dots(9\cdot1;\ \text{R. XIII. 4})
$$

or the formula quoted at the beginning of § 6, involve two functions only; the three formulae

$$
(1+x)^{2a} F(2a, 2a+1-\gamma, \gamma, x) = F\left(a, a+\tfrac{1}{2}, \gamma, \frac{4x}{(1+x)^2}\right), \quad \dots(9\cdot21;\ \text{R. XIII. 18})
$$

$$
(1+x)^{2a} F(a, a-\beta+\tfrac{1}{2}, \beta+\tfrac{1}{2}, x^2) = F\left(a, \beta, 2\beta, \frac{4x}{(1+x)^2}\right), \quad \dots(9\cdot22;\ \text{R. XIII. }20\cdot3)
$$

$$
F(2a, 2\beta, a+\beta+\tfrac{1}{2}, x) = F\{a, \beta, a+\beta+\tfrac{1}{2}, 4x(1-x)\}, \quad (9\cdot23;\ \text{R. XIII. 29})
$$

of Gauss, and Gauss's corollary

$$
F(2a, 2\beta, a+\beta+\tfrac{1}{2}, \tfrac{1}{2}) = \sqrt{\pi}\,\frac{\Gamma(\alpha+\beta+\tfrac{1}{2})}{\Gamma(\alpha+\tfrac{1}{2})\,\Gamma(\beta+\tfrac{1}{2})}. \quad \dots(9\cdot24;\ \text{R. XIII. 27})
$$

There is also Kummer's formula

$$
{}_1F_1(\alpha;\ 2\alpha;\ 2x) = e^x {}_0F_1(\alpha+\tfrac{1}{2};\ \tfrac{1}{4}x^2); \ \dots(9\cdot3;\ \text{R. XIII. 19})
$$

Clausen's formula

$$\{F(a,\ \beta,\ a+\beta+\tfrac{1}{2},\ x)\}^2 = {}_3F_2(2a,\ a+\beta,\ 2\beta;\ a+\beta+\tfrac{1}{2},\ 2a+2\beta;\ x); \ ...(9\cdot4;\ \text{R. xiii. } 30)$$

and Schäfli's formula*

$$J_\mu(x)\,J_\nu(x) = \frac{1}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\,(\tfrac{1}{2}z)^{\mu+\nu}\,{}_2F_3(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2},\ \tfrac{1}{2}\mu+\tfrac{1}{2}\nu+1;\ \mu+1,\ \nu+1,\ \mu+\nu+1:\ -z^2).$$
$$\dots\dots(9\cdot5;\ \text{R. xiii. } 21)$$

The formula

$${}_1F_1(-m,\ n+1,\ x)\,{}_1F_1(-m,\ n+1,\ -x) = {}_2F_3(-m,\ m+n+1;\ n+1,\ \tfrac{1}{2}n+\tfrac{1}{2},\ \tfrac{1}{2}n+1;\ -\tfrac{1}{4}x^2)$$
$$\dots\dots(9\cdot6;\ \text{R. xiii. } 24)$$

I have not been able to find elsewhere. It may be proved as follows. When we perform the multiplication, the coefficient of x^p is found to be (apart from sign)

$$\frac{m(m-1)\dots(m-p+1)}{p!\,(n+1)(n+2)\dots(n+p)}\,{}_3F_2\begin{pmatrix} -m,\ -p,\ -n-p \\ m-p+1,\ n+1 \end{pmatrix}.$$

The series here is of the type $(3\cdot2)$. Summing by means of $(3\cdot2)\dagger$, we obtain the result.

Another formula of essentially the same kind, but more complicated, is

$${}_0F_2(m+1,\ n+1;\ x)\,{}_0F_2(m+1,\ n+1;\ -x)$$
$$= {}_3F_8\left(\begin{array}{c} \tfrac{1}{3}(m+n+1),\ \tfrac{1}{3}(m+n+2),\ \tfrac{1}{3}(m+n+3) \\ \tfrac{1}{2}(m+n+1),\ \tfrac{1}{2}(m+n+2),\ m+1,\ n+1,\ \tfrac{1}{2}(m+1),\ \tfrac{1}{2}(m+2),\ \tfrac{1}{2}(n+1),\ \tfrac{1}{2}(n+2) \end{array};\ -\tfrac{27}{64}x^2\right).$$
$$\dots\dots(9\cdot7;\ \text{R. xiii. } 22)$$

The last series is $\Sigma\dfrac{(-1)^q}{q!}\,A_q x^{2q}$, where

$$A_q = \frac{(m+n+2q+1)(m+n+2q+2)\dots(m+n+3q)}{(m+1)\dots(m+q)(n+1)\dots(n+q)(m+1)\dots(m+2q)(n+1)\dots(n+2q)},$$

so that its form is somewhat obscured when it is written in hypergeometric form. The proof of $(9\cdot7)$ proceeds on the same lines as that of $(9\cdot6)$. In general, I imagine that Ramanujan regarded the formulae of this chapter as corollaries of those of Ch. xii. Thus Clausen's formula $(9\cdot5)$ may be proved by comparing coefficients and using $(2\cdot1)$, and this was no doubt Ramanujan's method.

It is hardly necessary to add that all these formulae are accompanied by a mass of corollaries and striking particular cases. I hope that it may be possible in the future to publish the note-book in full. These are the only two chapters which, up to the present, I have been able to subject to a really searching analysis.

* See Watson, *Bessel Functions*, 145 *et seq.*, where further formulae of this kind are given.

$\dagger$ **As** s is integral, we must replace $\Gamma(1-\tfrac{1}{2}s)/\Gamma(1-s)$ by its limiting value, 0 if p is odd, $(-1)^q\,2q\,!/q!$ if $p=2q$.

Capture and loss of electrons by α particles. By Professor Sir
ERNEST RUTHERFORD, F.R.S.

[*Read* 5 March, 1923.]

In a recent paper, Mr Henderson* drew attention to some
striking phenomena which attend the passage of α particles through
matter. A narrow pencil of α rays, after passing through mica,
was bent in a magnetic field in a high vacuum and fell on a Schu-
mann photographic plate. In addition to the main deflected band,
due to ordinary α particles carrying two positive charges, he found
evidence of a 'midway' band which suffered only half the deflection
of the main band. When the velocity of the issuing rays was
reduced by additional thicknesses of mica, it was found that the
midway band increased rapidly in intensity at the expense of the
main band and ultimately at low velocities became predominant.
At the same time a neutral band, which was not deflected by a
magnetic field, made its appearance.

Henderson interpreted these observations by supposing that
the α particle in passing through matter occasionally captures an
electron and the midway band is thus due to singly charged helium
atoms. The neutral band which appears at low velocities of the
α particles is ascribed to neutral helium atoms resulting from the
capture of two electrons.

It was very desirable to check these conclusions by means of
the scintillation method where the energy of the α particles can
be estimated by the brightness of the scintillations and their
number determined by direct counting. As Mr Henderson was
unable to complete this work in the Cavendish Laboratory before
leaving for Canada, I undertook to verify his conclusions by the
scintillation method and the present paper contains a preliminary
account of some of the results so far obtained.

The general method employed is shown in Fig. 1. The radio-
active source consisted of a fine platinum wire W (·3 mms. in
diameter) coated with radium B and C placed in an exhausted
box B. The rays passed through a fine slit S and fell on a zinc
sulphide screen placed inside the box. The scintillations were
viewed through a glass plate by a microscope in the usual way.
The microscope was arranged to have a vertical motion so that
any part of the screen could be brought into view. The slit was
equidistant, viz. 8 cms. from the source and screen and a magnetic
field of about 6000 gauss was ordinarily applied perpendicular to

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* G. H. Henderson, *Proc. Roy. Soc.* A, 1922, **102**, p. 496.

the path of the rays. In order to prevent contamination of the screen, the active wire was placed in a small chamber sealed with the requisite thickness of mica. During an experiment, this was exhausted by means of charcoal so all chance of contamination was avoided. The main box was initially exhausted by a Gaede pump and then by a Langmuir diffusion pump. The residual pressure of the gas in the box was measured by a Macleod gauge and two liquid air traps were in the circuit to prevent mercury vapour from the pump or gauge diffusing into the deflecting box. The field of view of the microscope (1·2 mms. wide) was more than sufficient to take in the whole width of the main band of homo-

Fig. 1

geneous rays. When a sheet of mica of about 6 cms. stopping power was placed over the source, the distribution of scintillations along the screen is shown diagrammatically in Fig. 2. The width of the deflected main band *MM* is much wider than with zero field (cp. dotted lines *P*) showing that the α rays after traversing the mica vary markedly in velocity. The midway band *LL* begins sharply at the midway point between *O* and *P* but extends with decreasing brightness of the scintillations to the main band where it can no longer be followed. A group of undeflected particles giving weak scintillations appears at *O* while between *O* and the midway band a small number of weak scintillations were counted. The origin of the latter was at first difficult to fix with certainty

but they were finally shown to be due to recoil atoms set in motion by collision with the α particles in their passage through mica. For example, these distributed scintillations were not observed when the rays passed through gold instead of mica, although the number of scintillations in the neutral and midway bands were about the same as for mica.

Incidentally it was noted that α particles of velocity $\cdot2V_0$, where V_0 is the maximum velocity of the α particles from radium C, could be readily counted, while α particles of velocity $\cdot15V_0$ gave visible scintillations but too weak to count with certainty.

Special experiments were made to prove definitely that the midway band consisted of singly charged helium atoms. By means of two parallel plates 4 mms. apart placed in the path of the rays between the source and slit, an electric field was applied of 7500 volts per cm. The inside edge of the two pencils of α rays deflected by a magnetic field was viewed successively in the microscope and the alteration of the deflection noted when the electric field was added. This method proved very simple and accurate, and it was found that the edge of the midway band was deflected by the electric field to exactly half the extent as the edge of the main band at M. This result, coupled with the observed magnetic deflexions in the ratio of 1 to 2, shows conclusively that the midway band consists of particles of mass 4 carrying a single charge.

Experiments were then made to examine quantitatively the ratio between the number of singly and doubly charged particles for different velocities of the α rays. It will be seen that the α particle in passing through matter occasionally captures an electron and this electron may be subsequently lost by collision with the molecules. We may suppose that for a definite velocity, the He_{++} particle has a mean free path λ_1 cms. in the material through which it passes before capturing an electron. Similarly the He_+ particle has a mean free path λ_2 cms. before it loses the captured electron. From the magnitudes given later for λ_1, λ_2, it is clear that, using a large number of α particles, a temporary equilibrium must exist for any given velocity between the number N_1 of He_{++} particles and the number N_2 of He_+ particles and that in traversing a distance dx, the number of captures is equal to the number of losses. Consequently

$$\frac{N_1 dx}{\lambda_1} = \frac{N_2 dx}{\lambda_2},$$

or

$$N_2/N_1 = \lambda_2/\lambda_1.$$

If the ratio N_2/N_1 is measured for particles over the same narrow range of velocity, then λ_2/λ_1 is known. This relation will obviously apply not only to a definite beam of homogeneous rays but to

α particles of all velocities present in a heterogeneous pencil of α rays such as is shown in Fig. 2. In this way, the ratio λ_2/λ_1 could be determined over a range of velocities from a single distribution curve of He_+ and He_{++} particles.

The mean free path λ_2 for loss of an electron from He_+ can be directly measured in the following way. Suppose the microscope is placed in such a position as to include the particles on the inside edge L of the midway band (Fig. 2) and the number of scintillations counted in a high vacuum. When a gas at low pressure is introduced into the box, some of the He_+ particles lose an electron by a collision and are deflected away from the field of view of the microscope. For example, the loss of an electron at P (Fig. 1)

CAPTURE OF ELECTRONS
BY
SWIFT α PARTICLES
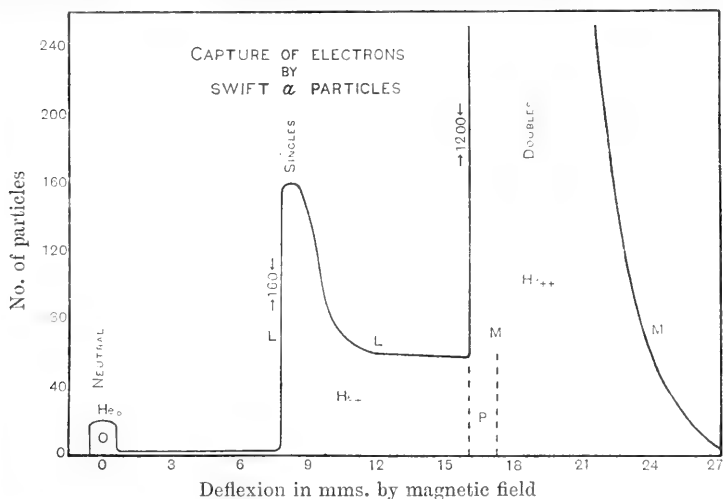
Fig. 2

will cause the particle to strike the screen at C instead of at B. It was found that the number of scintillations decreased nearly according to an exponential law with the pressure of the gas and ultimately became too small to count for a pressure of gas not sufficient to alter sensibly the velocity of the rays in traversing the gas*. Allowing for the finite width of the field of view of the microscope, the value of λ_2 could be easily deduced and in the tables is expressed in terms of mms. of air at N.P.T.

The removal of the electron from the He_+ particle is in some respects analogous to ionization and should obey similar laws. This was found to be the case for the value of λ_2 was approximately proportional to the velocity of the α particles between velocities

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* Under the conditions of the experiment, the conversion of He_{++} into He_+ in passing through the gas did not appreciably influence the results.

$\cdot 94 V_0$ and $\cdot 47 V_0$. Most of the experiments were carried out with dry air but several observations were made with purified hydrogen and helium. The mean free path in H_2 was between four and five times that observed for air and in He more than six times that in air.

It will be seen that the mean free path for loss is measured in air or other gas, while the ratio λ_2/λ_1 is measured in mica or other solid material. It was, however, found experimentally that the ratio λ_2/λ_1 observed with mica alone, for a velocity of rays of $\cdot 47 V_0$ changed only slightly when a thin layer of celluloid, aluminium, or silver was placed over the mica plate. The thickness of the layer—of the order of 1 mm. stopping power for α rays— was sufficient for a new equilibrium to be set up between He_{++} and He_+ particles but not enough to alter markedly the velocity of the issuing rays. Since little change was observed for such a wide range of average atomic weight of absorber, we may conclude that the ratio λ_2/λ_1, if it could be measured entirely in air, would be nearly the same as that observed for mica in which oxygen is an important absorbing constituent. The validity of this relation for all velocities requires further experimental confirmation but it is probably not seriously in error. The approximate constancy of the ratio λ_2/λ_1 for different elements is a matter of considerable interest but requires further verification by a more accurate method than that of counting scintillations.

The values of $\lambda_2/\lambda_1 = N_2/N_1$ for mica were directly measured for three velocities, viz. $\cdot 94$, $\cdot 76$, $\cdot 47 V_0$ in a high vacuum while the mean free path in air for conversion of He_+ into He_{++} was obtained for the same velocities. The results are given in the following table where the mean free path both for capture and loss is given in terms of mms. of air at N.P.T.

Velocity in terms of V_0	λ_2/λ_1 for mica	Mean free path for loss in air	Mean free path for capture in air
$\cdot 94$	1/200	$\cdot 011$ mm.	$2\cdot 2$ mm.
$\cdot 76$	1/67	$\cdot 0078$	$0\cdot 52$
$\cdot 47$	1/7$\cdot$5	$\cdot 0050$	$\cdot 037$

The velocity V_0 of the α particles from radium C is $1\cdot 922 \times 10^9$ cms. per second.

The midway bands for velocities $\cdot 94$ and $\cdot 76 V_0$ were sharply defined and no neutral particles were observed in either case. For the velocity $\cdot 47$, the issuing α rays were very heterogeneous and the distribution of scintillations is shown in Fig. 2. A well marked band due to neutral particles was also noted. By admitting gas

into the apparatus, this band decreased in intensity. The mean free path in air before the neutral particles were converted into He_+ particles was difficult to determine with accuracy, but was about $1·3/1000$ mm. or about $1/4$ of the mean free path for conversion of He_+ into He_{++} for a velocity $·47V_0$. No doubt the neutral particles had an average velocity much lower than $·47V_0$.

Similar results were observed when the mica was replaced by a gold sheet of equivalent thickness and the ratios λ_2/λ_1 were found to be of the same order of magnitude for all velocities examined.

A number of measurements have been made for α rays with velocities lying between $·30$ and $·20V_0$. At the lower velocities, accurate counting becomes difficult on account of the weakness of the scintillations. No measurements have yet been made of the mean free path for the conversion of He_+ to He_{++} or He_0 to He_- for these low velocities. Rough measurements indicate that the number of He_+ and He_{++} particles become equal for a velocity about $·30V_0$ and that the ratio of He_- to He_+ particles increases rapidly for velocities between $·25$ and $·20V_0$. It is desirable that the distribution of He_0, He_+ and He_- particles for low velocities should be determined for the rays after passing through absorbers of different atomic weight to test how far the distribution is dependent on the nature of the absorber.

Over the range of velocities examined, viz. $·94$ to $·47V_0$, the ratio λ_2/λ_1 varies roughly as V^{-5} where V is the velocity of the α particle. Since the mean free path for conversion of He_+ into He_{++} varies approximately as V, it follows that the mean free path for capture varies as V^6. These relations are only approximate; accurate values will require a large amount of careful counting of scintillations.

It is clear however from these results that the average α particle captures and loses an electron several hundred times before absorption. The process of capture and loss succeed each other rapidly for velocities below $·4V_0$. For velocities below $·2V_0$, the α particles, as Henderson showed, exist mainly in the state of singly charged or neutral helium atoms and no doubt there is a rapid process of interchange between these two types at such low velocities.

The results we have so far given refer to statistical averages for a large number of α particles. Certain observations have been made which seem to show that one α particle may differ from another in its power of capturing or losing an electron. As an example, we shall consider the effects observed on the main band due to the introduction of gas in the path of the rays. The inside edge of the main band for velocity $·47V_0$ was fairly sharply defined in the observing microscope. When a pressure of air was added sufficient, from the data given in the Table, to cause on an average several captures and losses for each particle, but yet not enough

to lower the velocity of the α particle sensibly, no certain displacement of the edge of the band as a whole was observed although a number of additional particles appeared close to the left of the main band shown in Fig. 2. Since the average charge on the α particle under these conditions was less than $2e$, it was to be anticipated that the band as a whole would be less deflected than in a vacuum. The calculated shift should easily have been detected under the conditions of the experiment. This failure to detect any shift as a whole is an indication that a considerable fraction of the α particles had either not captured an electron in their passage through the gas or, if they had done so, had lost it much more rapidly than the average.

A large amount of careful experiment is necessary to make certain of these points and of their interpretation, but the general evidence indicates that α particles of the same velocity may have somewhat different properties as regards ease of capture or loss or both. Mr Kapitza* has obtained more definite evidence of the individual differences between α particles of the same velocity by bending individual α particles of low velocity by a powerful magnetic field, and observing their track in a special Wilson expansion apparatus. He has suggested that the effects may be due to an asymmetry of the electric field round the helium nucleus.

The observations made by Mr Henderson on the capture of electrons by flying α particles has thus opened up a new and interesting field of enquiry. It is not easy to explain why an α particle moving with a velocity corresponding to a 1000 volt electron can capture and hold an electron. Until further evidence is available to show whether capture is dependent on the presence of tightly bound electrons in the atoms through which the α particle pass, it is not desirable at this stage to enter on a discussion of this difficult question. Experiments on this subject will be continued and I hope in a later paper to discuss in more detail the many interesting experimental and theoretical points involved.

* P. Kapitza, *Proc. Camb. Phil. Soc.*, following, 1923.

CAVENDISH LABORATORY,
    CAMBRIDGE.

March 1923.

Some Observations on α-Particle Tracks in a Magnetic Field. By
P. KAPITZA. (Communicated by Professor Sir E. RUTHERFORD,
F.R.S.)

(Plate V.)

[*Read* 5 March, 1923.]

The Wilson expansion chamber gives a powerful method of
studying the properties of single α-particles. In the recent experi-
ments of Blackett*, it was shown that by studying the forks and
the kinks which occasionally occur on the α-tracks, it is possible to
obtain the velocity of an α-particle at any point on the path. It is
obvious that for a further study of the phenomena along the track
it would be very convenient to observe some systematic changes
in the tracks due to a known cause. One of the most obvious of
such changes in a track will be produced by a magnetic field. Then
the tracks will be no longer straight, but curved. The radius r of
the curvature at each point of the track will be given by:

$$r = \frac{v}{e} \cdot \frac{m}{H},$$

where e and m are the charge and the mass of the α-particle taken
in suitable units, v is the velocity and H the applied magnetic field.

So from this expression, if r is measured along the track
and m and H are known, the ratio $\frac{v}{e}$ can be deduced. The main
difficulty in this experiment is to obtain a sufficiently strong
magnetic field, which will bend the track of the α-particle appre-
ciably. From the expression for r it is easily calculated that for a
doubly charged α-particle emitted from radium C', the radius of
curvature in a field of 75,000 gauss will be 5·3 cm. To produce
such a strong magnetic field special methods must be applied,
since it would be quite impossible to obtain such a field over a
suitable area with an ordinary electromagnet.

To obtain a strong field in a solenoid a large amount of energy
(several thousand kilowatts) is needed, also the current which is
required to produce such a field would fuse any coil in a few
seconds if it were maintained so long†. But this last difficulty can
be avoided if the field is produced only during a few thousandths
of a second, so as not to allow the current time to heat up the coil.
For an α-particle travelling with a speed of the order 10^9–10^8 cm.
per second we may regard such a field as constant. So only the
first difficulty, namely that of obtaining a store of energy of several
thousand kilowatts, has to be overcome.

* *Proc. Roy. Soc.* 1922, 294.
† See Fabry, *Journal de Physique,* 1910, 129.

It was found possible to obtain such a sudden kind of electrical discharge by means of a specially built accumulator. This battery was constructed in such a way as to have a very small resistance and capacity. By its means currents of 8000 amp. for 0·002 of a second could be obtained. In these experiments the field which was produced in a bobbin surrounding a circular Wilson chamber of 2·2 cm. in diameter, reached 75,000 gauss*.

It is not the subject of this paper to give a description of the apparatus used in these experiments. This will be done in another paper; here will be given only a short account of some phenomena observed on the photographs of curved α-ray tracks obtained in this magnetic field.

In Figs. 1 and 2 (Pl. V) are given examples of these photographs. Both were obtained in a magnetic field of the same magnitude, namely 75,000 gauss. It is easily seen from these photographs that the curvature is different for each individual α-track.

One immediately asks whether this difference in curvature is genuine or whether it is due to some stray effect. We shall shortly indicate the most probable sources of errors which were investigated.

From measurements it appeared that the magnetic field in the expansion chamber was uniform to 1 per cent. By means of an oscillograph the field is measured with an accuracy of 2 per cent. A special shutter which let the α-particles into the expansion chamber only during 0·001 of a second is timed with the current in the coil. Actually two oscillographs are used, one to record, on a falling photographic plate, the current, and the other the times of opening and closing the shutter. In this way it is possible to estimate the variation of the magnetic field in the time during which the α-particles enter the apparatus. This variation was not greater than 4 per cent. Up to now no stray effects due to the experimental arrangement have been found which could account for the difference in the curvature. But the experiment is a difficult one and it may be that in the course of the work some new stray effects which may be due to the unusual magnitude of the magnetic field will be discovered. It is possible that the kinks which often occur on the tracks are capable of producing this difference of curvature.

To observe a bend due to nuclear collisions on an α-ray track it is not sufficient to enlarge the photograph uniformly, because the angle remains always the same. But if the enlargement is done in the direction perpendicular to the track while there is no enlargement along the length of the track then the angle of a bend becomes sharper and more easily distinguished. The tracks *a* and *b* of Fig. 1

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* By means of this battery, a field of 500,000 gauss was obtained with a coil of 1 mm. inside diameter. In the near future an attempt will be made to measure magnetic susceptibilities in this field.

are enlarged 2·5 times in the vertical direction. It is easily seen that in the enlargement *b* there is a kink at the point indicated by the arrow. The upper track on the photograph has no kinks as is seen from the enlargement *a*. Considering only tracks without kinks, the extreme values of the radii of curvature of different tracks at equal distances from the ends are in the ratio of about 1 to 5. To show this difference more obviously the diagram of Fig. 3 is drawn. The curvature at various points of the last 1 cm. of the range of eleven good tracks was measured. It is assumed that the speed of each α-particle decreases in the same way. (This
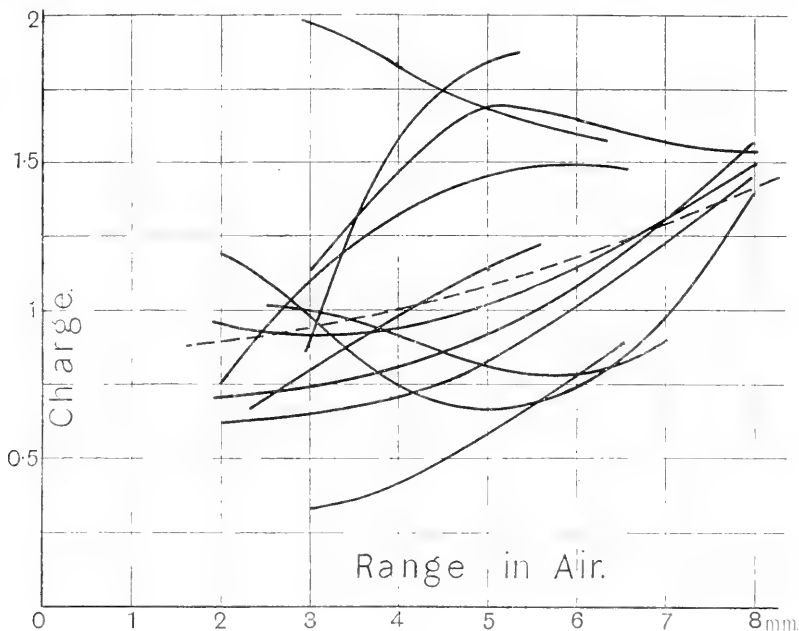
Fig. 3

assumption is probably not right as will be seen later.) The value for the speed of the α-particle when it is describing the end part of its track is taken from the curve given by Blackett*. Thus the average charge carried by an α-particle in different parts of the track could be obtained. In Fig. 3 the charge so obtained is plotted against the remaining range for each α-particle. On this diagram it is easily seen how differently the α's behave at the end of the track. Henderson† has shown that the α-particle in the end part of its range is recombining with the electrons and becomes singly charged or neutral. Prof. Rutherford‡ has shown that the loss

* *Loc. cit.*   † *Proc. Roy. Soc.* 1922, 496.   ‡ *Proc. Camb. Phil. Soc.*, this number.

and capture of an electron by an α-particle occurs more than 100 times along 1 mm. of the track in the end part of the range. So we could not expect to obtain whole numbers for the average charges in our experiments and in fact it is seen in Fig. 3 that we do not do so. From this diagram it could be seen that about 8 mm. from the end of the range all behave in a similar way and have nearly the same average charge $+1\cdot5e$.

Getting nearer the end the different α's are not equally capable of carrying an electron. Some of them, so to speak, prefer to travel doubly charged, some singly charged, and some neutral. This difference in behaviour on the part of the α's cannot be explained by some probability variation in the capture and loss of electrons along the path of the α's, because from the experiments of Prof. Rutherford the number of recombinations and ionisations of an α-particle is large. If there are no stray effects due to causes at present unknown we are led to the necessity of assuming that there are different kinds of α-particles.

We shall now give an explanation of a type of difference in α-particles which may be taken as a working hypothesis for explaining these facts.

From Fig. 3 it is seen that the difference in behaviour of α's is already large when the average charge is greater than unity. This means that some of the nuclei of the particles prefer to travel singly charged, others doubly charged. So we are led to the necessity of assuming that the difference in the α-particles is in the nucleus. Rutherford[*], from experiments on scattering of hydrogen atoms by collision with α-particles, came to the conclusion that the α-particle nucleus is an asymmetrical structure. Further experiments by Chadwick and Bieler[†] confirmed this point of view. The most probable shape of an α-nucleus is an oblate spheroid of semi-axes about 8×10^{-13} and 4×10^{-13} cm. Now if we take the α-particle nucleus as a disc and assume that the plane of this disc may have different orientations with respect to the direction of the motion of the α-particle, and further that this orientation remains fixed along a considerable portion of the range, we shall have provided different kinds of α-particles.

Before recombination, this difference in α's will not show itself. But it is probable that an α-particle, which has a different orientation of the plane of the nuclear disc from another α-particle, has also a different capacity for losing and gaining electrons. Very little is known about the mechanism of recombination and ionisation, so speculation is difficult. But still it is possible to sketch a picture of the phenomenon.

When an α-particle nucleus captures an electron the latter will describe an orbit round the nucleus. According to the quantum

[*] *Phil. Mag.* xxxvii, 537 (1919).   [†] *Ibid.* xlii, 923 (1921).

law the plane of the orbit of the electron in the non-uniform field of the nucleus must take up a definite orientation.

We must expect that the only position for the plane of the orbit in which the electron rotates is in the plane of the nuclear disc. So we anticipate that the plane of the moving electron, like the nuclear disc itself, must remain in a definite position relative to the velocity of the α-particle. Further, we may expect that the probability of the loss of an electron by an α-particle depends on the orientation of the orbit. To get some idea of how this may happen, let us imagine that the α-particle is at rest and the surrounding molecules have a velocity equal to that initially carried by the particle; then the α-particle will be placed in a parallel stream of electrons. According to the theory of Bohr*, a swiftly moving electron has its least chance of ionising an atom if its velocity is perpendicular to the plane of vibration of the electron which moves in the atom. So the α-particle, according to this picture, is least likely to lose an electron when the plane of the orbit is perpendicular to the stream of electrons. We see thus that an α-particle will have the best chance of travelling singly charged if the plane of the nuclear disc is perpendicular to the direction of motion. At least, these conclusions will be correct if we assume that the probability of the capture of an electron does not depend on the orientation of the nuclear disc. This picture is sketched only for the purpose of showing that it is possible to relate the fact that different α-particles have different powers of holding electrons with the asymmetrical structure of the α-particle nucleus found by Prof. Rutherford. More experiments are needed before we can regard this theoretical speculation as established.

It is worth while pointing out the relation of the phenomenon just described to other phenomena previously observed. It is known that α-particles emitted from a radioactive source have not exactly equal ranges in spite of the fact that their initial velocities are appreciably the same. This phenomenon is called straggling. In the recent measurement made by Mlle Irene Curie† in which the range of α's was measured by the length of the tracks in a Wilson chamber (probably one of the most exact methods of determining the range), it was found that the straggling of the ranges is about 3 mm. in air. The attempt to explain straggling on statistical grounds has failed ‡. From observations§ on a beam of α-rays deflected in a magnetic field it is known that the velocity is fairly uniform up to 1·5 cm. from the end of the range in air. This shows that the chief part of straggling occurs in the end of the range; it is here also that the phenomenon described in this paper appears.

   * *Phil. Mag.* xxv, 10 (1913).   † *Comptes Rendus*, 1923, 434.
   ‡ Henderson, *Phil. Mag.* xliv, 42 (1922).
   § Marsden and Taylor, *Proc. Roy. Soc.* 1913, 88. 443.

If we assume that a doubly charged α-particle has a different efficiency for ionising from a singly charged or neutral one, the speed of an α-particle which usually carries a double charge will diminish at a different rate from that of one which is for the most part singly charged. The different way in which the speed of different α-particles diminishes will provide an explanation of straggling. It is now clear that in drawing the Fig. 3 it was not legitimate to assume that all α's have the same speed in the same part of the range. So we cannot attach much importance to the increase in charge of some of the α-particles indicated in diagram towards the end of the range.

We expect that the experiments, now in progress, will clear up all the points now discussed and place the problem on a firmer basis.

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    CAMBRIDGE.

Fig. 1

Fig. 2

Scale

A Note on the Natural Curvature of α-Ray Tracks. By P. M. S. BLACKETT, B.A. (Communicated by Professor Sir E. RUTHERFORD, F.R.S.)

(Plate VI.)

[*Read* 5 March 1923.]

It has been mentioned by Shimizu[*] and also by the writer[†] that towards the ends of the tracks of α-rays an appearance of general curvature is noticeable that cannot be resolved by eye into isolated sharp bends. Examples of this are shown in Plate VI, Nos. 1 and 2. The question that immediately arises is whether this apparent curvature can be attributed to multiple scattering, that is, to the *chance* superposition of a number of small bends, or whether some additional mechanism must be postulated. An answer to this is made possible on account of an unexpected relation that has been observed between the *direction* of this apparent curvature, in the case of the tracks of those α-particles that have made certain types of nuclear collision, and the initial direction of the tracks before collision.

On examining carefully all the air forks that have so far been photographed by the writer, a strong impression was formed that there was a tendency for the particles to curl, after collision, away from their initial direction of motion (Nos. 3, 4 and 5). This effect appeared most marked when the remaining range of the α-particles lay between 0·4 and 2·0 mms. at N.T.P., and when the angles of deflection ϕ at the forks had values between 35° and 120°.

In many cases also the spurs due to the air recoil atoms showed a similarly directed curvature (Nos. 3, 4 and 6), while, on the other hand, the spurs due to the recoil atoms of helium showed curvatures in both directions (Nos. 7 and 8). In the latter case the plane of the curvature seemed to be inclined at an angle to the plane of the fork. However, as the data in these cases is very meagre, our attention will be chiefly directed to the discussion of the curvature of the α-ray tracks themselves. In order to obtain a numerical test of this phenomenon the following procedure was adopted.
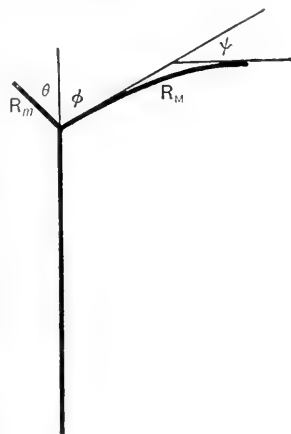
* Shimizu, *Proc. Roy. Soc.* vol. 99, p. 425 (1921).
† Blackett, *Proc. Roy. Soc.* vol. 102, p. 294 (1922).

The angle ψ between the initial direction of the α-particle after collision and the final direction* of the end of its track was measured in all the forks for which the deflection ϕ and the remaining range R_M lay between the limits mentioned. The positive direction of ψ is that of ϕ increasing. Eighteen forks were available for these measurements. The values of ψ were found to vary from $-20°$ to $+60°$, with a mean of $14°$, while the mean deviation from this mean value was $16°$. The average values of R_M and R_m were 1·1 and 0·26 mms. respectively. These results can be expressed as follows. On the average all the tracks suffered a *deflection* of $14°$ away from their initial direction, while their average angular *spread* was $16°$.

Rough as these results are they afford evidence of some other phenomenon than that of normal scattering. This latter is competent to explain the order of magnitude of the spread of the tracks but cannot possibly explain the average deflection in one direction. It is convenient to consider this average deflection as equivalent to a constant curvature, of an amount that can easily be calculated to be expressed by a radius of curvature of 4 mms.

It is difficult to calculate the average angle of scattering of such slow moving particles owing chiefly to the difficulty of estimating the effect of the electronic screening of the nuclei of the gas molecules, but by making some approximate assumptions and using the theory of scattering given by Thomson*[†] and Rutherford[‡], it is possible to show that the average deflection of the α-particles in travelling the last millimetre of their range is of the order of $10°$ or $20°$. This agrees roughly with the measurements.

In attempting an explanation of the average curvature away from the initial direction of the track it is necessary to distinguish two separate phenomena, the curvature itself, and the relation of the curvature to the previous collision. A few simple calculations show at once that the former effect cannot be due to any external electric or magnetic field. It is thus necessary to seek the mechanism of the curvature in an asymmetry of the α-particle itself. Now the work of Henderson has shown that at the velocities with which we are concerned, which are all less than 3×10^8 cms. per sec., the average charge of an α-particle is less than one. In addition, some recent work of Sir Ernest Rutherford, which he kindly allows me to mention, has shown that at velocities slightly greater than this the ratio of the numbers of singly to the numbers of doubly charged α-particles is about ten to one. At still lower velocities the ratio may be many times greater. It is thus probable that the

* The final direction is taken as the direction of the track at a distance of about 0·2 mm. from its extreme end.
† Thomson, *Proc. Camb. Phil. Soc.* 15, p. 415 (1910).
‡ Rutherford, *Phil. Mag.* p. 669 (1911).

particles remain usually in a singly charged or neutral state and
only rarely lose both electrons. Now an α-particle with a single
electron, revolving presumably in a 1-quantum orbit of radius
$2 \cdot 7 \times 10^{-9}$ cms. with an orbital velocity of $4 \cdot 4 \times 10^8$ cms. per sec.,
will possess a marked degree of asymmetry. If we suppose that
the plane of the orbit is parallel to the direction of motion, the path
of the electron in space will be a trochoid and there will be an
asymmetry in the electric field due to the complex system. Confining
ourselves to points in the orbital plane at distances from the particle
which are not great compared with the radius of the orbit, it can
be seen that, although the average amplitude of the electric inten-
sity is the same on both sides of the track, its variation with the
time is markedly different. On that side on which the electron is
moving slowly the electric forces will vary more slowly with the
time than on the fast side, and will, presumably, produce a
greater disturbance of the electronic systems of the gas molecules.
The singly charged α-particle can from this point of view be con-
sidered as moving faster on one side than on the other. Since the
energy lost to the gas molecules on the slow side, will on this view
be the greater, there is likely to be a resultant transverse component
of momentum which will tend to deflect the particle towards the
slow side, that is, the curvature of the α-ray track will be in the
same direction as the rotation of the electron.

It is hoped that it will prove possible to investigate precisely
the loss of energy and momentum of such a system, on the lines of
the theory of the decrease of velocity of charged particles given by
Bohr*. Failing such an investigation it is of some interest to calcu-
late the minimum radius of curvature that can be explained by
postulating the maximum degree of asymmetry, that is the case
when energy is only lost to molecules confined to one plane and to
one side of the track. If we suppose that all the energy is spent in
producing ions, and that in the formation of each ion there is a
transference of momentum equal to that possessed by an electron
of kinetic energy equal to the ionization energy W, then we can
derive† the expression

$$\rho = \left(\frac{W}{2m}\right)^{\frac{1}{2}} \Big/ \frac{dv}{dx}$$

for the radius of curvature of the track of the α-particle. In this
equation m is the mass of an electron and dv/dx the rate of decrease
of velocity of the α-particle. Taking W as $13 \cdot 5$ volts and dv/dx‡
as 2×10^9 cms. per sec. per cm., we find a value for ρ of $0 \cdot 5$ mm.
Comparing this with the observed value of $4 \cdot 0$ mms. it seems that

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* Bohr, *Phil. Mag.* 25, p. 10 (1913).

† It is necessary to assume that the total momentum transferred is considerably
larger than the loss of forward momentum of the α-particle.

‡ Blackett, *loc. cit.*

it will be found unnecessary to attribute any great degree of asymmetry to the singly charged α-particle to explain the observed curvatures.

It would be anticipated that the recoil atoms of air and other gases will also in general possess an asymmetrical arrangement of electrons, and so can be expected to travel in curved instead of straight paths. Thus this general type of explanation seems adequate to account for the marked curvature that is shown by the tracks of some air atoms.

It may be considered an objection to the theory that it is necessary to suppose that the α-particle rarely if ever loses its second electron along the last two millimetres of its track, although still able to produce several thousand ions. It must, however, be remembered that the second ionization potential of helium is very high (54 volts) and that the maximum velocity with which we are concerned is not much greater than that minimum velocity ($2 \cdot 2 \times 10^8$ cms. per sec.) that must be possessed by an α-particle in order that it may be able to impart this amount of energy to a *free* electron. It thus seems very probable that at these low velocities complete interpenetration of such heavily bound orbits may take place without energy transference.

The assumption that the plane of the orbit is parallel to the direction of motion was made expressly to give the maximum possible asymmetry. It may be, however, that this position is that of greatest stability.

Concerning the relation of the curvature to the preceding fork, it seems legitimate to picture an *immediate re-organization* of the electronic systems of the two particles as their respective nuclei separate after the collision. It appears reasonable to suppose that the mutual influence of the electric fields of the two separating particles will affect the chance of the capture of electrons in definite orbits. It can in fact be seen in the case of such a fork as is shown in the figure, that the large value of the field due to the air atom nucleus, will tend, as the particles separate, to rob the neighbourhood of the α-particle of those electrons whose positions and velocities are such that they are likely after capture to move in anticlockwise orbits (with reference to the figure). Thus there will be an increased chance that the electron actually captured will move in a clockwise orbit. The curvature produced by such an orbit will be in the direction of the observed curvature.

In conclusion, emphasis must be laid on the hypothetical nature of this theory of the asymmetric ionization by complex particles, and also on the inadequacy of the data on which the theory is based. The excuse for the latter is the rarity of the forks. Those on which the measurements were made are all that were obtained amongst the photographs of over six thousand tracks.

Plate VI
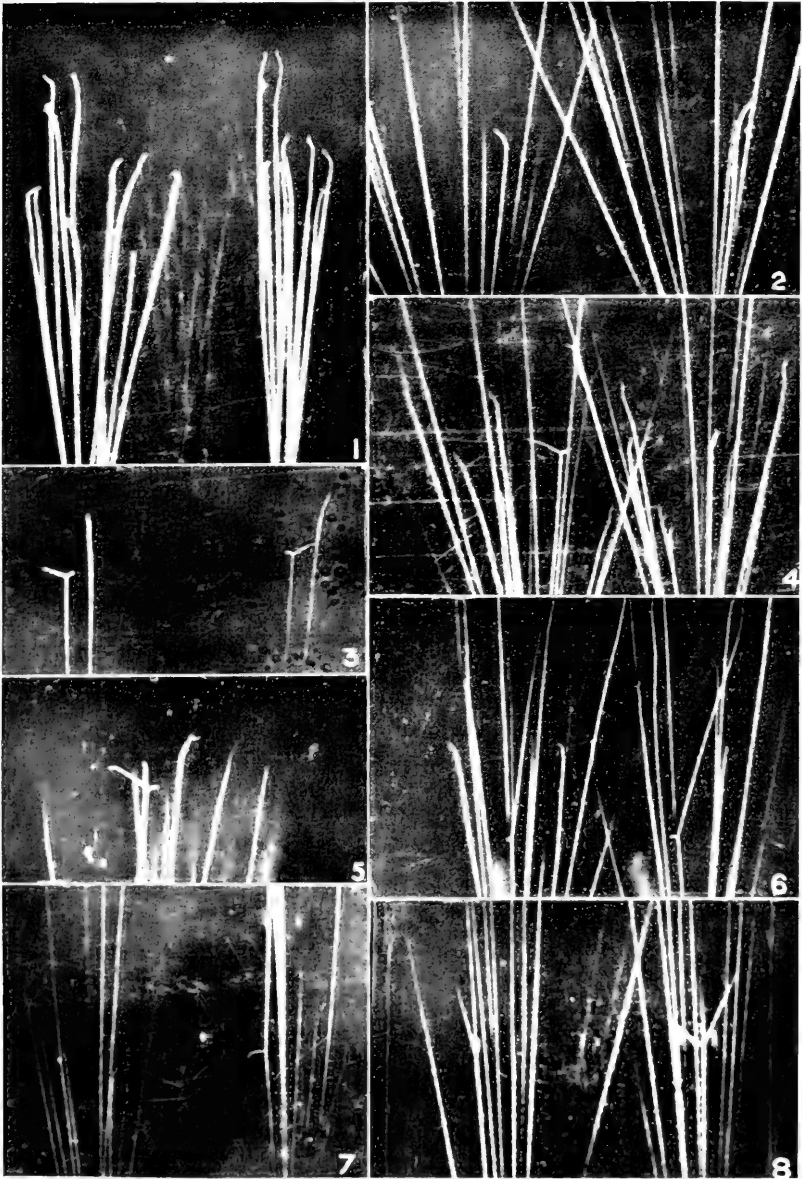
Contributions to the Theory of the Motion of α-particles through matter. Part I, Ranges. By R. H. FOWLER, M.A.

[*Read* 5 February, 1923.]

§ 1. *General introduction and summary.* The following paper (two parts) is definitely limited in scope and attempts only to throw light on certain special points of some importance in the theory of the retardation of an α-particle in passing through matter. The basis for the construction of a detailed theory of the whole phenomenon—more especially of the end of the range phenomena such as capture and loss of electrons by the moving α-particle—is hardly yet clear.

The theory of the retardation remains much as it was left by Bohr* in 1915, except for an interesting suggestion by Henderson†, that the whole effect of the α-particle in passing through matter might consist of ionization and transference, and be calculable at once when all the ionization and transference potentials are known. Henderson's theory, which will be described in greater detail in § 3 where it is critically discussed, is at first sight very attractive, and suggests that a general survey of the ranges of α-particles in various substances, based thereon, might be interesting and instructive; for, in general terms, all the ionization and more important transference potentials of most atoms are now at least roughly known as a result of X-ray and optical systematic spectroscopy. This survey occupies Part I of the paper, which is completed in Part II by a discussion of the ionization caused by the α-particle. It will be found that it is not possible to discuss these two problems entirely apart.

On a closer examination it is found that the simple form which Henderson gives to the theory cannot be fully accepted as an explanation of the stopping power of any atom for an α-particle. As we shall see in § 3, it accounts naturally only for about one-half of the observed stopping power of He and perhaps three-fifths that of Air. It is only by straining the facts concerning transference potentials quite unwarrantably that the theory can be made to agree with experiment for Air, which is the sole comparison made in the paper quoted. The failure of this theory to account for the full stopping power is itself a fact of some importance—either it drives us back on the theory of Bohr for at least the remainder of the stopping power, and shows that a considerable fraction of the energy lost by the α-particle must go in setting in vibration the electrons in their ordinary quantized orbits, without effecting transference from one orbit to another—or else we must suppose

* Bohr, *Phil. Mag.* vols. 24, p. 10 (1913), 30, p. 581 (1915).
† Henderson, *Phil. Mag.* vol. 44, p. 680 (1922).

that energy can be spent in ionization and transference much more freely than the classical calculations allow. Further light is thrown on these alternatives by the study of ionization in Part II.

Though we must admit the shortcomings of Henderson's form of the theory, a general survey on this basis is still of some interest, for there is a close parallelism between the atomic stopping powers of this theory and those of experience.

With regard to the ionization produced by an α-particle there do seem to be certain points at which Bohr's discussion can be carried somewhat further. Bohr* himself has pointed out that these assumptions, on which Henderson has attempted to calculate the total loss of energy in the early part of the range, can probably be correctly used to calculate the *ionization* there produced by the α-particle. Bohr's calculations are in need of revision in the light of more recent data, and are open to criticism on one theoretical point. Bohr assumes in calculating the secondary ionization that a slow electron moving in a gas of ionization potential λ with an energy between $s\lambda$ and $(s + 1) \lambda$ will make exactly s ions. But this is equivalent to assuming that at each effective encounter it either transfers the whole of its energy or exactly λ to the other electron, an assumption which seems hardly warranted by existing evidence. By introducing considerations from classical dynamics, admittedly inadequate in such cases for a full account of the exchange, simply as a guide to the *relative* frequencies of various types of effective encounter a more satisfactory figure for the number of ions made can be obtained which roughly speaking is less than Bohr's by a factor $\frac{3}{4}$. The agreement between theory and experiment in Air and H_2 is considerably improved. This discussion occupies §§ 1–5 of Part II.

The outstanding discrepancy of the ionization in He is left untouched by these adjustments, or even rendered more acute. By comparing the ionization and the resulting stopping power it is found that the *efficiency* thus calculated of the α-particle as an ionizing agent in Air and H_2 is greater than that observed, but correct in the case of He. Thus while the extra stopping power of Air and H_2 can come from the vibrations set up in the atomic or molecular systems according to Bohr's theory, and also from dissociation, in the case of He no such explanation is possible. The extra expenditure of energy must be spent with much the same efficiency on ionization, over and above the maximum ionization allowed by the classical calculations. The experiments of Millikan† on the proportion of doubly charged ions left by an α-particle in He throw light on these points. That these experiments may be

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* Bohr, *loc. cit.* § 6, extending previous work by J. J. Thomson, *Phil. Mag.* vol. 23, p. 449 (1912).

† Millikan, *Phys. Review*, vol. 18, p. 456 (1921).

properly interpreted, account must be taken of the secondary ionization. *It then appears that at the point of maximum ionization at least three-fourths of the* He *atoms effectively hit by the α-particle must have both electrons removed, and over one-third at the higher velocity used by Millikan.* Allowing for some selective recombination, these facts suggest the tentative hypothesis that the anomalous ionization in He is largely due to a very large proportion of primary ions being doubles. The stopping power of, and ionization in, He are then satisfactorily accounted for, with very little surplus energy loss to be assigned to vibrations. The completed group of two 1-quantum electrons in He (and the K-levels of other elements) is a unique electronic structure. The two electrons are closely interconnected and appear to respond in a unique way to the passage of the α-particle. Further experimental confirmation of the facts on which these deductions are based is highly desirable. This discussion occupies § 6.

These calculations were just finished when a very interesting paper by Svein Rosseland* came to hand. This paper also discusses incidentally Millikan's observations of doubles in He and relates them to the occurrence of non-diagram lines in X-ray spectra. No allowance is, however, made for secondary ionization, which, I still believe, is essential for the interpretation of Millikan's results on α-particles and for applications to α-particle phenomena. This does not in any way affect the main part of Rosseland's paper which is concerned with electron impacts.

It is necessary before starting to marshal the existing evidence on ionization and transference potentials for the various levels in the atom, which must be done somewhat systematically if the exact position of Henderson's theory is to be made properly clear. The necessary table is given in § 2, and is more extensive than is required for immediate use. But its contents may prove to be of use in any further discussion of ionization and stopping powers.

§ 2. *The assignment of ionization and transference potentials.* An approximate knowledge of all the energy levels in the atom, that is, of all the ionization and transference potentials, is now for the first time becoming available. Existing information deduced from X-ray spectra has been systematized from this point of view in a recent paper by Bohr and Coster†, which embodies and refers to all the available evidence. For the purpose in hand here this must be supplemented to some extent for light elements and valency electrons from other sources. Not only are the various energy levels known, but the number of electrons associated with each level is assigned with considerable certainty by Bohr's theory of the structure of the atom. The only remaining doubt concerns the exact

* Rosseland, *Phil. Mag.* vol. 45, p. 65 (Jan. 1923).
† Bohr and Coster, *Zeit. f. Physik,* vol. 12, p. 342 (1923).

distribution of the electrons between the two sub-levels into which Bohr's main levels, characterized by given values of the quantum numbers n and k, must be supposed to be split. Even here there is little room for uncertainty. For the latest exposition of Bohr and Coster, which exhibits a beautiful and complete formal analogy between X-ray and optical levels, seems certainly to require that it is in all cases the upper main levels which are split into doublets, the lowest level ($k = 1$) remaining undivided. As a working hypothesis, which is eminently reasonable, I have supposed that the sub-levels divide equally between themselves the electrons of the main level to which they belong.

Ionization and transference potentials of the valency electrons are known for many gases and metallic vapours either from direct experiments with electron impacts such as those of Franck, Hertz, Horton and their collaborators, or from the optical spectrum. But further information is required in this region for Kr, Xe, N_2 and O_2.

In the intermediate region of voltages (10–50), especially for light elements, the levels are still somewhat uncertain. The X-ray data can be supplemented to some extent from the work of Millikan in the extreme ultra-violet and McLennan on electron impacts, but the interpretation of their results has not yet reached satisfactory certainty.

Reviewing the whole range of levels one may say that these can now be specified nearly enough to start on applications to theories of the ranges and ionization of the α-particle. For a preliminary survey in this connection we confine attention to

$$H_2, \text{ He, } N_2, O_2, \text{ Ne, Al, A, Cu, Kr, Ag, Xe, Au.}$$

Table I contains this information, which is probably the best yet available. The sources of information used are given in a footnote. The main entries are ionization potentials in volt, preceded by a number ( ) giving the number of electrons in the group or group to which the potential refers. A number of levels may here be grouped together when their potentials are not seriously different. Entries in [ ] at the end of each list are the first transference potentials for the valency electrons in the vapour state. The double vertical lines mark off the $K, L, \dots$ levels from one another. It must be assumed when necessary that the potentials for O and N are the same as those for O_2 and N_2 with half the electrons in each group.

A word should be said as to the values assigned to He in the table, which are less by 0·7 than the usual values. They are those preferred by Hertz, *loc. cit.*, on the basis of Lyman's observation of the true principal series of singlets in the far ultra-violet. They are used by Hertz as reference points relative to which he determines the potentials for Ne and A. The values for the L-levels and

valency electrons in O_2 and N_2 are unfortunately doubtful. It is not possible to fit together in a convincing way the somewhat inconsistent requirements of the diagrams of Bohr and Coster, McLennan's voltages and Millikan's "L-spectra". It is not really certain *a priori* that the assigned numbers of electrons are reliable in these two cases.

TABLE I. Atomic and Molecular Energy Levels (Ionization and Transference Potentials) in (electron) Volts, with the corresponding numbers of electrons.

1. H_2 (2) 11·5 ‖ [(2) 9·0?]
2. He (2) 24·6 ‖ [(2) 19·75]
7. N_2 (4) 405 ‖ (8) 32, (2) 17 ‖ [(2) 8]
8. O_2 (4) 540 ‖ (8) 35, (4) 15·5 ‖ [(4) 8]
10. Ne (2) 880 ‖ (4) 40, (4) 21·5 ‖ [(4) 16·65]
13. Al (2) 1550 ‖ (4) 106, (4) 69 ‖ (2) 11, (1) 5·9 ‖
18. A (2) 3190 ‖ (4) 325, (4) 250 ‖ (4) 30, (4) 15·3 ‖ [(4) 11·55]
29. Cu (2) 8950 ‖ (4) 1070, (4) 955 ‖ (6) 122, (6) 73, (6) 11 ‖ (1) 8 ‖
36. Kr (2) 14200 ‖ (4) 1900, (4) 1700 ‖ (6) 310, (6) 200, (6) 108 ‖ (4) 30, (4) 13? ‖ [(4) 10·5?]
47. Ag (2) 25400 ‖ (4) 3780, (2) 3520, (2) 3350 ‖ (6) 720, (3) 590, (3) 560, (3) 380, (3) 370 ‖ (6) 100, (6) 55, (6) 10 ‖ (1) 7·5 ‖
54. Xe (2) 34600 ‖ (4) 5450, (2) 5090, (2) 4780 ‖ (6) 1150, (3) 1050, (3) 940, (3) 690, (3) 675 ‖ (6) 217, (3) 174, (3) 159, (6) 70 ‖ (4) 30, (4) 11? ‖ [(4) 8·5?]
79. Au (2) 80400 ‖ (4) 14350, (2) 13720, (2) 11880 ‖ (6) 3420, (3) 3180, (3) 2740, (3) 2290, (3) 2210 ‖ (8) 785, (4) 664, (4) 579, (4) 357, (4) 338, (8) 96 ‖ (6) 107, (6) 76, (6) 11 ‖ (1) 7·3 ‖

X-ray Levels. Bohr and Coster, *loc. cit.* In certain cases the numerical values for upper levels in their tables are somewhat ragged (*e.g. O*-levels of Au) and I have then considered it best to take smoothed values from their diagrams, which provide a rational means of extrapolation and interpolation and have been closely followed throughout.

Ionization and Transference Potentials. Hertz, *Proc. Sec. Sciences, Amsterdam,* vol. 25, p. 179 (1922); Franck and Reiche, *Zeit. fur. Phys.* vol. 1, p. 154 (1920); Franck and Kipping, *ibid.* p. 320; Horton and Davies, *Phil. Mag.* vol. 39, p. 592 (1920), etc.; Olmstead, *Phys. Rev.* vol. 20, p. 613 (1922); Foote and Mohler, "The Origin of Spectra," *Amer. Chem. Soc.* (1922). This systematizes most of the preceding work.

Optical data. Fowler, "Report on Series in Line spectra," *Phys. Soc. Report* (1922).

Intermediate levels. Millikan, *Proc. Nat. Acad. Sciences,* vol. 7, p. 289 (1921); Hopfield, *Phys. Rev.* vol. 20, p. 573 (1922); McLennan and Clark, *Proc. Roy. Soc.* A, vol. 102, p. 389 (1922).

§ 3. *A critical examination of Henderson's theory.* It is assumed in this theory that when an *α*-particle encounters an electron bound in an atom or molecule, the electron is at the end of the encounter completely unaffected unless sufficient energy can be communicated to it to lift it right out of the atom—that is, to ionize the atom—or at least to transfer it to an orbit of higher quantum number, *which in view of X-ray results must be assumed to be a previously vacant orbit.* Unless this can be done the *α*-particle loses no energy in the encounter. If enough energy can be supplied for transference, then precisely the required amount is exchanged; if enough or more than enough can be supplied for ionization, ionization occurs and

the surplus energy is also transferred to the electron and carried
off by it as its kinetic energy. One further assumption must be
made, which is really equivalent to a law of probability for the
various types of encounter that occur. Henderson assumes that the
frequency of any type of effective encounter can be calculated on
the basis of the ordinary laws of dynamics for an encounter between
an α-particle and a free electron initially at rest—or, in other words,
that ionization or transference will occur if the energy so calculated
is sufficient. In general the transfer of energy so calculated will be
an upper limit for the actual transfer, for any lack of response of
the electron due to the forces binding it in the atom will usually
though not always diminish the possible effect of and on the
α-particle.

These assumptions lead at once to the formula

$$q = \frac{2E^2e^2}{mV^2\left(p^2 + E^2e^2/m^2V^4\right)} \qquad \ldots\ldots(1),$$

where q is the energy transferred during an encounter between an
α-particle (mass M, charge E, velocity V) and an electron (mass m,
charge e) initially at rest in an encounter in which the α-particle
is aimed to pass at a distance p from the electron. Putting $E = 2e$
we obtain

$$p^2 = \frac{8e^4}{mV^2}\left(\frac{1}{q} - \frac{1}{2mV^2}\right) \qquad \ldots\ldots(2).$$

In passing through a thickness dx of matter containing N atoms or
molecules per c.c. each with n electrons of any one specified type
the number of encounters in which p lies between p and $p + dp$ is

$$2\pi Nn\, pdp\, dx = \pi Nn\, dx\, \frac{8e^4}{mV^2}\, \frac{dq}{q^2} \qquad \ldots\ldots(3),$$

and the loss of energy by the α-particle due to all encounters with
such electrons is given by

$$-MV\frac{dV}{dx} = \frac{8\pi e^4Nn}{mV^2}\int\frac{dq}{q} \qquad \ldots\ldots(4).$$

In equation (4) q is an energy, but since dq/q is a pure number the
unit of energy is immaterial, and it is convenient for practical
calculations to express all energies in "equivalent voltages", that
is, in the potential drop in volts necessary to generate the same
velocity *in an electron*. If W is the voltage equivalent to V^2 and λ
is the ionization potential in volts, the limits of the integral in (4)
are $4W$, which is the maximum transfer of energy possible $(p = 0)$
and λ, and we have

$$-MV\frac{dV}{dx} = \frac{8\pi e^4Nn}{mV^2}\log\frac{4W}{\lambda} \qquad \ldots\ldots(5).$$

Formula (5) gives on Henderson's theory the rate of loss of energy due to ionization from one particular level in the atom or molecule. We have also to take account of transference. In general, for most atoms and molecules, the transference potentials are probably distributed at fairly close intervals after the first one. Their combined effect can therefore be sufficiently accurately allowed for on this theory by substituting τ, the first transference potential, for λ in (5). It is unnecessary to use the accurate summation of Henderson, which has the disadvantage of obscuring the physical meaning of the formula. In any case the use of this approximation will give an upper limit to the stopping power on Henderson's theory, for we are assuming that once the first transference potential is reached the whole transferable energy is transferred, instead of for a time only a portion of it. We have, therefore, for the rate of loss of energy, due to a single atomic level,

$$- MV \frac{dV}{dx} = \frac{8\pi e^4 N n}{m V^2} \log \frac{4W}{\tau} \qquad \ldots\ldots(6),$$

and in general $$- MV \frac{dV}{dx} = \frac{8\pi e^4 N}{m V^2} \Sigma_s n_s \log \frac{4W}{\tau_s} \qquad \ldots\ldots(7).$$

In Henderson's form of (7) $\log \tau_s$ is replaced by

$$\log \lambda_s - \Sigma_r \left\{ 1 - \frac{(\tau_s)_{r+1}}{(\tau_s)_r} \right\},$$

which is easily seen to be equivalent when the $(\tau_s)_r$ are close together.

Henderson obtains his agreement with experiment for air by taking two levels in the "air molecule" one with 4 electrons at 200 volts and one with 10·4 electrons at 15 volts. It is clear from Table I that this underestimates the ionization potentials and therefore over-estimates the stopping power. But even then he assumes in addition that for each level $\Sigma_r (1 - (\tau_s)_{r+1}) (\tau_s)_r) = 2$, which is practically equivalent to assuming that

$$\tau_s = \lambda_s / e^2 = \lambda_s / 7\cdot 4.$$

This is completely at variance with all the known facts about transference potentials.

The matter can be made clearer by direct calculations with the data of Table I and formula (7). It is better for this purpose to avoid the use of the function $Ei\,(x)$, and use approximate numerical integrations, which are of ample accuracy. We have from (7)

$$\frac{V_0^4 - V^4}{X} = \frac{32\pi e^4 N}{mM} \Sigma_s n_s \log \frac{4\overline{W}}{\tau_s} \qquad \ldots\ldots(8).$$

where X is the range in cm. and $\overline{W}$ is a suitable mean value of W for the velocity drop V_0, V. If we take $V_0 = 1\cdot 922 \times 10^9$ cm./sec.

(RaC), $V = (0.9)\,V_0$, then $\overline{W} = 940$ approximately, and $X = 1.90$ cm ·
in air at 760 mm. Hg and $15°$ C.   From Table I the value of Σ_s
can easily be calculated for N_2 and O_2 and the properly weighted
mean taken for Air. To do this we use ionization potentials for all
levels except the highest and for that the transference potential,
for since the electron must go to a vacant orbit the first transfer-
ence potential for an X-ray level will in general not differ much
from the ionization potential. We find

$$\Sigma_s n_s \log \frac{4\overline{W}}{\tau_s} = 61.5 \qquad \qquad \ldots\ldots(9).$$

Using known values for the atomic constants in (8) we find that
to give the observed range we ought to have

$$\Sigma_s n_s \log \frac{4\overline{W}}{\tau_s} = \frac{(1.922)^4\,10^{36}\,0.344 \cdot 81 \cdot 4 \cdot 1840 \cdot 10^{-56} \cdot 288}{1.90 \cdot 32\pi \cdot (4.774)^4\,10^{-40}\,2.7 \cdot 10^{19} \cdot 273}$$
$$= 110.$$

To give the correct range we must not use the genuine λ's and τ's,
but must assume, as Henderson does, that the transference poten-
tials for each level start at, say, λ/α or τ/α where

$$14.4 \log \alpha = 110 - 61.5, \ \ \alpha = 29.$$

This agrees nearly enough with Henderson's figure 7.4 allowing
for the different ionization potentials we have used. It is impossible
to believe that these innumerable transference potentials can exist
in view of the overwhelming negative evidence of all direct lines of
attack. What is, or may be, of value however in Henderson's dis-
cussion is the fact that ionization and transference calculated
according to modern ideas of atomic structure accounts at most
for 61.5 parts out of 110 in stopping power, or 56 per cent. Either
the rest of the stopping power must arise out of disturbances of the
atomic system by the passing α-particle which do not result in a
change of orbit on the lines of Bohr's theory; or else ionization and
transference can take place much more freely than these calculations
by classical dynamics can allow, although they have taken approxi-
mately the most favourable case.

A still clearer example is provided by He, for in this case we
have only one energy level and we know with some certainty that
the first transference potential is 19.75 volts and the ionization
potential 24.6 volts. For the same velocity drop the range* in He
is about 4.71 times the range in air. To give the observed range we
must have

$$2 \log 4\overline{W}/\tau = 23.4,$$

while the actual calculated value is

$$2 \log 4\overline{W}/\tau = 10.5.$$

* Taylor, *Phil. Mag.* Sept. 1913, p. 402.

Then Henderson's theory gives less than half the stopping power or more than twice the range. No possible appeal can be made to errors in the potentials used, for to give the required result we must take $\tau = 0.03$ instead of 19.75, a quite impossible value. A similar discrepancy is exhibited by H_2.

It can, I think, still be maintained that calculations of the partial stopping power which can be regarded as due to ionization and transference are not entirely without importance, and a short table of these values compared to observation is not without interest. We take in all cases the velocity drop from 1.0 to 0.9 of the velocity of the a-particle from RaC; the observed atomic or molecular stopping power is defined as the coefficient S in the equation

$$\frac{V_0^4 - V^4}{X} = \frac{32\pi e^4 N}{mM} S \qquad \ldots\ldots(10):$$

the stopping power calculated as due to ionization and transference is of course

$$\Sigma_s n_s \log 4\overline{W}/\tau_s.$$

TABLE II. Observed atomic stopping powers compared to the stopping powers due to ionization and transference on the simple theory, for the velocity drop 1.0 to 0.9 of

$$1.922 \times 10^9 \text{ cm./sec.}$$

Atomic No.	Atom	S observed	S calculated	Difference	Observer
79	Au	233	167·1	66	M. and R.*
54	Xe	—	131·2	—	—
47	Ag	177	124·4	53	M. and R.
36	Kr	—	102·7	—	—
29	Cu	146	94·2	52	M. and T.†
18	A	—	63·4	—	—
13	Al	77, 86	50·2	27, 36	M. and R.; M. and T.
10	Ne	—	42·8	—	—
8	$\frac{1}{2}O_2$	61·5	35·0	26·5	T.
[7·2]	$\frac{1}{2}$Air	55·0	30·8	24·2	T.
7	$\frac{1}{2}N_2$	—	29·7	—	—
2	He	23·4	10·5	12·9	T.
1	$\frac{1}{2}H_2$	12·3	6·05	6·2	T.

The observed and calculated values run closely parallel, with the biggest abnormalities for Al and Cu which are probably the least reliable observations. An interesting feature is the small differences

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* Marsden and Richardson, *Phil. Mag.* Jan. 1913, p. 184.
† Marsden and Taylor, *Proc. Roy. Soc.* A, vol. 88, p. 443 (1913).
‡ Taylor, *loc. cit.*

in S for Xe and Ag and for Kr and Cu compared with their differences in atomic number. These small differences result from the large groups of loosely bound electrons present on Bohr's theory in the noble metals. It would on any theory be directly confirmatory of the theoretical numbers in the groups if the smallness of these differences could be substantiated by experiment. Another feature of interest is the reproduction by the theory of the surprisingly large difference in S between O_2 and Air. This almost amounts to a proof that the extra electrons in O_2 compared to N_2 are added in the most lightly bound orbits, as was assumed in Table I.

It is hardly legitimate to press these correspondences any further, but they may serve to raise interest in a more systematic experimental study of stopping powers.

I take this opportunity of expressing my indebtedness to Sir E. Rutherford for his help and encouragement, without which these notes would not have been written.

Contributions to the Theory of the Motion of α-particles through matter. Part II, Ionizations. By R. H. FOWLER, M.A.

[*Read* 5 February, 1923.]

§ 1. *The ionization produced by the passage of an α-particle through a gas. Introductory.* We have commented on stopping powers in Part I. We now turn to the ionization produced by an α-particle, and here, as Bohr* and J. J. Thomson* have pointed out, the basis of calculation which is inadequate for stopping powers should be adequate to calculate the primary ionization. In the very intense encounters which give rise to primary ionization—that is, the ionization caused directly by the α-particle itself—it is reasonable to suppose that the forces binding the electron to the atom can as a first approximation be neglected during the encounter. As was first pointed out by Bohr, this is only a fraction of the total ionization produced, for these primary encounters give rise to some electrons with sufficient speed to cause secondary ionization. To calculate this Bohr assumes that in a gas with a single ionization potential λ an electron with energy between $s\lambda$ and $(s+1)\lambda$ will form s secondary pairs of ions. This ignores the possibility that a secondary ion may carry off with it some kinetic energy $x\lambda$ say, after absorbing originally $(1 + x)\lambda$. It assumes in fact that the secondary either absorbs the whole of the primary's energy or exactly λ. It is more natural and probably more in accordance with experiments on electron impacts to assume that the secondary can absorb any possible amount of energy according to some definite distribution law, and this introduces a non-negligible correction into the calculated ionization.

Bohr points out that the number of ions produced by these slow electrons is not properly calculable if we neglect the binding forces during an encounter—the condition for this, on Bohr's theory, is that the energy of the electron should be large compared to the ionization potential. This is undeniable. But it is in entire accord with all the very fundamental work on the ionization and transference potentials of gaseous atoms and molecules by slow electron impacts to assume that the electron never loses energy perceptibly in an atomic encounter except by causing ionization or transference. To calculate accurately the average ionization due to a slow electron all that we require to know further is a distribution law for the velocities of the ejected electrons after collision and the relative frequency of the different types of ionization and transference. It seems to me, though Bohr originally discarded such a course, that calculations on the basis of classical dynamics ignoring binding forces can probably be safely used to calculate the required dis-

* *Loc. cit.* in Part I.

tribution law. It is to be remarked that only *relative* frequencies
of the different types of collision are required—absolute frequencies
will be found to be irrelevant. The point at which this method is
most likely to be in error is in the relative frequency which it pre-
dicts for ionization and transference at velocities of the electron for
which either is possible. Its results on this point should be used
with great caution. Though the problem has been formally solved
on these lines for the general case of any number of ionization and
transference potentials, it has only as yet been reduced to exact
numerical results in the simplest case of a single ionization potential
—this case, however, is far the most important.

§ 2. *The average ionization produced by a single slow electron in
a gas with a single ionization potential.* The actual ionization produced
by a single slow electron of given initial energy will vary greatly
from electron to electron. We are only concerned, however, with
the average ionization, for in applications to α-particles we have
always to deal with a large number of primary electrons in any
given energy range.

Applying the classical calculation to the encounter of a primary
electron with another electron initially at rest we have

$$\tan \tfrac{1}{2}\phi = \frac{e^2}{\mu V^2 p}, \quad \left(\frac{1}{\mu} = \frac{1}{m} + \frac{1}{m} = \frac{2}{m}\right),$$
$$= \frac{2e^2}{m V^2 p}.$$

The energy lost by the primary can at once be written down if we
remember that the tracks of the electrons after the encounter must
on the present basis be at right angles. If V' and v' are the respective
velocities after the encounter

$$v' = V \sin \tfrac{1}{2}\phi, \quad V' = V \cos \tfrac{1}{2}\phi \qquad \ldots\ldots(11).$$

If λ is the ionization potential nothing happens unless
$\tfrac{1}{2} m V^2 \sin^2 \tfrac{1}{2}\phi \geqslant e\lambda$, but when this is satisfied ionization occurs and
a new free electron goes off, the square of whose velocity is
$V^2 \sin^2 \tfrac{1}{2}\phi - 2e\lambda/m$.

This quantity v^2 we can express in terms of p; we find

$$v^2 = \frac{4e^4/m^2 V^2}{p^2 + 4e^4/m^2 V^4} - \frac{2e\lambda}{m},$$
$$p^2 = \frac{4e^4}{m^2 V^4}\left[\frac{V^2}{v^2 + 2e\lambda/m} - 1\right] \qquad \ldots\ldots(12).$$

This is the relation which must be satisfied by p when, after the
encounter, the primary electron (V^2) has given rise to two in all
with squared velocities v^2 and $V^2 - v^2 - 2e\lambda/m$.

In a long track of length L each primary would, if not slowed
up, make $2\pi N n L p\, dp$ encounters in which p lies between p and

$p + dp$. Hence the first effective encounter will have a chance proportional to pdp of lying in this region, and therefore *on the average for many primaries* the number of such first encounters will be $kpdp$. To determine k we must integrate over all possible values of p which give an effective encounter; the limits are 0 and the value obtained by putting $v^2 = 0$ in (12). If we write $V^2 = z$, $v^2 = q$, and $2e\lambda/m = \rho$ so that z, q and ρ represent squares of electron velocities and determine k, the chance that the first encounter of a primary with $V^2 = z$ leaves electrons with $z - q - \rho$ and q can be put in the form

$$\frac{m^2 z^2 \rho}{4e^4 (z - \rho)} \, 2pdp \qquad \qquad \text{......(13)},$$

or in terms of q instead of p

$$\frac{\rho z}{z - \rho} \frac{dq}{(q + \rho)^2} \qquad \qquad \text{......(14)}.$$

In (14) units are obviously irrelevant and everything may again be expressed in volts.

Suppose now that $g(z)$ is the average total number of electrons set free by one primary, with $V^2 = z$, the primary itself being counted as one. It is then obvious that

$$g(z) = 1, \quad (0 < z < \rho). \qquad \qquad \text{......(15)}$$

For any greater value of z we can obtain an integral equation for $g(z)$ by the use of (14). For the average total number of electrons set free by the two after the first encounter must be

$$g(q) + g(z - q - \rho)$$

and $g(z)$ itself must be obtained by integrating

$$\frac{\rho z}{z - \rho} \frac{dq}{(q + \rho)^2} [g(q) + g(z - q - \rho)]$$

over all possible values of q, that is, from 0 to $z - \rho$. We find, therefore, that

$$g(z) = \frac{\rho z}{z - \rho} \int_0^{z - \rho} \frac{dq}{(q + \rho)^2} [g(q) + g(z - q - \rho)],$$

$$= \frac{\rho z}{z - \rho} \int_0^{z - \rho} g(\rho) \left[\frac{1}{(q + \rho)^2} + \frac{1}{(z - q)^2} \right] dq, \quad (z > \rho), \quad (16),$$

$$g(z) = 1, \quad (0 < z < \rho) \qquad \qquad \text{......(16)}'.$$

It is important to observe that on any theory whatever $g(z)$ must be given by the equation

$$g(z) = \frac{\int_0^{z - \rho} f(q, \rho) \, dq \, [g(q) + g(z - q - \rho)]}{\int_0^{z - \rho} f(q, \rho) \, dq} \quad :$$

the assumptions we have made simply determine $f(q, \rho)$ and it is hard to suppose that any application of classical arguments will give an $f(q, \rho)$ of essentially different form.

§ 3. *Numerical study of* $g(z)$. It is apparent from equation (16) that when $g(z)$ is known for all values from 0 to $z - \rho$ it can be determined at once by simple integration for all values from $z - \rho$ to z. Since $g(z) = 1$ when $0 < z < \rho$, it follows at once that $g(z)$ can thus be determined uniquely step by step for all values of z. We have at once when $\rho < z \leqslant 2\rho$

$$g(z) = \frac{\rho z}{z - \rho} \int_0^{z-\rho} \left[\frac{1}{(q + \rho)^2} + \frac{1}{(z - q)^2} \right] dq$$
$$= 2, \quad (\rho < z \leqslant 2\rho), \qquad \qquad \dots\dots(17),$$

in agreement with the self-evident fact that in this range of velocity every primary makes just one secondary and no more. Further, when $2\rho \leqslant z \leqslant 3\rho$

$$g(z) = \frac{\rho z}{z - \rho} \left\{ \int_0^\rho + 2 \int_\rho^{z-\rho} \right\} \left[\frac{1}{(q + \rho)^2} + \frac{1}{(z - \rho)^2} \right] dq$$
$$= \frac{7}{2} - \frac{\rho}{2} \frac{z + \rho}{(z - \rho)^2}, \quad (2\rho \leqslant z \leqslant 3\rho), \qquad \dots\dots(18),$$

and so on step by step. In practice it is, however, desirable to have a more powerful method of obtaining $g(z)$ for large z, which can be obtained as follows.

We observe first that $g(z) = A(z + \lambda)$ satisfies (16) for all values of A. This is easily verified. Further, since the factor multiplying $g(q)$ in the integrand is always positive, if

$$g(z) \leqslant A(z + \rho), \quad (0 < z < y - \rho),$$

then by integration of (16)

$$g(z) \leqslant A(z + \rho), \quad (y - \rho < z < y),$$

and therefore, step by step, for all values of z. The same facts hold for the other inequality $g(z) \geqslant A'(z + \rho)$. These facts take us a certain distance, but the inequalities are needlessly strict. It is only necessary to assume:

(i) *that for all values of* $z \geqslant y + \rho$

$$\int_0^y g(q) \left[\frac{1}{(q + \rho)^2} + \frac{1}{(z - q)^2} \right] dq \leqslant \int_0^y A(q + \rho) \left[\frac{1}{(q + \rho)^2} + \frac{1}{(z - q)^2} \right] dq$$
$$\dots\dots(19),$$

and (ii) *that when* $y \leqslant z \leqslant y + \rho$,

$$g(z) \leqslant A(z + \rho)$$

in order to be able to infer that

$$g(z) \leqslant A(z + \rho), \quad (z \geqslant y).$$

A similar result holds for the other inequality $g(z) \geqslant A'(z + \rho)$.

By taking $y = 3\rho$ we can obtain a very satisfactory approximation for the general form of $g(z)$ without undue labour. We have therefore to determine values of A for which the inequalities

$$\int_{0}^{3\rho} g(q) \left[\frac{1}{(q+\rho)^2} + \frac{1}{(z-q)^2} \right] dq \lessgtr A \int_{0}^{3\rho} (q+\rho) \left[\frac{1}{(q-\rho)^2} + \frac{1}{(z-q)^2} \right] dq$$

are satisfied for all values of $z \geqslant 4\rho$. The form of $g(q)$ is given by equations (16)′, (17) and (18). A careful numerical study shows that to satisfy condition (i) the value of A required for an upper limit may be taken to be $0.758/\rho$ and for a lower limit $0.754/\rho$. Similarly when $3\rho \leqslant z \leqslant 4\rho$ $g(z)$ lies between $0.758(z+\rho)/\rho$ and $0.744(z+\rho)/\rho$. It follows, therefore, that for *all* values of z greater than 3ρ

$$0.744(z+\rho)/\rho \leqslant g(z) \leqslant 0.758(z+\rho)/\rho, \quad (z \geqslant 3\rho).$$

As a working approximation it is clear that

$$g(z) = \frac{3}{4} \frac{z+\rho}{\rho}, \quad (z \geqslant 3\rho), \qquad \ldots\ldots(20)$$

will be sufficiently near the truth. Little serious error will result in most calculations if this approximation is used for $z \geqslant 2\rho$, or even for all values of z.

§ 4. *Extensions to include more than one ionization potential and transference potentials.* To extend the foregoing arguments which determine equation (16) satisfied by $g(z)$, to the case in which there are more than one ionization potential and transference potentials is fairly simple, though it is not so easy to discuss the equation numerically, and it is convenient to insert this extension here. If τ is the squared velocity which corresponds to the first transference potential for the electrons of the group whose ionization is at ρ, we have in addition to the condition (12) for ionization with emergent q which is

$$p^2 = \frac{4e^4}{m^2 z^2} \left(\frac{z}{q+\rho} - 1 \right) \qquad \ldots\ldots(12 \ bis),$$

the further condition

$$p^2 = \frac{4e^4}{m^2 z^2} \left(\frac{z}{q} - 1 \right), \quad (\tau \leqslant q < \rho),$$

under which transference occurs. The greatest effective value of p^2 for these electrons is

$$P^2 = \frac{4e^4}{m^2 z^2} \left(\frac{z}{\tau} - 1 \right) \qquad \ldots\ldots(21).$$

If there are then a number of groups of removable electrons to be considered, the total possible area per atom or molecule for a first effective encounter is

$$\pi \left(n_1 P_1^2 + n_2 P_2^2 + \ldots \right)$$

and each element of this is equally likely to contribute the first
effective encounter, if we assume that all the electrons are equally
accessible. So the chance that the first effective encounter shall be
with an electron of the first group with p between p_1 and $p_1 + dp_1$ is

$$\frac{n_1 2p_1 dp_1}{n_1 P_1{}^2 + n_2 P_2{}^2 + \ldots} \qquad \ldots\ldots(22).$$

The fraction (22) gives the chance that the first effective encounter
leaves two electrons with q and $z - q - \rho_1$ and this chance leads to
an average production of $g(q) + g(z - q - \rho_1)$ free electrons. It
also, for the larger values of p_1, gives the chance that the first
effective encounter shall leave only one electron free with $z - q$
$(\tau_1 \leqslant q < \rho_1)$, and this leads to an average production $g(z - q)$ of
free electrons.

Taking into account all groups, expressing everything in terms
of z and q and integrating we find

$$\left[\Sigma_s n_s \frac{z - \tau_s}{\tau_s z} \right] g(z) = \Sigma_s n_s \left[\int_0^{z - \rho_s} g(q) \frac{dq}{(q + \rho_s)^2} \div \int_0^{z - \tau_s} g(q) \frac{dq}{(z - q)^2} \right]$$
$$\ldots\ldots(23).$$

Everything can again be supposed expressed in volts. In (23)
each integral only enters with the corresponding term on the left
when its upper limit is positive, and by definition $g(z) = 1$ when z
is less than τ_1, the least of the transference potentials. We verify
at once that $g(z) = 1$ so long as $z < \rho_1$ the least ionization potential.
The proper numerical discussion of (23) cannot be entered into in
this paper.

§ 5. *Application to α-particles.* We are now in a position to
revise Bohr's calculations of the ionization produced by an α-
particle. On the present theory (which is also Bohr's) the distribu-
tion of velocity among the primary electrons can be obtained at
once from (3) (Part I). The number of encounters in a length of
track dx which result in a total transfer of energy between q_0 and
$q_0 + dq$ is there shown to be

$$\pi N n dx \frac{8e^4}{m V^2} \frac{dq}{q_0{}^2},$$

and therefore the number which give rise to primaries with energies
after escape between q and $q + dq$ is

$$\pi N n dx \frac{8e^4}{m V^2} \frac{dq}{(q + \rho)^2}.$$

Each of these when secondaries are included gives rise to $g(q)$ free

electrons, or pairs of ions, in all. The total amount of primary ionization in a length dx of track is therefore

$$\pi N dx \frac{8e^4}{mV^2} \Sigma_s n_s \int_0^{2mV^2 - \rho s} \frac{dq}{(q + \rho_s)^2}$$

......(24) **Primary Ionization,**

and the total ionization is

$$\pi N dx \frac{8e^4}{mV^2} \Sigma_s n_s \int_0^{2mV^2 - \rho s} \frac{g(q)\, dq}{(q + \rho_s)^2}$$

......(25) **Total Ionization.**

The corresponding loss of energy by the α-particle has already been shown to be

$$\pi N dx \frac{8e^4}{mV^2} \Sigma_s n_s \int_0^{2mV^2 - \rho s} \frac{dq}{(q + \rho_s)}$$

......(26) **Energy loss.**

These integrals can be at once evaluated, including (25) if $g(q)$ is given by (20).

We have determined $g(q)$ in the simplest case of a single ionization potential. A preliminary investigation of (23) in the case of a single ionization together with a transference potential shows that for reasonably small values of $(\rho - \tau)/\rho$ the values of $g(q)$ are not much affected. In the general case one can see that Bohr's suggestion to regard all the secondary ionization as made from electrons of the least ionization potential must be fairly near the truth, especially if the electrons in the highest levels are all grouped together, and a mean taken for their ionization potentials. We can in this way evaluate (25) with sufficient exactness for a first discussion, without an elaborate investigation of (23), which might be needed later.

If we approximate to (25) in this way we put $g(q) = \frac{3}{4}(q + \bar{\rho})/\bar{\rho}$ and obtain after integration

$$\pi N dx \frac{8e^4}{mV^2} \frac{3}{4\bar{\rho}} \Sigma_s n_s \left\{ \log \frac{2mV^2}{\rho_s} + (\bar{\rho} - \rho_s)\left(\frac{1}{\rho_s} - \frac{1}{4mV^2}\right) \right\} \quad (27).$$

The table on p. 538 gives the numerical results of these equations for velocities of the α-particle equivalent to 1000, 800, 600 and 400 volts, at distances from the end of the range in air of about 7·0, 5·0, 3·25 and 1·8 cms., and compares them with the observed values.

In view of the uncertainties in the precise ionization potentials which should be used for H_2 and Air in these calculations the agreement between theory and observation for these gases is satisfactory. [It is well recognised that no theory of this type gives the proper variation with velocity.] There would be distinct disagreement for H_2, for which there is the least uncertainty if the

TABLE III. Showing T the total ionization per cm. track of the
α-particle at normal temperature and pressure both calculated
and observed (after Geiger* and Taylor*), and T/P the calcu-
lated ratio of the total to primary ionization.

Gas	V^2, α-particle volts	1000	800	600	400
H_2	$T \times 10^3$ Calc.	5·28	6·35	8·04	11·13
	Obs.	4·6	4·8	5·6	8·3
	T/P	4·4	4·2	4·0	3·7
He	$T \times 10^3$ Calc.	2·15	2·56	3·22	4·40
	Obs.	4·6	4·8	5·6	8·3
	T/P	3·8	3·6	3·4	3·1
Air	$T \times 10^4$ Calc.	1·6	1·9	2·3	3·0
	Obs.	2·2	2·3	2·7	3·95
	T/P	5·8	5·4	4·9	4·4

factor $\frac{3}{4}$, introduced by the present theory but omitted by Bohr,
is neglected. Further experiments for the heavier monatomic gases
are highly desirable.

A more direct and still more satisfactory verification or refuta-
tion of the present theory and the factor $\frac{3}{4}$ to which it leads would
be obtained from measurements of the total ionization $g(q) - 1$
produced by slow electrons. The results of Kossel† and Glasson‡ to
which Bohr refers, do not seem to provide any such absolute total
values. It is not possible to make comparisons on a basis of ions
per cm. for these electrons, for this introduces the absolute fre-
quency of the various types of encounter in which no confidence
can or need be placed. The theory demands rather more than
double the ionization in H_2 to the ionization in He due to a slow
electron, and Kossel's observations confirm this.

There is a total disagreement between observation and calcula-
tion for α-particles in He, which we discuss in the following section.

§ 9. *The passage of α-particles through* He. Table II (Part I)
shows that the stopping power of He resulting from ionization
(and transference, but the effect of this is slight) is, like the ioniza-
tion, considerably less than one-half of the observed value. Now it
is a well-known fact that the average energy spent per pair of ions
made is about 34 volts in all three gases, H_2, He and Air. The

* Geiger, *Proc. Roy. Soc.* A, vol. 82, p. 486, 1909; Taylor, *loc. cit.* No attempt
has been made to discriminate between the ionization produced at given total range
in the three gases.
* Kossel, *Ams. der Physik*, vol. 37, p. 393 (1912).
† Glasson, *Phil. Mag.* vol. 22, p. 647 (1911).

precise values are* 33 volts for H_2 and Air and 31·5 for He. If we compare, on the basis of the present theory, the ionization made with the energy spent in making it we see at once that it is about $\frac{4}{3}\bar{\rho}$ in every case. More precisely it is for H_2 15·35 volts, for Air 25 volts and for He 33 volts. The values for Air and H_2 come at once to approximately 33 volts when we compare the computed ionization with the total (observed) loss of energy per cm., but the figure for He is then about 73 volts per ion.

These figures show clearly that while a loss of energy other than that involved in ionization (and transference) may (and in fact must) be supposed to occur in Air and H_2, it cannot occur in He provided that the actual total ionization in He really is approximately the same as that in the other gases. This point is of the utmost importance and the experiments of Taylor should certainly be repeated for confirmation. We have calculated the ionization in roughly the most favourable way using classical dynamics on the assumption that the two electrons can be considered independently, and arrived at a result which gives the right expenditure of energy per ion made, but less than half the proper totals of each. The rest of the energy must therefore also be spent on ionization with much the same efficiency.

We can approach this question from another direction, that of Millikan's† experiments on the doubly charged ions left by α-particles in He. Millikan finds that of the positive ions left by an α-particle from Polonium at the end of its track where the ionization is a maximum 1 in 6 is doubly charged. Millikan concludes that the α-particle removes both electrons from the He atom once in every 6 effective hits. This is a beautiful result of the utmost importance for atomic theory, but the true conclusion is, I believe, even more striking, for Millikan has omitted to take into consideration the difference between primary and total ionization. Let a fraction x of primaries result in double ionization. Suppose, as is most probably correct in view of Kossel's results, that no secondaries are double, and further that the velocities of the secondaries from a double are distributed on the average in much the same way as those from a single. Then the total number of pairs of ions made by the α-particle is increased in the ratio $x + 1$ to 1 and the ratio of doubly charged to singly charged positives is $x/(x + 1)\, y$ where y is the ratio of total to primary ionization as calculated above. It is this fraction which Millikan observes so that

$$(x + 1)\, y = 6x,$$

and at this point (200 volts) calculations analogous to Table III show that $y = 2·6$ approximately. Therefore $x = 0·76$, or prac-

* Geiger, *loc. cit.*; Taylor, *loc. cit.*, corrected to $e = 4·774 \times 10^{-10}$.

† Millikan, *Phys. Review*, vol. 18, p. 456 (1921).

tically three-fourths of the primaries must be doubles. In the middle of the polonium α-ray range (400 volts) Millikan gives the fraction as $\frac{1}{12}$, so here $y = 3 \cdot 15$, $x = 0 \cdot 36$, or more than one-third of the primaries are doubles. Perhaps these figures for x should even be increased slightly if there is any selective recombination. Perhaps there is nothing inconsistent with Millikan's results in supposing that *near the maximum ionization practically all the primaries are doubles.* If this is true it is a very important fact, for it shows that the two electrons in He are so closely interconnected that the passage of an α-ray near enough to remove one of them is bound at the same time also to remove the other.

These results have an obvious application to the range and ionization of the α-particle in He, but it is to be emphasized that the application here made is highly speculative. If we tentatively suppose that every encounter of an α-particle with He results in double ionization with the usual distribution of velocity among the secondaries, the ionization produced in He will be just double that given in Table III in good agreement with experiment. Moreover, the expenditure of energy will be slightly more than double for about 80 volts instead of 49·2 must be spent on the double extraction. The extra 30 volts is distributed on the average between about 6·8 ions, so that the expenditure of energy per ion made would now be about 37 volts. The actual rate of loss of energy due to ionization is now increased in the ratio 2 (37/33) to 1 so that the atomic stopping power of Table II becomes (ignoring all transferences) 22·7 instead of 10·5, in close agreement with the experimental value 23·4.

The ionization in He by slow electrons may thus remain at one-half that of H_2, consistently with α-particle phenomena. Nor is it inconsistent with these ideas that the usual type of ionization in H_2 and other gases by an α-particle is single and not double. We may have to suppose that the α-particle, when it removes a K-electron from O_2 or N_2 usually removes both, but this will not result in any appreciable percentage of doubles in the total. The 2-electron K-level is a unique structure and it is not impossible to suppose that it reacts differently to the α-particle to any other electron or group of electrons.

It is again emphasized that this view of the matter is highly speculative, but it may serve to show the fundamental importance of further and more accurate investigations of the ionization in He both in absolute magnitude and relative to H_2 and Air, of the percentage of primary doubles and of the total ionization due to a slow electron. It will then have served its purpose.

Chemical Constants of Diatomic Molecules. By R. R. S. Cox, B.A., Christ's College. (Communicated by Mr R. H. Fowler.)

§ 1. *Introduction.* The statistical theory of the chemical constants of monatomic substances originated by Stern and Tetrode leads to the expression

$$\Gamma = \log \frac{(2\pi m)^{\frac{3}{2}} k^{\frac{5}{2}}}{h^3} \qquad \ldots \ldots (1 \cdot 1)$$

for the chemical constant. This value has been compared with the values calculated from experimental data for a variety of substances, and very good agreement is found: in the monatomic case, the theory is well supported.

In the case of diatomic substances the rotations of the molecule have to be taken into account; but in many instances the rotations reach their equi-partition value at such low temperatures that in any temperature region in which experimental measurements are available, they may be regarded as "fully-excited." The assumption in these instances that the rotations have been fully excited from absolute zero upwards leads to a different value of the chemical constant, the theoretical expression for which has been given by Ehrenfest and Trkal* and Fowler†, namely:

$$\Gamma \,(\text{diatomic}) = \log \frac{(2\pi m)^{\frac{3}{2}} \cdot 8\pi^2 J \cdot k^{\frac{7}{2}}}{\sigma h^5}, \qquad \ldots \ldots (1 \cdot 2)$$

where J is the moment of inertia of the molecule, and σ its "symmetry number," that is, the number of orientations in space in which it is statistically equivalent. Γ (diatomic) as thus defined is the same as the "chemische Konstant im normalen Zustande" of Langen‡, and its constancy depends on the constancy of

$$\log T - \log B\,(T),$$

where $B\,(T)$ is the partition function for the rotations as defined by Darwin and Fowler§.

The present paper is an attempt to calculate the values of Γ (diatomic) for those diatomic gases for which suitable data are available, with the object of testing the theoretical expression (1·2). The substances discussed are the halogens: Iodine, Bromine, and Chlorine; and Nitrogen. It is found that, except in the case of Nitrogen, the values obtained by different methods are discordant

* Ehrenfest and Trkal, *Proc. Amst. Acad.* vol. 23, p. 162.
† Fowler, *Phil. Mag.* 45, p. 32 (1923).
‡ *Zeits. f. Elektrochem.* 25, p. 28 (1919).
§ *Phil. Mag.* 44, p. 450 (1922). See also Fowler, *loc. cit.*

with each other and with the theoretical values, and that the discordance is probably due to the insufficiency of the experimental data at present available. Since the work was commenced some valuable measurements of the vapour pressures of solid Bromine and Chlorine by Henglein, Rosenberg, and Muchlinski * have been published: and Henglein † has also calculated the chemical constants. We used these experimental results, but have introduced some modifications into the calculation which materially alter the values of Γ deduced, for we do not think Henglein's procedure entirely justifiable.

§ 2. *The equations.* The following units will be used throughout the paper: pressures in atmospheres, quantities of heat in calories, and temperature in degrees absolute; atomic and molecular heats are all at constant pressure, and logarithms are taken to the base 10.

The chemical constant can be determined: (a) from the values of the dissociation constant, and (b) from the vapour pressure.

(a) For the dissociation $X_2 \rightleftarrows 2X$:

$$\frac{d \log K_p}{dT} = -\frac{Q}{2 \cdot 303 R T^2} = -\frac{Q}{4 \cdot 571 T^2}, \qquad \ldots\ldots(2 \cdot 1)$$

$$Q = Q_0 + \int_0^T [C(X_2) - 2C(X)] \, dT, \qquad \ldots\ldots(2 \cdot 11)$$

where the C's are molecular and atomic heats. Since we are to assume that the rotations of the molecule are always fully excited, we have:

$$C(X_2) = \tfrac{7}{2} R + C_{\text{vib}},$$

where C_{vib} is an addition due to internal vibrations to be discussed below. Also $C(X) = \tfrac{5}{2}R$ and is constant. Thence:

$$\log K_p = -\frac{Q_0}{4 \cdot 571 T} + 1 \cdot 5 \log T - \frac{1}{4 \cdot 571} \int_0^T \frac{dT}{T^2} \int_0^T C_{\text{vib}} \, dT + 2\Gamma(X) - \Gamma(X_2)\ldots\ldots(2 \cdot 12)$$

(b) For the vapour pressure:

$$\frac{d \log p}{dT} = \frac{\lambda}{4 \cdot 571 T^2}, \qquad\qquad \ldots\ldots(2 \cdot 2)$$

$$\lambda = \lambda_0 + \int_0^T [C_{\text{vapour}} - C_{\text{condensed}}] \, dT, \qquad \ldots\ldots(2 \cdot 21)$$

$$\log p = -\frac{\lambda_0}{4 \cdot 571 T} + 3 \cdot 5 \log T + \frac{1}{4 \cdot 571} \int_0^T \frac{dT}{T^2} \int_0^T [C_{\text{vib}} - C_{\text{condensed}}] \, dT + \Gamma.\ldots(2 \cdot 22)$$

In these equations, K_p and p are known from experiment, $C_{\text{condensed}}$ in the case of a solid is known in a Debye form, while Q_0 and λ_0 have almost inevitably to be regarded as adjustable constants. The remaining quantity, C_{vib}, requires further consideration.

* *Zeits. f. Physik*, 11, I, p. 1 (1922).　　　　† *Ibid.* 12, v, p. 245 (1922).

§ 3. *The molecular heat of the vapour.* If the rotations are considered to be fully excited at all temperatures, then the molecular heat of constant pressure at the absolute zero is $\frac{7}{2}R$. It is known, however, to rise above this value to $\frac{9}{2}R$ at high temperatures, an increase which has either been ignored or accounted for empirically in most previous calculations of the chemical constant. The increase is presumably due to the vibrations of the atoms along the axis of the molecules, the vibrations of the electrons being relatively unexcited at the temperatures considered. We shall assume, as a sufficient approximation, that the vibrations are independent of the rotations, and that the molecules behave as simple Planck oscillators with frequency ν_0, where ν_0 should correspond to the difference between the centres of successive bands of the same family in the band spectrum. The molecular heat due to vibration is then given by a Planck term:

$$C_{\text{vib}} = R \left(\frac{h\nu_0}{kT}\right)^2 \cdot \frac{e^{-h\nu_0/kT}}{(1 - e^{-h\nu_0/kT})^2}.$$

If ν_0 cannot be deduced from the band spectrum, it must be determined by analysis of the curve of specific heats. To form an estimate of its values for the halogens, we take Strecker's[*] values of the molecular heat, and adjust ν_0 to correspond to them.

	T° (abs.)	$C_{\text{vib}}/\frac{1}{2}R$	$h\nu_0/k$
Iodine	570	1·924	400
Bromine	470	1·856	446
Chlorine	450	1·25	1093

These values are necessarily very rough, especially in the case of Iodine and Bromine, because at the given temperatures the vibrations are nearly fully excited, and we are on the "flat" part of the molecular heat curve, where a small error in C_{vib} would make a considerable difference to $h\nu_0/k$. This consideration does not apply to Chlorine, since in this case the temperature 450° corresponds to the steepest part of the curve. A test is possible in the case of Iodine, whose resonance spectrum has been measured by R. W. Wood[†] and calculated by Lenz[‡]. Lenz gives for Iodine $h\nu_0/k = 306$, with which, considering the possibility of error, the value above is

* *Wied. Ann.* 13, p. 41 (1881). The values are mean values over a considerable temperature range; we have taken the temperature at the middle of the range.
† *Researches in Physical Optics*, Part II, Adams Fund Publication, No. 8, Part 2.
‡ *Physikal. Zeits.* 21, p. 691 (1920). Lenz calculates C_{vib} from his value of ν_0 and finds agreement with Strecker's measurement; but it is to be noted that in this part of the curve, almost any value of ν_0 would give a good agreement. The test in the opposite direction, of finding ν_0 from C_{vib}, is, for this very reason, far more sensitive.

in very fair agreement. The succession of values for the three halogens agrees with what might be expected; ν_0 being larger for the lighter, more tightly bound molecules. A similar succession of values for $\beta\nu$ in the Debye expression for the solid has been noted by Henglein*, who discusses also other such relations between the halogens. In the following we shall take $h\nu_0/k$ to be 306, 446, 1093 for Iodine, Bromine, and Chlorine respectively.

§ 4. *Dissociation*. Substituting $C_{\text{vap}} = \frac{7}{2}R + C_{\text{vib}}$ in (2·11) and (2·12), we obtain:

$$Q = Q_0 + \frac{h\nu_0}{k} R \cdot \frac{e^{-h\nu_0/kT}}{1 - e^{-h\nu_0/kT}} - 3T, \qquad \ldots\ldots(4\text{·}1)$$

$$\log K_p = \frac{Q_0}{4\text{·}571 T} + \frac{3}{2}\log T + \log(1 - e^{-h\nu_0/kT}) + 2\Gamma(X) - \Gamma(X_2). \ldots(4\text{·}2)$$

(a) Iodine. Bodenstein† gives experimental values for K_p, and also calculates Q for various temperatures from the approximate equation:

$$Q = \frac{4\text{·}571 (\log K_1 - \log K_2) T_1 T_2}{T_2 - T_1}. \qquad \ldots\ldots(4\text{·}3)$$

Using his values for Q, we have from (4·1):

T	$-Q$	$\dfrac{306R}{e^{306/T} - 1}$	$-Q_0$
1123	35670	1939	34240
1223	37000	2136	35467
1323	37840	2332	36243
1423	36940	2534	35205

The mean value of Q_0 is -35290: substituting this, and the values for K_p in (4·2), we obtain:

T	$\log K_p$	$\log(1 - e^{-306/T})$	$2\Gamma(I) - \Gamma(I_2)$
1073	$-1\text{·}945$	$-\text{·}605$	$1\text{·}308$
1173	$-1\text{·}325$	$-\text{·}639$	$1\text{·}292$
1273	$-0\text{·}782$	$-\text{·}670$	$1\text{·}295$
1373	$-0\text{·}309$	$-\text{·}700$	$1\text{·}306$
1473	$+0\text{·}091$	$-\text{·}727$	$1\text{·}306$

The mean value of $2\Gamma(I) - \Gamma(I_2)$ is 1·302.

(b) Bromine. Bodenstein‡ has also measured the dissociation of Bromine, taking a large number of readings, and constructing

* *Loc. cit.*   † *Zeits. f. Elektrochem.* 16, p. 966 (1910).
‡ *Ibid.* 22, p. 327 (1916).

an empirical formula which fits them very closely. From his formulae:

$$Q = 46160 + 3\cdot5\,T - \cdot001869\,T^2 + 4\cdot319 \times 10^{-7}\,T^3,$$

$$\log K_p = -\frac{10100}{T} + 1\cdot75 \log T - \cdot000409\,T + 4\cdot726 \times 10^{-8}\,T^2 + 0\cdot548,$$

we calculate the following, using $(4\cdot1)$ and $(4\cdot2)$:

T	$-Q$	$\dfrac{446R}{e^{446/T}-1}$	$-Q_0$
1000	48223	1572	46790
1200	48414	1967	46780
1305	48506	2175	46770
1430	48607	2418	46740
1520	48680	2596	46720

Thus $Q_0 = -46760$: and thence

T	$\log K_p$	$\log(1 - e^{-446/T})$	$2\Gamma\,(\mathrm{Br}) - \Gamma\,(\mathrm{Br}_2)$
1200	$-2\cdot902$	$-\cdot508$	$1\cdot512$
1300	$-2\cdot224$	$-\cdot537$	$1\cdot513$
1400	$-1\cdot640$	$-\cdot564$	$1\cdot513$
1500	$-1\cdot134$	$-\cdot589$	$1\cdot512$

We take $2\Gamma\,(\mathrm{Br}) - \Gamma\,(\mathrm{Br}_2) = 1\cdot513$.

(c) Chlorine. Measurements of the dissociation of Chlorine have been made by Trautz and Stäckel*, and Henglein†; the former at approximately atmospheric pressure, the latter at much lower pressures. The use of lower pressures enabled Henglein to measure K_p at considerably lower temperatures: but the two sets of measurements are not in agreement, and it may easily be seen by plotting them roughly on a graph that it is impossible to represent them both by one formula: we take the former values, as giving a more probable result for the chemical constants. Trautz and Stäckel also give several different determinations of Q_0, and conclude that $Q_0 = -71000$ is the most probable value. Using this, and also their measurements of K_p, we have:

T	$\log K_p$	$\log(1 - e^{-1093/T})$	$2\Gamma\,(\mathrm{Cl}) - \Gamma\,(\mathrm{Cl}_2)$
1473	$-3\cdot88$	$-\cdot28$	$2\cdot19$
1513	$-3\cdot38$	$-\cdot29$	$2\cdot41$
1553	$-3\cdot05$	$-\cdot30$	$2\cdot46$

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* *Zeits. f. Anorg. Chem.* **122**, p. 112 (1922).   † *Ibid.* **123**, p. 137 (1922).

Henglein's results, using the same value of Q_0, give

$$2\Gamma \,(\text{Cl}) - \Gamma \,(\text{Cl}_2) = 5 \text{ approximately}$$

which is almost certainly too high. A further investigation of this discordance might be of some interest. We take

$$2\Gamma \,(\text{Cl}) - \Gamma \,(\text{Cl}_2) = 2{\cdot}35.$$

§ 5. *Vapour Pressure.* The vapour pressure equation is:

$$\log p = -\frac{\lambda_0}{4{\cdot}571\,T} + 3{\cdot}5 \log T - \log\left(1 - e^{-h\nu_0/kT}\right)$$
$$-\frac{1}{4{\cdot}571} \int_0^T \frac{dT}{T^2} \int_0^T C_{\text{condensed}}\, dT + \Gamma.$$

(*a*) Iodine. Vapour pressure measurements by Ramsay and Young[†], Baxter, Hickey, and Holmes[‡], and Haber and Kerschbaum[§] are available, and also specific heat measurements of the solid by Günther[¶]. The molecular heat can be fitted to a Debye function with $\beta\nu = 106$, with a correction term for the difference between C_p and C_v:

$$C_{\text{condensed}} = C_v\,(\text{Debye}) + 2\,.\,10^{-4}T^{\frac{3}{2}}.$$

Thence we find $\lambda_0 = 16100$ gives the best constancy for Γ, and calculate the following table:

T	$-\log p$		$-\log(1 - e^{-306/T})$	$\Gamma\,(\text{I}_2)$
224·8	7·12 (H. and K.)		·129	3·50
226·8	6·96	,,	·131	3·53*
240·8	6·13	,,	·143	3·48
244·3	5·94	,,	·146	3·49
252·2	5·48	,,	·153	3·49
273·0	4·40 (B., H. and H.)		·172	3·56
288·0	3·76	,,	·184	3·57
298·0	3·40	,,	·193	3·55
313·0	2·87	,,	·205	3·55
323·0	2·55	,,	·213	3·52
331·1	2·19 (R. and Y.)		·220	3·63*
337·5	2·10	,,	·225	3·52*
339·3	2·08	,,	·226	3·49
348·2	1·82	,,	·233	3·50
359·0	1·55	,,	·241	3·49
364·9	1·41	,,	·246	3·47
375·7	1·18	,,	·254	3·45
386·8	0·94	,,	·262	3·44

[†] *Journ. Chem. Soc.* 49, p. 453 (1886).
[‡] *Journ. Amer. Chem. Soc.* p. 134 (1907).
[§] *Zeits. f. Elektrochem.* 20, p. 296 (1914).
[¶] *Ann. der Physik.* 51, p. 828 (1916).

The three values marked with an asterisk are probably discordant because of experimental error, since on plotting the vapour pressures we find the corresponding points considerably off the smooth curve. The results from Baxter, Hickey, and Holmes' measurements are slightly discordant with the others, as has already been noted by Henglein*; but it should be observed that by the introduction of the vibration term, and a change in λ_0, we have practically removed this discordance. We take $\Gamma\,(I_2) = 3\cdot49$.

(b) Bromine. Taking the results of Henglein, Rosenberg and Muchlinski† for solid Bromine, and Henglein's‡ values of the integral of $C_{\text{condensed}}$, we find the best constancy for Γ given by $\lambda_0 = 10930$, and obtain the table:

T	$\log p$	$-\log(1 - e^{-446/T})$	$\Gamma\,(Br_2)$
177·6	− 5·539	·037	1·82
210·0	− 3·483	·055	1·85
222·8	− 2·860	·063	1·84
227·0	− 2·619	·066	1·87
241·0	− 2·077	·074	1·83

We take the mean value $\Gamma\,(Br_2) = 1\cdot84$.

(c) Chlorine. In this case, where $h\nu_0/k = 1093$, the value of the vibration term at the temperatures considered is negligible, and it will be ignored. λ_0 may be found from the value of λ at the melting point given by Henglein§, namely 6960. Then:

$$6960 = \lambda_0 + \int_0^{170} \left[\tfrac{7}{2}R - C_{\text{condensed}}\right] dT,$$

whence $\lambda_0 = 6970$. Using the same values for the double integral as Henglein, we obtain from his vapour-pressure results:

T	$\log p$	$\Gamma\,(Cl_2)$
120	− 5·677	0·48
140	− 3·881	0·41
160	− 2·506	0·40
170	− 1·932	0·38

The value of λ_0 taken gives about as good constancy for $\Gamma\,(Cl_2)$ as can be obtained, and is not very different from the mean of 5 determinations given by Trautz and Stäckel¶. We take $\Gamma\,(Cl_2) = 0\cdot42$.

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* *Loc. cit.* p. 250.   † *Loc. cit.*   ‡ *Loc. cit.*
§ Henglein, Rosenberg and Muchlinski, *loc. cit.*   ¶ *Loc. cit.*

(d) Nitrogen. The excellent low temperature measurements made at the laboratory at Leiden are available for vapour pressure* and atomic heats†, and there are also atomic heat measurements by Eucken‡. From the latter's results a table of

$$\int_0^T C_p \, dT \quad \text{and} \quad \int_0^T \frac{dT}{T^2} \int_0^T C_p \, dT$$

has been calculated by Frl. Miething§: the Leiden measurements are slightly different from those of Eucken, but the discrepancy is small enough to be ignored for our purpose. From the two readings of the vapour pressure of solid nitrogen:

T	57·89	59·95
p	·03792	·06178

we calculate λ from the approximate equation, analogous to (4·3),

$$\lambda = \frac{4 \cdot 571 \left(\log p_1 - \log p_2 \right) T_1 T_2}{T_1 - T_2}.$$

Thence $\lambda_{58 \cdot 92} = 1633$

$$= \lambda_0 + \int_0^{58 \cdot 92} (6 \cdot 95 - C_{\text{condensed}}) \, dT,$$

neglecting the effect of atomic vibrations, which is certainly inappreciable. The integral of $C_{\text{condensed}}$ is found from Miething's tables to be 431·6. Thus:

$$\lambda_0 = 1655.$$

We can now construct the following table:

T	$\log p$	$\dfrac{1}{4 \cdot 571} \int_0^T \dfrac{dT}{T^2} \int_0^T C_p \, dT$ (Miething)	Γ (N$_2$)
59·95	− 1·2091	1·227	− 0·165 Solid
63·1	− 0·9064	1·313	− 0·155 Liquid
70·0	− 0·4153	1·552	− 0·149 ,,
77·4	0	1·785	− 0·148 ,,

Thence, the mean value of Γ (N$_2$) = − 0·154.

(e) Oxygen. An attempt was made to calculate Γ (O$_2$) from the Leiden measurements of the vapour pressures¶ (of the liquid only in this case), and Miething's tables of the atomic heats which are based on Eucken's measurements. But the value of λ_0 calculated from the empirical equation for $\log p$ by equation (2·21) showed

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* Communications from the Physical Laboratory at Leiden, No. 152.
† *Ibid.* No. 149.   ‡ *Verh. d. D. Physikal. Ges.* 18, p. 4 (1916).
§ *Tabellen zur Berechnung des gesamten und freien Wärme-inhalts fester Körper.*
¶ *Comm. Phys. Lab. Leiden*, No. 152.

such a considerable increase with the temperature taken, that no certain result could be obtained. The discrepancy probably arises in connection with the specific heat of the solid.

§ 6. *Consideration of results.* The values of Γ (diatomic) are collected together in the table; those from the dissociation being obtained by substituting the theoretical value of $\Gamma(X)$ in $2\Gamma(X) - \Gamma(X_2)$.

	Γ (monatomic)	$\Gamma(X_2)$ (Dissociation)	$\Gamma(X_2)$ (Vapour Pressure)	$\Gamma(X_2)$ (Theoretical)
Iodine	1·57	1·84	3·49	2·42
Bromine	1·26	1·01	1·84	[1·8]
Chlorine	0·72	− 0·91	0·42	[0·9]
Nitrogen	—	—	− 0·15	− 0·16

The theoretical value for Iodine is calculated from (1·2), using $J = 2 \times 10^{-38}$, which is given by Lenz* from the resonance spectrum; that for Nitrogen from Heurlinger's band spectrum data, which give $J = 14·2 \times 10^{-40}$. Unfortunately, the band spectra of Bromine and Chlorine have not yet been measured, but we have made a rough estimate of their moments of inertia in comparison with that of Iodine, by assuming the distance between the atoms to be proportional to the molecular diameters as given by Rankine†, or Bragg‡. The resulting values of Γ (which are approximately the same, whether calculated from Rankine's or Bragg's data) are bracketed to indicate that they are only rough estimates.

Thus, except in the case of Nitrogen, in which there is very good agreement, the values of Γ obtained from dissociation, vapour pressure, and theory are in considerable discordance with each other; those from the vapour pressure being systematically much higher than those from the dissociation. In particular, it is impossible to draw any conclusion with regard to one point of interest which it was hoped to test, namely, the validity of the introduction of σ, the symmetry number into the formula for the chemical constant except again in the case of Nitrogen, where the omission of σ would quite destroy the agreement. The possible causes of disagreement are:

(i) Uncertainty as to the values of $h\nu_0/k$. As stated in § 3, the values used are necessarily very rough, and a certain amount of error is probably introduced. It cannot be very great, however, because the value of the vibration term, $\log(1 - e^{-h\nu_0/kT})$, in all the above calculations is never greater than 0·7, and a variation in ν_0 results in a relatively small change in Γ. The possible error from

* *Loc. cit.* † *Phil. Mag.* 29, p. 552 (1915).
‡ *Ibid.* 40, p. 169 (1920).

this source is insufficient to account for the discordance we are considering.

(ii) Uncertainty as to the values of Q_0 and λ_0. This is in reality the main difficulty in the calculation, and the most fruitful source of error. The terms containing Q_0 and λ_0 are almost always the largest terms in the equations (4·1) and (4·2), and at the same time they are the most difficult to estimate accurately. The values of λ_0 for Chlorine, for example, given by Trautz and Stäckel from various vapour pressure measurements vary from 6600 to 7628; and the chemical constant deduced correspondingly varies from -0.17 to 1.44. It is to be noted that the method of obtaining Q_0 and λ_0 by adjusting them so as to give the best constancy for Γ in (2·12) and (2·22) is really equivalent to finding Q or λ from experimental values and equations (2·1) or (2·2), and then using (2·11) or (2·21) to find Q_0 or λ_0. Thus, the accuracy with which Q_0 and λ_0 can be determined depends on the accuracy with which dp/dT and dK_p/dT are known; and it seems, therefore, that more numerous and accurate experimental values of p and K_p are necessary before the chemical constants can be found with any certainty.

This point is illustrated by the discordance between our results and those of Henglein[*]. Henglein does obtain good agreement between the values of Γ deduced from dissociation and vapour pressure, and it is unfortunate that this agreement is not supported by our calculations. In the case of dissociation, the introduction of the vibration term in equation (2·12) tends to decrease Γ directly, and also indirectly by decreasing Q_0 (see (2·11)). Further, Henglein takes the empirical formulae of Bodenstein for K_p, which apply at temperatures above $1000°$ abs., and uses them to find Γ at temperatures below $273°$[†], and also to find Q_0, while we have used them only in the experimental range to which they apply.

In the vapour pressure calculations the vibration term is small, and the difference here arises chiefly from different values of λ_0, which we have taken to give the best constancy for Γ; the value so obtained does not, in the case of Chlorine, for example, differ from other determinations more than these differ among themselves. We may note, however, that, for Iodine, the inclusion of the vibration term allows of a much better agreement between the two sets of measurements above and below $T = 273$ than can be obtained otherwise[‡].

(iii) Uncertainty as to transformation points in the solid at low temperatures. This is a source of error which may have a considerable effect on the results: we proceed to estimate the effect due to an ignored transformation point at a low temperature. Let the

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[*] *Loc. cit.*

[†] Henglein (p. 252) notes an "increase of the chemical constant" with increasing temperature.   [‡] Henglein (p. 250).

heat of transformation be q, and the temperature T_1. We suppose that the Debye function used is fitted to the specific heats at higher temperatures, and assume that it does represent the specific heats sufficiently accurately both above and below the transformation point. This latter assumption is found to be justified when q is not too great*. On calculating λ_0 from (2·21), the omission of the transformation point will evidently give $\lambda_0 - q$ instead of the true value λ_0. The true vapour pressure equation will be:

$$\log p = -\frac{1}{4\cdot571}\left[\frac{\lambda_0}{T} + \int_0^{T_1}\frac{dT}{T^2}\int_0^T CdT + \int_{T_1}^T\frac{dT}{T^2}\left\{\int_0^T CdT + q\right\}\right] + \text{(unaltered terms)} + \Gamma;$$

while the equation actually used is:

$$\log p = -\frac{1}{4\cdot571}\left[\frac{\lambda_0 - q}{T} + \int_0^T\frac{dT}{T^2}\int_0^T CdT\right] + \text{(unaltered terms)} + \Gamma'.$$

$$\text{Thence}\qquad \Gamma - \Gamma' = \frac{1}{4\cdot571}\left[\frac{q}{T} + q\left(\frac{1}{T_1} - \frac{1}{T}\right)\right]$$

$$= \frac{q}{4\cdot571\,T_1}.$$

The value of Γ found will thus be too low by a constant quantity; and the constancy of Γ as determined at higher temperatures will be unaffected by the existence of the unknown transformation point. It is easily seen also that any error in the specific heats makes a constant difference to Γ, provided Γ is calculated at temperatures above that at which the error occurs. The magnitude of the error may be seen in the case of Nitrogen which has a transformation at $T_1 = 35\cdot5$, whose molecular heat is 200. The error due to ignoring this would be:

$$\Gamma - \Gamma' = 1\cdot2.$$

There is very probably a transformation point for Iodine at a low temperature†, which Eucken was unable to measure on account of the slowness of the change. This would indicate that the value $\Gamma(I_2) = 3\cdot49$ obtained above from the vapour pressure is too low.

§ 7. *Conclusion.* It appears finally, therefore, that no certain conclusion as to the chemical constants of diatomic molecules can at present be drawn. It seems quite possible that the discordance we have found may be a genuine one having a theoretical basis; but this cannot be definitely asserted until further experimental data are available. Accurate and numerous measurements of p and K_p; further information as to the specific heats of the vapour and of the solid at low temperatures, and measurements of the band spectra are necessary before much further progress can be made.

I would like to take this opportunity of thanking Mr R. H. Fowler for his suggestion and advice throughout the work.

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* See *Comm. Phys. Lab. Leiden*, No. 152, in the case of Nitrogen: also Eucken, *loc. cit.* † Eucken, *loc. cit.*

A Note on the Electromagnetic Mass of the Electron. By E. C. STONER, B.A., Emmanuel College.

The object of this note is to direct attention to a curious apparent discrepancy existing between the value obtained for the mass of the electron by the ordinary methods, and that obtained by making use of the relativity relation between energy and mass. A tentative resolution of the difficulty is given.

Consider the Lorentz electron with a surface charge of uniform density. Let e_0 be the total charge, a the radius. Then the electrostatic energy W_E in the field of such an electron is given by

$$W_E = \int_0^\pi \int_a^\infty \left(\frac{e_0{}^2}{8\pi r^4}\right) 2\pi r \sin\theta r d\theta dr$$
$$= \frac{1}{2}\frac{e_0{}^2}{a} \qquad\qquad \ldots\ldots(1).$$

In the well-known method for finding the electromagnetic mass, the total magnetic energy in the medium due to the electron moving with a velocity v is equated to $\frac{1}{2}m_0 v^2$ with the result $\left(\text{for } \dfrac{v}{c} \text{ small}\right)$

$$m_0 = \frac{2}{3}\frac{e_0{}^2}{ac^2} \qquad\qquad \ldots\ldots(2),$$

or

$$m_0 = \frac{4}{3}\frac{W_E}{c^2} \qquad\qquad \ldots\ldots(2a).$$

A similar result is obtained from considerations based on the conception of electromagnetic momentum in the medium. If m_l is the longitudinal, m_t the transverse mass, the result may be put in the form

$$m_l = \frac{4}{3}\frac{W_E}{c^2}\kappa^3,$$
$$m_t = \frac{4}{3}\frac{W_E}{c^2}\kappa,$$

where

$$\kappa = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

For $\dfrac{v}{c}$ small both of these expressions reduce to

$$m_0 = \frac{4}{3}\frac{W_0}{c^2} \qquad\qquad \ldots\ldots(3),$$

as in (2).

Now, on the basis of the relativity theory, the kinetic energy of a material point, moving with a velocity v, is given by

$$\frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + C = m\kappa c^2 + C.$$

C is an arbitrary constant which it is convenient to take as zero. Then, expanding, for $\frac{v}{c}$ small, this is equal to

$$mc^2 + \tfrac{1}{2}mv^2.$$

If energy E is absorbed, from radiation or otherwise, the velocity remaining constant, the energy becomes

$$\left(m + \frac{E}{c^2}\right)c^2 + \tfrac{1}{2}mv^2.$$

This illustrates the generalisation, that the laws of conservation of mass and of energy become one. To secure complete conservation we must suppose that all forms of energy possess mass $\frac{E}{c^2}$; or conversely that a mass m possesses intrinsic energy of amount mc^2.

On this basis, for the electron, if W is the energy of the electron at rest

$$m_0 = \frac{W}{c^2} \qquad\qquad \text{......(4)}.$$

If $W = W_E$, this is in conflict with the values given by (2) and (3).

This would suggest that the electron has energy other than the electrostatic energy of the external field.

Consider the work done in the fictitious process of building up an electron by the gradual accretion of charge. If the radius remains constant, the work done during the accretion is

$$\int_0^{e_0} \frac{e}{a}\, de = \frac{1}{2}\frac{e_0^2}{a} \qquad\qquad \text{......(5)},$$

in agreement with (1). If, however, we suppose the surface density of charge to remain constant, the radius depending on the charge so that

$$\frac{e}{r^2} = \frac{e_0}{a^2}.$$

the work done is given by

$$\int_0^{e_0} \frac{e}{r}\, de = \int_0^{a} \frac{r^2 e_0}{a^2 r} 2r \frac{e_0}{a^2}\, dr$$
$$= \frac{2}{3}\frac{e_0^2}{a} \qquad\qquad \text{......(6)}.$$

If this expression could be used in the relativity formula for W, the discrepancy between (2) and (4) would be removed. The electrostatic energy of the field, though, is still $\dfrac{1}{2}\dfrac{e_0^2}{a}$ and the reason for the larger value obtained in (6) is that, although the electrostatic forces tend to expand the electron, we have supposed that no work is gained by mere expansion. This will only be true if the electrostatic forces are balanced by a system of tensions.

The force due to the tensions will then have to be equal to $2\pi\sigma^2$ per unit area, σ being the surface density of charge. The total work done in building up the electron will still be $\dfrac{2}{3}\dfrac{e_0^2}{a}$; part of this — $\dfrac{1}{2}\dfrac{e_0^2}{a}$ — appears as external electrostatic energy; while the internal potential energy of the electron, due to expansion against the tensions, will be

$$W_I = \int_0^a 4\pi r^2 \times 2\pi\sigma^2 \, dr$$

$$= \int_0^a 4\pi r^2 \times 2\pi \left(\frac{e_0}{4\pi a^2}\right)^2 dr$$

$$= \frac{1}{6}\frac{e^2}{a} = \frac{1}{3}\,W_E \qquad\qquad \dots\dots(7).$$

Thus $\qquad\qquad W = W_E + W_I \qquad\qquad \dots\dots(8),$

and using the values of (1) and (7), and substituting in (4), (2) and (4) are reconciled.

The tensions supposed, of course, are precisely those which are required to maintain the electron in equilibrium. The fact that consistent relations are obtained by assuming these tensions is very striking, and seems to provide some evidence for their existence.

The question has here been considered from a very simple standpoint. I have since noticed that Jeans* considers what is essentially the same problem, though rather differently and in a more general manner. The argument adopted here, however, is so obvious and direct, that it seems worth while to give it, if only to serve as an introduction to more general treatment.

To summarize. — If W_E is the electrostatic energy of the electron, the ordinary methods give for the mass

$$m_0 = \frac{4}{3}\frac{W_E}{c^2}.$$

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* Jeans, *Electricity and Magnetism*, p. 590 (1920).

The relativity theory, on the other hand, gives

$$m_0 = \frac{W}{c^2},$$

where W is the total energy of the electron. This suggests that the electron possesses energy other than the external electrostatic energy of amount $\frac{1}{3} W_E$. It is shown that if the electron is held in equilibrium by a system of tensions producing a force $2\pi\sigma^2$ per unit area, it would possess internal potential energy of precisely this amount.

Infra-red spectra: (1) *infra-red emission spectra of various substances, and* (2) *infra-red absorption spectra of benzene and some of its compounds.* By J. E. PURVIS, M.A.

[*Received* 12 February 1923.]

Coblentz in his treatise, *Investigations of infra-red spectra, Part I. Infra-red absorption spectra,* published by the Carnegie Institution of Washington in 1905, describes a series of investigations on the infra-red absorption of a number of liquid organic substances. It is well known that the absorption bands in the ultra-violet regions of many organic liquids, or solutions of these liquids, break up into a series of narrow well defined lines when they are in the vaporous condition. For example, benzene, chlorobenzene and bromobenzene as liquids or in solution show a number of wide diffuse bands, whereas, as vapours, each of these bands breaks up into a series of narrow fine lines*. On the other hand iodobenzene shows no similar selective absorption either as a liquid, or in solution, or as a vapour†.

The absorption spectra of the vapours of these substances in parts of the infra-red regions have been investigated by the author. The absorption spectra of liquid iodobenzene has also been studied. This substance was not investigated by Coblentz.

APPARATUS.

The spectrometer, supplied by Messrs Bellingham and Stanley, London, was arranged so that all the optical parts were entirely enclosed in an air-tight metal case, Fig. 1. The optical train consists of a concave mirror, a plane mirror, and a 30° prism of clear rock salt. The rays from the entrance slit, Fig. 2, are rendered parallel by the concave mirror and reflected back over their original path by the "silvered" back surface of the prism. This gives a dispersion equal to that of a 60° prism while maintaining the conditions for minimum deviation. The damp atmosphere of the laboratory was a serious drawback to the use of a rock salt prism. After two or three months' work the surfaces had to be repolished. The table on which the prism rests can be rotated by a micrometer screw on which is mounted the wave-length indicator. The latter is a flat disc on which is cut the spiral slot guiding the index arm; and the spiral slot is divided and figured to indicate wave-lengths directly. The entrance and exit slits have jaws of platinoid with divisions for setting. The focus of the reflecting mirror is 598·5 mm.,

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* Pauer, *Wied. Ann.* 1897, 61. 363; Hartley, *Phil. Trans.* A, 1907, 208. 475; Purvis, *Trans. Chem. Soc.* 1911, 99. 811.

† Purvis, *Trans. Chem. Soc.* 1911, 99. 2318.
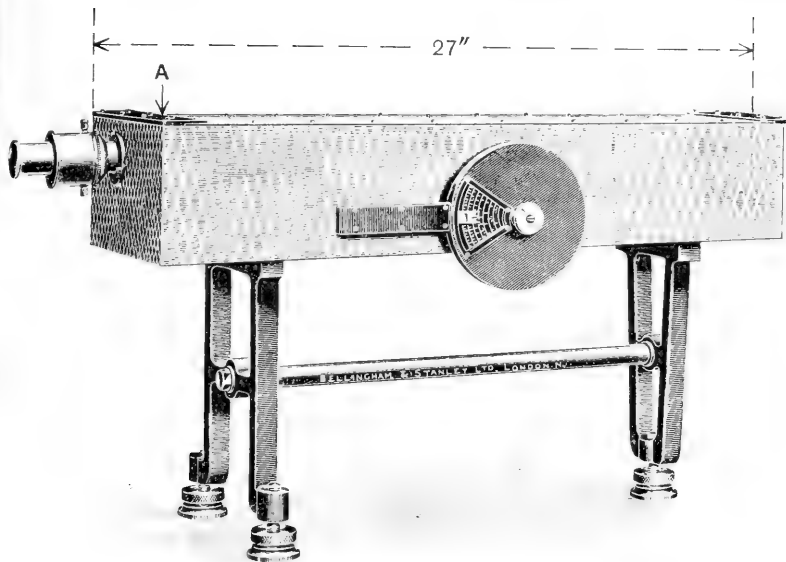
Fig. 1
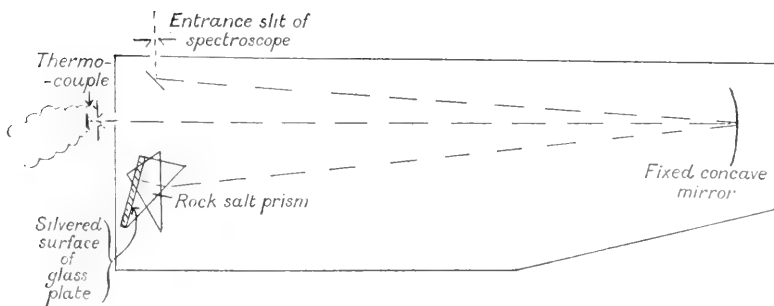
Fig. 2
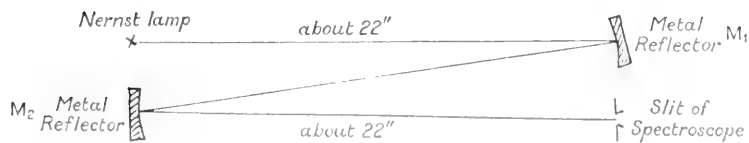
Fig. 3

Figs. 1–3. Infra-red Spectrometer

and the reflecting surface of the prism is $2\frac{3}{4}'' \times 2\frac{1}{2}''$. The latter is made of a glass plate silvered and polished on the front surface.

The thermopile has a receiving surface of ten separate rectangles of thin silver foil, to which bismuth and silver wires are soldered. The cover of the receiving surface is 20 mm. $\times$ 2 mm. The radiant energy was focussed on the slit of the spectrometer by the arrangement shown in Fig. 3. The concave metallic disc M_1 was adjusted so that it was exactly in line with the slit tube of the spectrometer. The mirror M_1 was in the same plane as the slit and adjusted so that M_2 was evenly illuminated.

A "Broca" galvanometer, supplied by the Cambridge Scientific Instrument Co., Ltd., was levelled on a brick column with a slate top. The foundations of this column were laid 5 feet deep in the earth, and were $4\frac{1}{2}$ feet above the ground, and $1\frac{1}{2}$ feet square. The galvanometer was covered, and surrounded, by five soft iron cylinders with covers of similar metal. The working distance of the mirror was one metre from the scale.

Results.

Various sources of radiation were investigated, and the Figs. 4 to 8 give their emission curves. The ordinates represent the deflection of the galvanometer in mms. and the abscissae the wavelengths. Fig. 4 shows the curves for a Nernst "heater" and a Nernst "filament" requiring 108 volts. These two sources of radiation were used in the earlier experiments. Fig. 5 shows the curve of a Nernst "filament" (also requiring 108 volts) used in later experiments. Infra-red emission spectra of a Nernst "glower" have been studied by Coblentz*. He shows that the energy curves undergo
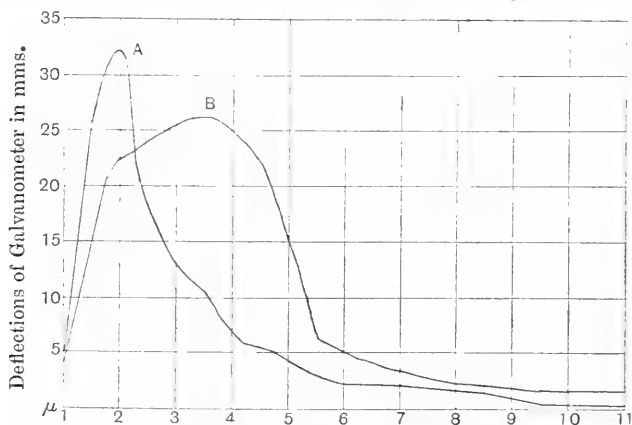
Fig. 4. Emission spectra: A. Nernst filament; B. Nernst heater.
Width of slit of spectroscope = 0·5 mm.

* *Supplementary Investigations of Infra-red spectra*, Pt VII, 1908.
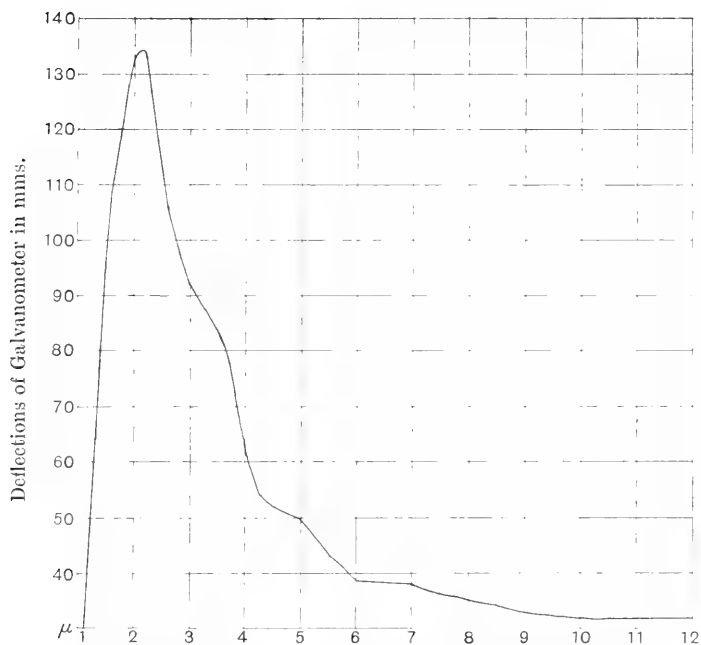
Fig. 5. Emission spectrum: Nernst filament. Width of slit of
spectroscope = 0·5 mm.
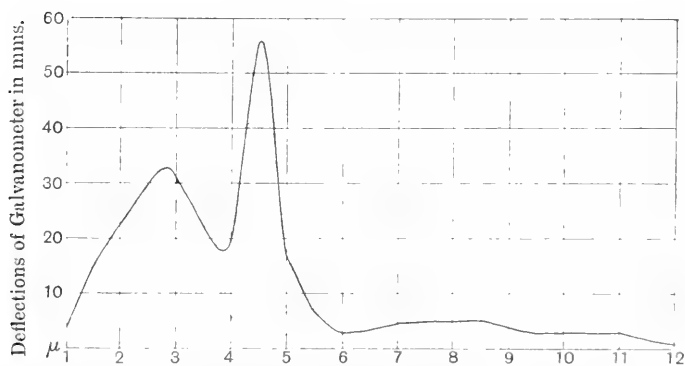
Fig. 6. Emission spectrum of Welsbach light. 12 ins. from slit of
spectroscope: slit 1 mm. wide
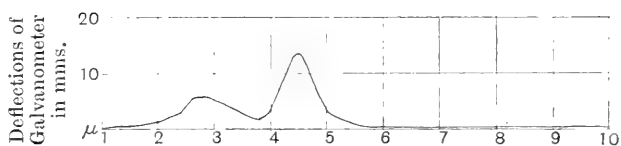
Emission spectrum: Fish tail burner—broad side on—middle of jet. 10 ins. from slit of spectroscope. Slit 1 mm. wide
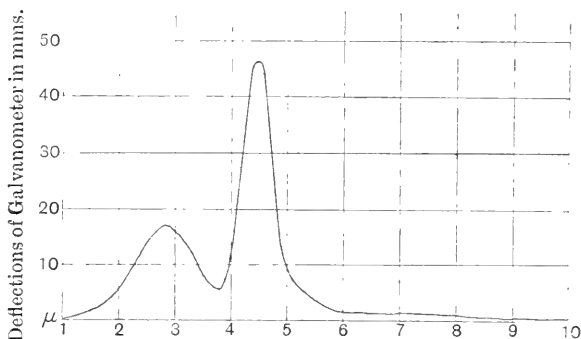
Emission spectrum: Large Bunsen burner. 12 ins. from slit. Slit 1 mm. wide

Fig. 7
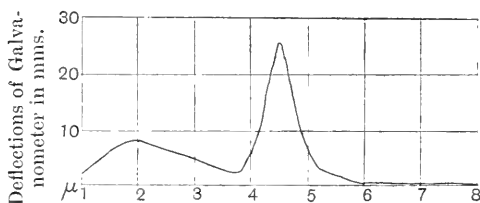
Emission spectrum: Acetylene jet. 10 ins. from slit. Slit 1 mm. wide
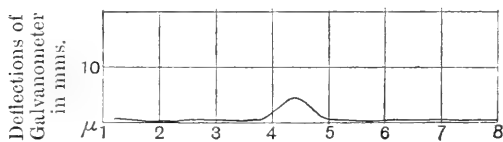
Emission spectrum: Carbon monoxide jet. 4 ins. from slit. Slit 1 mm. wide

Fig. 8

great variations in appearance with rise in temperature. The emission
curves of various other sources have been drawn (Figs. 6 to 8). Those
appertaining to a Welsbach light (Fig. 6), a "fish-tail" burner, and
a large Bunsen burner (Fig. 7) and an acetylene jet (Fig. 8, upper
curve) are very similar. Fig. 8 (lower curve) is the emission curve
of a small jet of burning carbon monoxide. The radiation of a
Welsbach light was used from time to time in the earlier work on
these absorption spectra.

Iodobenzene (liquid). The absorption cell was made of two
polished rock salt plates 0·165 mm. apart. This was fixed on a
stand which could be worked by a lever upwards and downwards
in front of the slit of the spectrometer. A single rock salt plate of
the same thickness was fixed on the stand just below the cell.
Two readings of the galvanometer movements were taken—one

C_6H_5I—continuous line. C_6H_5Br—dot and dash line. C_6H_6—dotted line.
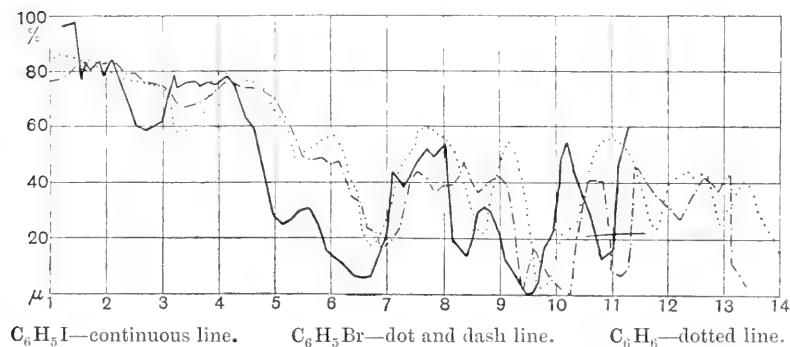
Fig. 9. Transmission through liquid C_6H_5I compared with the transmission
through C_6H_6 and C_6H_5Br (Coblentz); cells 0·165 mm. thick for C_6H_5I; cells
0·160 mm. thick for C_6H_6 and C_6H_5Br.

when the cell was in front of the slit and the other when the plate
took its place. The percentage amount of the transmission was
calculated from these readings. The slit of the spectrometer was
1 mm. wide for the readings between $\mu1$–$\mu9\cdot2$; 1·5 mm. wide for
the observation between $\mu9\cdot2$–$\mu9\cdot6$, and 2 mm. wide between
$\mu9\cdot6$–$\mu11\cdot3$. Fig. 9 shows the infra-red absorption of this iodo-
benzene to just beyond $\mu11$. Coblentz curves for benzene, and
bromobenzene are also reproduced for comparison. The general
forms of the three curves are fairly comparable, but there are
some differences. The bands of iodobenzene show some signs of
a rhythm, *e.g.* the two bands between $\mu5$–$\mu7$ may be compared
with those between $\mu8$–$\mu10$. These may be compared with the
two bands of benzene and bromobenzene—between $\mu5$–$\mu7$ and
$\mu8$–$\mu10$ respectively. In iodobenzene the "shift" of the bands is

distinctly in the direction of the visible regions of the spectrum. The bands of iodobenzene between $\mu 2–\mu 3$, $\mu 5–\mu 6$, $\mu 6–\mu 7$, $\mu 8–\mu 9$, $\mu 9–\mu 10$ are fairly comparable with those of benzene and bromo-benzene between $\mu 3–\mu 4$, $\mu 5–\mu 6$, $\mu 6–\mu 7$, $\mu 8–\mu 9$, and benzene $\mu 9–\mu 10$, but bromobenzene gives two bands between $\mu 9 \cdot 5–\mu 10 \cdot 3$. Again the position of the bands do not show that those of the heavier molecule are always shifted towards the region of greater wave-lengths. This is noticeable on examining the curves of benzene and bromobenzene between $\mu 3–\mu 4$, $\mu 8–\mu 9$, and $\mu 9–\mu 10$, and it is better marked in the curve of iodobenzene. These differences may be explained by considering the absence of absorption bands in the ultra-violet region of iodobenzene (*loc. cit.*). Some portion of the radiant energy would be more available and this would tend to speed up the vibrations and the bands would be in the direction of the regions of the shorter wave length. Similarly, there are fewer solution and vapour bands in bromobenzene than in benzene, and a similar explanation may be suggested.

Vapours of benzene and some of its compounds. The tube, with entrance and exit side-tubes, containing the vapour was 120 mm. long, and the ends were covered with polished rock-salt plates. This tube was used for exploring the vapours beyond $\mu 2 \cdot 8$. The ends of a similar tube were covered with plates of polished quartz. This tube was used for exploring the regions between $\mu 1–\mu 2 \cdot 8$. Beyond this, quartz stops all radiations in the infra-red regions. The tubes were placed in a copper vessel with open ends and embedded in asbestos wool to keep the temperature constant. The tube was filled with the vapour of each substance, and attached to a mercury pump by one of the side tubes, so that varying volumes of vapour could be examined at varying pressures.

In order to compare the vapour curves with the liquid curve of benzene, Fig. 10 gives the absorption between $\mu 1–\mu 2 \cdot 8$ described by Coblentz (*loc. cit.*) through a quartz cell 0·2 mm. thick. The liquid benzene curve shows three well-marked bands at $\mu 1 \cdot 7$, $\mu 2 \cdot 08$, and $\mu 2 \cdot 49$, and these bands show signs of division when the radiations pass through the vapour (Fig. 11, left-hand curve). The curves of chlorobenzene (Fig. 11, right-hand curve), bromobenzene and iodoben-zene (Fig. 12) also show signs of division. The liquid band at $\mu 1 \cdot 7$ was investigated in greater detail. A considerable number

Fig. 10. Transmission through liquid C_6H_6. Cell 0·2 mm. thick (Coblentz).

of readings between $\mu 1 \cdot 5–\mu 1 \cdot 9$ were taken and the curve plotted (Fig. 13). It will be seen that this liquid band shows signs of being divided when the substance is a vapour.
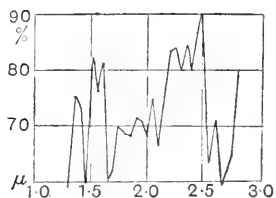
Transmission through vapour of C_6H_6.
$t = 21°$ C. Cell 120 mm. long
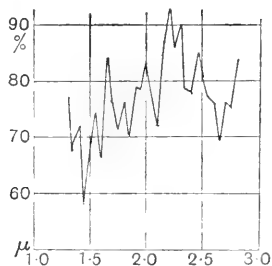
Transmission through vapour
of C_6H_5Cl. $t = 21°$ C. Cell
120 mm. long

Fig. 11
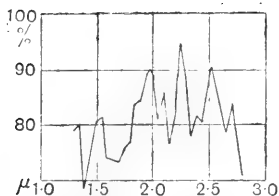
Transmission through vapour
of C_6H_5Br. $t = 21°$ C. Cell
120 mm. long
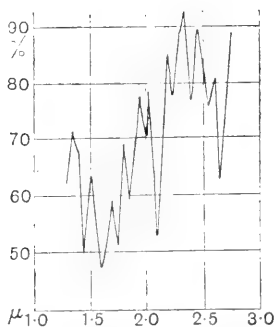
Transmission through vapour
of C_6H_5I. $t = 21°$ C. Cell
120 mm. long

Fig. 12
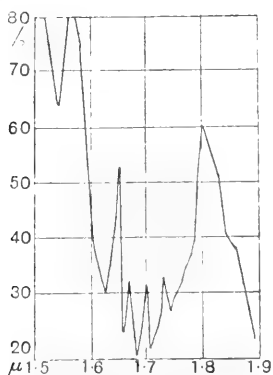
Fig. 13. Transmission through
C_6H_6 vapour. $t = 21°$ C. Cell
120 mm. long

Fig. 14 is a comparison of a liquid band of benzene between $\mu 3-\mu 4$ (Coblentz, *loc. cit.*) and the absorption of a small quantity of the vapour of the same substance in the same region. The division of the liquid bands into a series of separate vapour bands is suggested by comparing the curves. In the liquid condition the bands are crushed together to form one large band. In the vaporous condition there is more freedom and a widening out so that separate bands are fairly well indicated.

It is obvious, however, that a strict comparison of these phenomena with those in the ultra-violet regions is not possible under the conditions of the experiments. In the first place, a photographic plate is more sensitive, and more exact in its records, than a galvanometer. A greater dispersion would also be required to separate bands in the infra-red regions. The separation of a liquid band extending over $\mu 3-\mu 4$, for example, would require a much greater
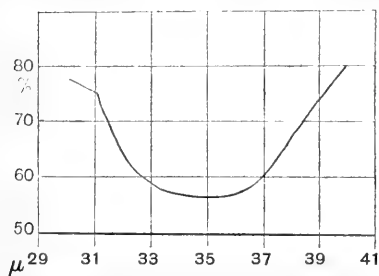
Transmission through liquid C_6H_6. Cell 0·16 mm. thick, between $3\mu-4\mu$ (Coblentz)
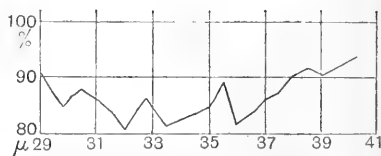
Transmission through C_6H_6 vapour. Tube 120 mm. long. $t = 19\cdot6°$ C.

Fig. 14

dispersion than that produced by a single prism. In the ultra-violet regions there are 84 vapour bands of benzene extending over a region between $\mu 0\cdot2277-\mu 0\cdot2745$[*].

It has been suggested that the atomic vibrations are responsible for the absorption bands in the less refrangible regions, and that the bands in the ultra-violet regions owe their origin to electronic oscillations. It has been proved (*loc. cit.*) that iodobenzene, as a liquid or a gas, or in solution, has no bands in the ultra-violet regions. Again, aniline vapour has a considerable number of bands in the ultra-violet regions, whereas monomethylaniline has none[†]. It might have been expected that iodobenzene would show similar absorption bands to those of bromobenzene; and it is not clear why the introduction of a CH_3-group into aniline should eliminate all the ultra-violet vapour bands of the latter. The discovery of

[*] Hartley, *loc. cit.*
[†] Purvis, *Trans. Chem. Soc.* 1910, 97. 1546.

isotopes brings in another factor: but the slow movements of these comparatively large masses would produce wide and diffuse bands. Their vibrations would not be comparable with the rapid oscillations which produce the narrow bands in the ultra-violet regions.

The absorption of the ultra-violet rays by phosphorus and some of its compounds. By J. E. PURVIS, M.A.

The absorption spectra of phosphorus and of the hydride of phosphorus (PH_3) in the ultra-violet regions do not appear to have been investigated. The absorption of the chloride of phosphorus (PCl_3) has been studied by Martens[*]. He found no bands, but the rays were transmitted to about λ 2630.

The absorption of triphenyl phosphine [$P(C_6H_5)_3$] has been investigated by the author[†]. This substance showed a weak wide band in very dilute solutions, and also great general absorption. The centre of the band was about λ 2630. The vapour at 140° C. and 740 mm. showed a very weak band between about λ 2500 and λ 2300.

The author has studied the absorption capacity of the vapours of phosphorus, the hydride of phosphorus (PH_3), and the trichloride of phosphorus. To hold the vapours two brass tubes of 100 mm. length and 200 mm. length respectively were employed. The internal diameter of each tube was 20 mm. Both tubes could be unscrewed for the introduction of solids, and each had an inlet tube and an exit tube for gases. Polished quartz plates were fitted to the ends. For temperatures above 100° C. the tubes were completely buried in sand, except for a few mm. at the ends. The ends were well within the walls of the sand holder and the temperature was not much lower than that of the body of the tube. Thermometers were placed in contact with the tubes, so that the temperature just outside could be easily read. The source of radiation was a condensed cadmium spark, and exposures of three and five minutes were given. The following notes describe the phenomena on the photographic plates:

Phosphorus. The M.P. of this substance is 44·5° C., and the B.P. 290° C. A small portion of phosphorus was introduced in the 100 mm. tube, the latter was filled with nitrogen, and the entrance and exit tubes closed. The tube was placed in the sand bed and heated. The phosphorus melted when the thermometers registered 95°. There were no absorption bands, and the following numbers give the position of general absorption at various temperatures:

150° C. general absorption began at about λ 2500
160°   ,,   ,,   ,,   2580

$170°$ C. general absorption began at about λ 2690

$180°$	,,	,,	,,	2750
$190°$	,,	,,	,,	2820
$200°$	,,	,,	,,	2850
$210°$	,,	,,	,,	2900
$220°$	,,	,,	,,	2960

In another series of experiments the phosphorus was introduced in the 100 mm. tube and heated in an atmosphere of nitrogen. There were no absorption bands. The rays were transmitted at $120°$ to λ 2100, at $160°$ to λ 2500, and at $200°$ to about λ 2740.

In a third series of experiments the 200 mm. tube was immersed and heated in a water bath. The phosphorus melted at $50°$. At $70°$ the rays were transmitted to about λ 2130; at $100°$ to about λ 2420, but the strong Cd lines 2321 and 2313 were just visible.

Phosphorus hydride (PH_3). The gas was made from sodium hydroxide and phosphonium iodide. The absorbing tube was exhausted to 30 mm. pressure and then filled with the gas. In the 100 mm. tube at temperatures ranging between $30°$ and $100°$ the rays were transmitted to about λ 2230. In the 200 mm. tube, between $30°$ and $100°$, the rays were transmitted to about λ 2240. No absorption bands appeared.

Phosphorus trichloride (PCl_3). In the 200 mm. tube, and at temperatures ranging between $16°$ and $100°$, no bands were seen; at $16°$ the rays were transmitted to λ 2480, and at $100°$ to λ 2590. This confirms Martens' observation (*loc. cit.*). In the 100 mm. tube at $30°$ the rays were transmitted to λ 2130 and at $100°$ to λ 2230.

The observations prove, therefore, that neither phosphorus nor any of these compounds of phosphorus, exhibit any absorption bands in the ultra-violet regions. The oscillations or vibrations of electrons are supposed to be the origin of specific absorption in these regions; but, in the molecule of phosphorus, the four atoms and their electrons have not sufficient freedom to produce any specific effect. Nor does the introduction of hydrogen or chlorine relax this rigidity. In triphenyl phosphine, the seven solution bands and the large number of vapour bands of benzene, are fused into one weak and wide band. In this substance the atomic energy of phosphorus appears to be partly spent on the benzene residues producing considerable changes in, but not completely destroying, the vibrations of the constituent atoms of benzene.
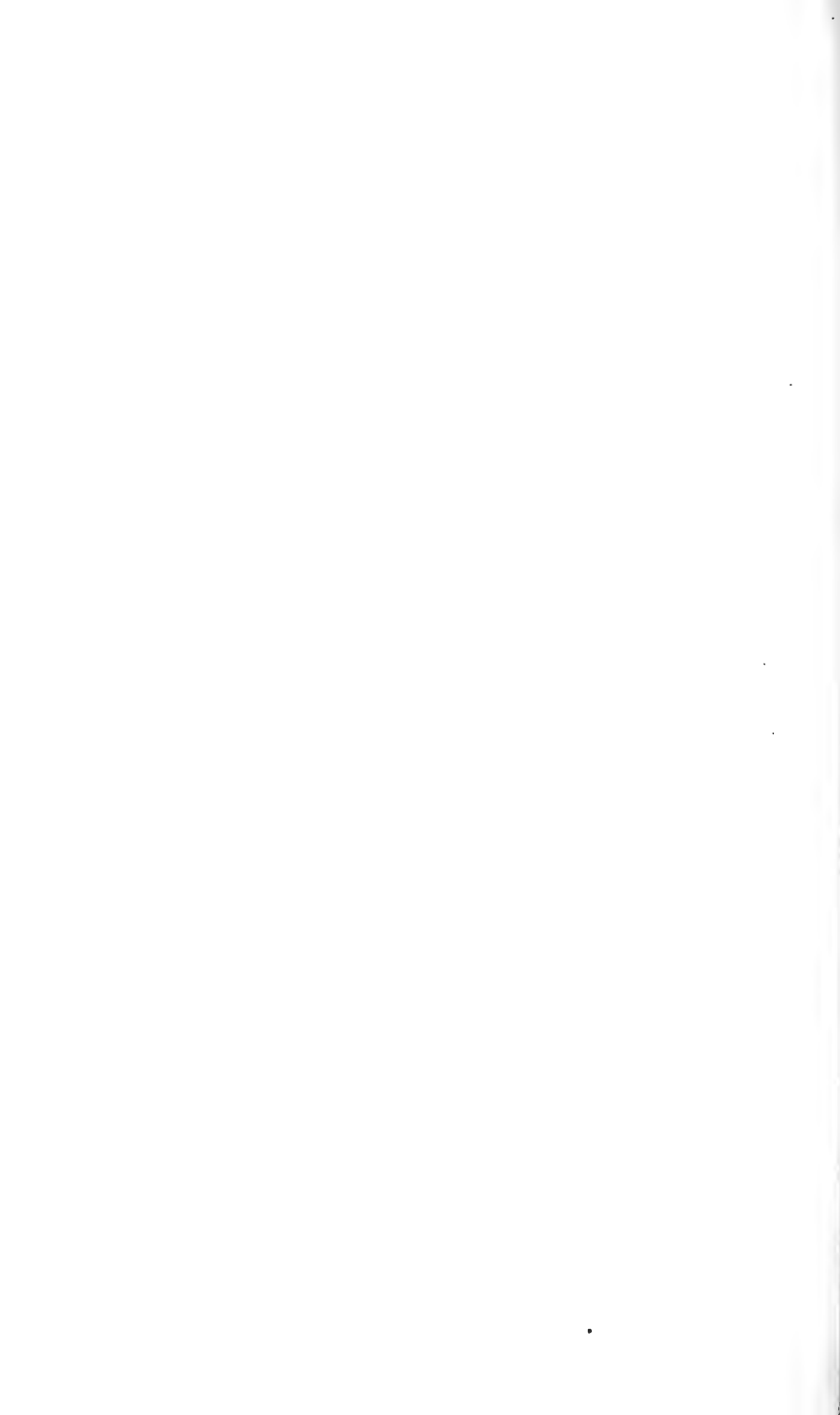
PROCEEDINGS

The Recuperation of Energy in the Universe. By Dr G. D.
LIVEING.

[*Read* 7 May 1923.]

The dissipation of energy in the universe has long been recog-
nised: an attempt to explain how the reverse process, namely, its
recuperation, comes about, is developed in this essay.

The key to the theory herein advanced lies in the difference
between the two recognised types of energy, namely (1) the trans-
lational, and (2) the vibrational. The former is well illustrated in
a bullet discharged from a rifle. It carries its energy with it from
the rifle to the object struck, or, if you please, its energy carries it:
at all events, they are translated together, including the energy
of the spin in which every particle of the bullet traverses a different
path. Besides these movements there is that which is called its
heat-energy, and, on the kinetic theory of gases (liquids and solids),
belongs to every material object. On this theory every material
substance is made up of a multitude of molecules which are
separate from each other in the mass, and have some freedom of
movement, and, by reason of their heat-energy, are in continual
motion among themselves. They repeatedly collide and, being
perfectly elastic, forthwith rebound. Their energies are redistributed
between them during their collision, but no energy is lost. The
average kinetic energy of all its molecules constitutes the measure
of the temperature of the mass. The continual movement of the
molecules brings about an equalisation of temperature throughout
the mass; and if that be a solid in air of a different temperature,
the molecules of the one will collide with those of the other and
their temperatures be equalised. The energies are still translational.
There is, however, another way in which a material object may lose
its heat-energy; and that leads to the second form of energy.

Light is the typical case of energy of this form, which is often
called radiant energy: a name which will include light of a kind to
which our eyes are not sensitive.

If you take an ordinary small electric lamp with a wire of some
very refractory metal in a glass globe from which all gas has been
removed as completely as possible, and pass an electric current
through it, gradually increasing the strength of the current, the
wire becomes hot, soon red-hot, then yellow and then white-hot.
The energy of the electric current at first takes the form of the heat-

agitation of the molecules of the wire, then of light successively red, yellow, and white, as the temperature rises. To bring out the difference between these different energies, we must consider the, now generally accepted, undulatory theory of light. That theory presupposed the existence of a special substance named the ether; this substance completely fills the universe, and has no crack or void place in it, and does not consist of separate, or separable, parts, but is uniform throughout: identical at all points, everywhere, and in every direction. Having no parts it is incapable of heat motion: it can never be hot or cold. Having no internal energy of its own it has no weight, and can offer no resistance to the passage of material objects through it. The only energy which it can have consists in strains (which may be reckoned as potential energy) started by extraneous energy moving in it. It is not unreasonable to suppose that, in the collision of two molecules, the ether adjoining them, about the point of collision, may be strained for a short distance thereabout, and the strained ether, being perfectly elastic, in reacting will strain the next thin layer of ether in form of a hollow sphere around it, and so on. The wave will thus always be in form of a hollow sphere growing in diameter about a fixed centre, and dwindling in intensity as the energy becomes distributed in a volume which increases as the square of the diameter. Molecules which collide probably recover their normal state after a series of vibrations which succeed one another at uniform rate, so that a *series* of strains will be produced in the ether after each collision, and will succeed one another at equal intervals of time. And since the rate at which light travels in the ether is independent of its intensity, the distance between the heads of the successive waves produced by one collision will all be at the same distance one from another throughout their course. This distance is the wave-length, and is longer for red than for yellow light, and for yellow longer than for blue and for blue than for violet. Plenty of light evolved from the collisions of hot molecules is longer in wave-length than red light, and still more is shorter in wave-length than violet; but our eyes are not sensitive to them. The longer waves have little chance of penetrating far into a dense mass of molecules. The molecules take the energy and relieve the strain in the ether. The added energy partly increases the rate of movement of the molecules and so becomes heat and is no longer vibratory but translational energy. The waves of shorter length have a better chance of penetrating a mass of molecules.

To speak of the "penetrating power" of radiant energy, and of "hard" and "soft" rays, as if these qualities of the rays held good in regard to molecular masses in general, is apt to mislead. Really a transparent material is one of which the molecules cannot easily be set vibrating at the same rate as the light passing through

the mass vibrates, and therefore does not relieve the ether of the strain. Opaque material stops the passage of light by vibrating in concord with the light passing through it, and converting it into translational forms of energy. This puts the light out: it ceases to be light. It might be said that diamond transmits light because it is so hard that it is difficult to set its molecules vibrating, and that black lead absorbs light because it is so soft that its molecules are easily set vibrating to any pitch. However, there is other radiant energy besides light: for example, the rays given off by radium pass through a considerable thickness of lead. It is well ascertained that the vibrations of light are always at right angles to the direction in which the light is passing, and it is conceivable that there may be radiant energy for which the vibration is in other directions: but light has been much the most completely studied, and most certainly pervades the whole universe and it would be distracting to discuss the characters of any other sort here.

It is a well-established law of nature that a molecular mass which readily emits light of a particular wave-length, will also readily absorb light, of the same wave-length, passed through it from another source. Consider the light of the sun. The temperature of the sun is extremely high; iron, titanium, and other refractory substances, are in the state of vapour in its atmosphere, and it emits light of all sorts of wave-lengths and of great intensity. There is a limit to the shortness of the wave-length of the light to which ordinary photographic plates are sensitive, nevertheless there is no indication that any limit of the wave-length of the light has been reached, but only a limit of the absorbent power of the chemicals. There is, however, for every substance which emits radiant energy when heated, one substance which will absorb that energy. It is the same substance at a somewhat lower temperature. For example, the intense yellow light emitted by the flame of an oil lamp, in which some common salt is held, is much less intense after it has passed through the colder flame of a spirit lamp with salt in it. The flame has absorbed some of the energy, and has so become hotter; and this will go on so long as the temperatures of the two flames are unequal.

Attempts to estimate the heat equivalent of the sunshine at the surface of the earth by absorbent actinometers are fallacious in that they do not take into account the very short wave-length light, which makes its way through all ordinary materials and penetrates the interior of the earth.

Stars.

There is good reason for supposing that the sun is one of the stars, and not one of the larger, or hotter, of them. All that we know about them is derived from the light they send to us. They are

all emitting radiant energy incessantly; and though their spectra show that some are hotter than the sun, and that their atmospheres are not all alike, yet in general characters they closely resemble it.

We believe that they have been pouring forth their energies in radiation in enormous quantities for untold ages, because the changes in the configuration of the heavens, during the time in which these have been recorded, are so small that we may say that the configuration is steady.

Whence is all this inexhaustible energy drawn, and whither does it go?

Every spherical wave of light, once started, will go on expanding in the ether until it meets with some molecular mass which will relieve the strain and take the energy. Even such a mass (unless it be as big as the whole wave) will only make a hole in it, and then every point in the edge of the hole will become the centre of a new wave. It has been calculated that only one hundredth of a millionth $(\frac{1}{100,000,000})$ of the radiance of the sun is taken up by the planets, meteorites and other molecular substances within the solar system; all the rest is dispersed in stellar space. This is probably far too large an estimate of the absorption, because no account is taken of the radiance of very short wave-length, which cannot be caught by the actinometers used to measure the amount of the solar radiation.

Some think that the question of the limitation of our universe need be determined before we can give any reply about the supply and disposal of all this energy. It may be as well to review the alternatives. Suppose

1. That the universe is an immense mass of ether bounded by vacuum. This will not preclude the existence of other similar universes bounded in the same way: we should never find out their existence, they could take no energy from us, nor give us any. The boundary of our ether would bound the expansion of every wave of light in it, but could destroy no energy. Every point of the wave on reaching the boundary of the ether would become the origin of a new wave which would expand in the ether in the only way open to it, and add its strength to the store of our universe.

2. The only alternative is to suppose our ether to be unbounded. In that case we must suppose that there are dark stars beyond our universe, or some means of returning the energy which leaves it. It cannot be accepted that our universe should be always losing energy and getting none back. The life of the world is always a circuit of changes. "All the rivers run into the sea, yet the sea is not full" (Eccles. i. 7).

Considering the number of the stars, their wide distribution in all directions and distances, and the rapidity with which their radiance is dispersed owing to the character of its movement in

ever growing spherical waves propagated with enormous velocity, and further considering the ages during which this has been going on, with hardly any permanent change, we cannot easily avoid the conclusion that the accumulation of this sort of energy in our universe must be enormous. It comes from every quarter of the heavens, the waves crossing and re-crossing one another in every direction without interference, so that they still form sharp images of the stars from which they come.

The sun and stars have remained for ages, and since this radiant energy has proceeded from them we feel sure that their molecules are concordant with it, and will become heated by absorbing it. The ether cannot be either hot or cold; but the hotter the molecules are the more intense will be the radiance which they generate, so that they compensate by the intensity of what they emit for the dissipated state of that which they absorb. If the star is more cooled by the emission than it is heated by the absorption, the emitted rays will become less intense: while, on the other hand, if it is more heated by the absorbed rays, those emitted will be more intense. The universal radiance is so vast, and is diffused with such rapidity, that the supply must be very steady. To this steady supply the general temperature of each star must adjust itself. That is to say, if the mass of the star is steady, the star will adjust its temperature so that the radiant energy it emits is equivalent to that which it absorbs.

If there should be an accession of mass to the star by the falling into it of some extraneous matter, the star would absorb so much more energy from the universal supply, and its temperature would rise until the intensity of its emitted radiance balanced the amount absorbed.

Another adjustment remains to be considered, namely, that of volume. No doubt the outer part, at least, of the sun and stars is in the gaseous state, so that their volume will easily respond to pressure. The only pressure, so far as we know, to which they are regularly subject is that of the universal radiance, together with the reaction of the radiance of the star itself, which is a steady uniform compression of the whole mass*; and the volume of the star will expand and contract under this pressure as the temperature rises and falls, and be steady as long as that temperature is steady.

The general result may be stated, that the larger the mass of the star, the hotter will it be, and the more intense the radiant energy that it emits.

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* This is on the supposition that the star has no proper motion. If it have such motion, the pressure on that side of the star which goes to meet the universal radiance will be slightly greater than it is on the other side which moves in the same direction as the radiance. The difference will be very minute because the star's velocity cannot be comparable with that of light. Nevertheless it may have a cumulative effect, and may apply to planetary motions as well as to those of stars.

The earth, so small in comparison with the sun, is not cooler than might be expected. Probably all the universal radiance of wave-length long enough to be photographed which falls on the earth is taken up by a very thin layer at the surface and soon radiated away again; while the greater part, being of shorter wave-length, penetrates the interior, and is only gradually absorbed as it meets with material which can vibrate in concord with it, and so carries its energy into the interior, compressing and heating it.

Mars, still smaller than the earth, is seen to have much larger accumulations of snow at its poles, during their respective winters, than the earth has: and is probably colder.

Jupiter, with a mass 300 times as great as the earth, and a density only one-quarter of that of the earth, is believed to be nearly, or quite, red-hot: and has been called a semi-sun. This is quite in agreement with the theory here maintained.

So far no mention has been made of gravitation except in relation to the orbits of double stars which follow the law of gravitation. There was no need to mention it. Newton never believed in mechanical action at a distance, and only used this metaphysical theory of gravitation for convenience of calculation, in the same way as the mathematicians of our day, for the investigation of the kinetic theory of gases, used metaphysical molecules, hard, impenetrable, perfectly elastic, spheres, for this purpose of calculation. Natural philosophers are fast coming to the belief that gravitation must have its seat in the field rather than in the sun and stars, and that means that it must be radiant energy, because no other known form of energy of sufficient intensity can exist in stellar space or where there are next to no molecules to carry it.

Perhaps some may object that it is not likely that radiant energy should be intense enough to do the work ascribed to gravitation; but no reason can be given for this. We have not found any method of measuring the intensity of the universal radiance which is distinct from that employed to measure the intensity of gravitation. But there is one distinction between the two: we are quite sure that the universe is filled with the visible light from the stars, but have no suggestion of any other source of gravitation. When gravitation is put into the field its sign must be changed, it becomes repulsive instead of attractive, and universal radiance can do all that gravitation can.

It may be well now to turn to the surface of the earth and see what confirmation of the theory of universal radiation can be found in the course of nature as we see it at close quarters and within reach of experiments. Fortunately, there is one branch of physics which has been explored by the ablest mathematicians and experimenters: it is capillarity, the rise and fall of liquids in narrow

tubes, the curvature of their surfaces, and other connected appearances.

Laplace devised a theory to account for the observed facts, in which he postulated an attractive force between every two molecules of the liquid, proportional to their masses, and varying with the distance between them in some undefined way but quite independent of any mutual attraction between them arising from gravitation. An important feature of this attraction was that it was sensible when the molecules were at infinitesimal distances apart and became zero when the distance became finite. The results deduced on this theory were found to agree closely with the observed facts. Subsequently Gauss, working by general principles, was able to deduce results in agreement with experiment without reference to any gravitational attraction between the molecules. Of course it is not meant that the weight of the liquid or other effects usually ascribed to gravitation were neglected, but that gravitation does not interfere with capillarity.

If now we substitute the universal radiance for gravitation what difference will it make? The radiation comes from every side in all directions: most of it will fall at some inclination to the normal on the surface of the liquid. It is a character of radiation that the pressure which it exerts on a mass of material is always in the direction in which the wave is moving; hence only the resolved part of the pressure which is in the direction of the normal compresses the mass. The other part of the full pressure will act tangentially on a very thin layer of material at the place of incidence. This tangential pressure will come from all sides, and the united effect will be to push closer together the molecules in a very thin layer of the exterior of the mass. That is to say they will produce a surface tension all over the surface. The normal pressure will ensure the cohesion of the liquid (or solid), and the tangential pressure will make the surface tension, in a far more satisfactory way than the old theory of gravitative attraction can provide.

Sur la représentation analytique des congruences de coniques. Par L. GODEAUX, professeur à l'Ecole Militaire (Bruxelles). (Communicated by Professor H. F. BAKER.)

Dans un article très intéressant publié sous le même titre dans les *Proceedings of the Cambridge Philosophical Society* (Vol. XXI, Part 3, 1922), M. James applique les méthodes de M. Stuyvaert pour l'étude des congruences linéaires de variétés algébriques au cas où ces variétés sont des coniques de l'espace*. M. James utilise les représentations analytiques suivantes de la conique dans l'espace :

$$\begin{Vmatrix} a_x^2 & a_x' & A \\ b_x^2 & b_x' & B \end{Vmatrix} = 0,$$

$$\begin{Vmatrix} a_x^2 & b_x & c_x \\ a_x' & B & C \end{Vmatrix} = 0.$$

Il étudie ensuite en détail les congruences linéaires obtenues en supposant les éléments d'une ligne ou de deux colonnes fonctions linéaires de deux paramètres. Dans deux courtes notes publiées en 1908†, nous avions, à la suite de l'étude des travaux de M. Stuyvaert, fait les mêmes recherches que M. James, en utilisant les mêmes représentations analytiques de la conique, mais en nous bornant à la détermination de la configuration des lignes singulières des congruences rencontrées.

Qu'il nous soit également permis de compléter la bibliographie des congruences linéaires de coniques donnée par M. James au début de son travail. Toutes ces congruences ont été déterminées en 1895 par M. Montesano au moyen d'une méthode géométrique‡. Nous avons plus tard cherché à déterminer ces congruences par une autre méthode, mais celle-ci n'est applicable qu'aux cas où le nombre des lignes singulières est peu élevé; nous nous sommes précisément borné aux cas où ce nombre est un ou deux§.

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* Dans ses travaux, M. Stuyvaert s'est particulièrement attaché aux congruences linéaires de cubiques gauches; ses premières recherches sur ce sujet forment le chap. III de sa dissertation: Etude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre (Gand, 1902). Voir ensuite: *C. R.* 1905; "Cinq études de Géométrie analytique" (*Mém. de la Soc. Roy. des Sciences de Liége*, 1907); "Algèbre à deux dimensions" (Gand, 1920); "Congruences de cubiques gauches" (*Mém. in-8º de l'Acad. Roy. de Belgique*, 1920), et d'autres travaux que l'on trouvera cités dans ces deux derniers Mémoires.

† "Sur la représentation analytique de la conique dans l'espace" (*Bull. Acad. Roy. Belgique*, 1908); "Sur quelques congruences linéaires de coniques" (*Archiv der Math. und Phys.*, 1908).

‡ "Su le congruenze lineari di coniche nello spazio" (*Rend. Ist. Lomb.* 1893); "Su i varii tipi di congruenze lineari di coniche dello spazio" (*Rend. Accad. Napoli*, 1895).

§ "Recherches sur les systèmes de coniques de l'espace" (*Mém. Soc. Roy. Sc. Liége*, 1911).

The motion of a neutral ionised stream in the earth's magnetic field. By S. CHAPMAN, M.A., Trinity College.

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Introduction.

(1)   The object of this paper is, mainly, the examination of the mode of incidence upon the earth of a neutral ionised stream of matter from the sun. The question arises in connection with the theory of magnetic storms and aurorae, in view of a recent suggestion by Prof. Lindemann* that the solar agent which produces terrestrial magnetic storms consists of such a stream. In various other recent theories on the subject the solar stream has been supposed to contain electric corpuscles mainly of one sign of charge, but all such theories hitherto have encountered the objection that the mutual repulsion of the particles would disperse the streams laterally, while on their way from the sun, so much as to preclude the incidence upon the earth of an adequate amount of charge. This criticism was directed by Sir A. Schuster against the theory of (amongst others) the late Prof. Birkeland, and by Prof. Lindemann against a theory (similar only in the respect mentioned) proposed by the writer†. In a rejoinder‡ I assented to Prof. Lindemann's criticism, though, as he suggested, the detailed figures from my papers, on which he based his criticism, were very tentative. I also proposed an alternative to his suggestion concerning the solar agent, as to which I am now doubtful. Prof. Lindemann's proposal was to modify my theory, while retaining its distinctive feature of the radial motion of charge within the earth's atmosphere, by supposing that the solar stream is neutral but ionised, the two sets of charges becoming separated in the earth's atmosphere on account of their different powers of penetration, and afterwards approaching and combining with one another by a motion mainly radial. He developed this view in detail as regards the emission, propagation, energy and mass of the stream, but without considering the mode of incidence of the stream upon the earth, or how (in detail) the magnetic storm would then be produced. In this paper I show that aurorae cannot be explained by a neutral stream. Prof. Lindemann's suggestion seems, therefore, to fail, while my theory of magnetic storms must be modified so as to be consistent with a less heavily charged stream than at first appeared necessary.

(2)   The close connection between aurorae and magnetic storms makes it almost certain that both are associated with the same solar

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* F. A. Lindemann, *Phil. Mag.*, Dec. 1919.

† S. Chapman, *Proc. Roy. Soc.* A, 95 (1918).   ‡ *Id., Phil. Mag.*, Nov. 1920.

agent. If it were established that magnetic storms are caused by neutral ionised streams, the mathematical theory of aurorae would then have to deal with the motion of such streams in the earth's magnetic field. At present the principal theory of aurorae discusses the motion of a stream of corpuscles of like sign, the element considered by Prof. Störmer* in his valuable researches being, for the most part, a single electric corpuscle. Even in this case the apparently simple equations of motion cannot be integrated completely, but they indicate that a charged stream would impinge upon the earth only in two limited zones, one around each magnetic pole, like the observed auroral zones. By laborious numerical quadratures Prof. Störmer has worked out many possible paths of the single corpuscle; these paths are of very varied types, some of them being extremely complicated. In the problem of the motion of a neutral ionised stream, considered in this paper, the equations are still more complex, and numerical calculation has to be resorted to at an early stage; the actual stream-lines, however, are much less intricate in form than some of the possible paths of a single charged corpuscle.

The motion of a single charged particle.

(3) In order to facilitate some comparison between the two problems, a few of the chief features of Prof. Störmer's theory will be briefly indicated. Let M be the magnetic moment of the earth, regarded as a uniformly magnetised sphere, and let $\mathbf{H}$ be the magnetic intensity at the point (r, θ, ϕ), so that $H = (M/r^3) f(\theta)$. Let the mass, charge (in E.M. units) and velocity of the corpuscles be $m, e, \mathbf{V}$ respectively. The force on the particle due to the magnetic field is $[e\mathbf{V}, \mathbf{H}]$, using the ordinary right-handed vector-product notation. This being normal to $\mathbf{V}$, the velocity varies in direction only, and is constant in magnitude. The curvature $1/\rho$ is in the plane of $\mathbf{V}$ and $[e\mathbf{V}, \mathbf{H}]$, and is given in magnitude by the equation $V^2/\rho = \dot{V}$, where $m\dot{\mathbf{V}} = [e\mathbf{V}, \mathbf{H}]$. Introducing the constant scalar (linear) magnitude l† defined by $l^2 = Me/mV$, the equation of motion may be written

$$\frac{r}{\rho}\frac{\dot{\mathbf{V}}}{\dot{V}} = \frac{Me}{mVr^2}\left[\frac{\mathbf{V}}{V}, \frac{\mathbf{H}}{H}\right] f(\theta) = \frac{l^2}{r^2}\left[\frac{\mathbf{V}}{V}, \frac{\mathbf{H}}{H}\right] f(\theta).$$

Here the quotients $\dot{\mathbf{V}}/\dot{V}$, $\mathbf{V}/V$, $\mathbf{H}/H$ are unit vectors which merely determine the direction of the vectors on the two sides. Equating the scalar magnitudes of the vectors, we see that

$$r/\rho = (l^2/r^2) f(\theta) \sin(V, H).$$

The constants of the particle enter into these last two equations only through l. Clearly, for two particles of like sign but with

* A résumé of Prof. Störmer's theory, with references, is given in *Terrestrial Magnetism*, vol. XXII, p. 101.

† l is the same as Prof. Störmer's constant c.

different values of l, there is a complete correspondence between all possible paths in the two cases, except for a scale difference of which l is the parameter. For oppositely charged particles with equal constants l, the two families of paths are mirror images of one another in meridian planes of the sphere.

From a certain first integral of the equations of motion Prof. Störmer has drawn conclusions concerning the regions of space within which all paths reaching the centre must lie; this is important because only those corpuscles with trajectories in the vicinity of the said paths can reach the earth*—the other paths return into space. It appears, in fact, that particles from the sun cannot reach the earth except within two narrow zones, one centred at each magnetic pole. Particles may fall on the dark as well as on the sunlit portion of these zones. The angular radius of the zones depends on l, which in the case of α-particles, β-particles, and cathode-rays, has the respective values (roughly) of

$$25a, \ 250a, \text{ and } 1000a,$$

a being the earth's radius; the corresponding values of the angular radius of the zones are roughly

$$18°, \ 5°, \ 3°.$$

These zones are naturally identified, on this auroral theory, with the zones of maximum auroral frequency; the latter, however, are generally supposed to have an angular radius of about 23°, while, moreover, aurorae are visible at much greater distances from the poles at times of magnetic disturbance. One of the chief difficulties of the theory is this necessity of accounting for the precipitation of corpuscles in regions beyond the theoretically-calculated zones of precipitation: and the difficulty may be even greater than auroral observations indicate, because the study of magnetic storms suggests that corpuscles may enter the atmosphere even in the tropical zones and over the equator, though less intensely than in the zones where aurorae and magnetic disturbances are most frequent. The theory seems directly at variance with this possibility, as it indicates that the paths from infinity cannot approach the equator in the equatorial plane within a distance of many diameters of the earth†.

In considering these difficulties, the simplified nature of the theory must, of course, be remembered, and, in particular, that the mutual influence of the corpuscles on each other's paths is neglected. Prof. Störmer has pointed out that the theory predicts a streaming of corpuscles round the earth on one or other side of the axis (on the post meridiem side, if the corpuscles are negative, and on the

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* *Terrestrial Magnetism*, vol. XXII, p. 101.

† Cf. the diagram on p. 100, vol. XXII, *Terrestrial Magnetism*: the paths from infinity are those for which $\gamma > -1$.

opposite side, if they are positive), and has suggested that the magnetic field of this stream might greatly modify the position of the auroral belts. By discussing the analogous but more tract-able problem of the effect of a closed ring of corpuscles circulating round the equator, he has found that such a ring could draw the belts of precipitation down from their theoretical position (even for cathode-rays) to the belt of maximum auroral frequency, without exercising greater magnetic effect on the earth than about 0·00030 c.g.s. (30γ). The magnetic effect of a ring which could draw the zone down to $45°$ would be 1200γ. But the arguments which seem to preclude solar streams of corpuscles of one sign from being regarded as the direct cause of magnetic storms apply also to this part of Prof. Störmer's auroral theory.

The path of a neutral ionised stream.

(4) In considering the motion of an ionised neutral stream in a magnetic field, the corpuscles will be supposed (merely for mathe-matical convenience) to have no random motion additional to their streaming motion. The magnetic field will deflect the oppositely-charged corpuscles in opposite directions, a tendency which will be counteracted by the electric field set up by the relative displace-ment of the two sets of charges: a kind of polarisation will thus result. It will be supposed that this equilibrium state has been attained, and that it is constant in time in any given region of space specified relative to the axis of the earth and the line joining the centres of the earth and sun. The tendency to displacement for the particles of the two kinds will vary, at any point, inversely as their masses, and also inversely as the total charge density of either sign.

In an isolated stream the surface electrifications resulting from the polarisation would tend to leak away, the leakage being more rapid on the side of the charges of smaller mass, until the stream acquired an opposite charge sufficient to neutralise this differential effect. This leakage will in the first instance be neglected.

In the case of a uniform stream moving with velocity $\mathbf{V}$ through a uniform magnetic field of intensity $\mathbf{H}$, the polarisation $\mathbf{P}$ would be uniform and equal to $(3/4\pi c)\,[\mathbf{V},\mathbf{H}]$, c being the velocity of light. When the magnetic field is not uniform the polarisation will still be equated to $(3/4\pi c)\,[\mathbf{V},\mathbf{H}]$, though this is now only an approxi-mation and no longer strictly true.

The electromagnetic force on any volume-element of the stream can be analysed as follows:

(a) The polarisation will result in a non-uniform distribution of charge, of amount $-\operatorname{div}\mathbf{P}$ per unit volume, together with a surface distribution of amount P_n per unit area, where P_n denotes the component of $\mathbf{P}$ normal to the surface. The motion of these charge distributions, with the velocity $\mathbf{V}$ appropriate to each point, is

equivalent to a non-uniform set of electric currents **i**, which will be acted on by the magnetic field with the force [**i**, **H**]. Thus this part of the force on the element is

$$\frac{1}{c}\iiint(-\operatorname{div}\mathbf{P})\,[\mathbf{V},\,\mathbf{H}]\,dv + \frac{1}{c}\iint P_n\,[\mathbf{V},\,\mathbf{H}]\,dS\,;$$

when the latter integral is transformed to a volume integral it becomes

$$\frac{1}{c}\iiint\left[\frac{\partial}{\partial x}\{P_x\,[\mathbf{V},\,\mathbf{H}]\} + \frac{\partial}{\partial y}\{P_y\,[\mathbf{V},\,\mathbf{H}]\} + \frac{\partial}{\partial z}\{P_z\,[\mathbf{V},\,\mathbf{H}]\}\right]dx\,dy\,dz$$

$$= \frac{1}{c}\iiint(\operatorname{div}\mathbf{P})\,[\mathbf{V},\,\mathbf{H}]\,dv + \frac{1}{c}\iiint\left(P_x\frac{\partial}{\partial x} + P_y\frac{\partial}{\partial y} + P_z\frac{\partial}{\partial z}\right)[\mathbf{V},\,\mathbf{H}]\,dx\,dy\,dz.$$

The first term cancels out with the force on the volume-charge current, so that this part of the force reduces to the second term of the last expression; this may be written:

$$\frac{1}{c}\iiint(\mathbf{P},\,\operatorname{grad})\,[\mathbf{V},\,\mathbf{H}]\,dv.$$

(b) The polarisation is supposed constant at any point, but, as an element of the stream moves along, it traverses regions of different polarisation, and there must therefore be cross-currents in the element, sufficient to produce this change of polarisation. In time dt the element suffers the displacement $\mathbf{ds} = \mathbf{V}\,dt$, and if $\mathbf{dP}$ is the corresponding change of polarisation, the cross-current per unit volume is in the direction $\mathbf{dP}$ and of amount $\mathbf{dP}/c\,dt$ or $V\,(dP/c\,ds)$, the differentiation being performed in the direction of $\mathbf{V}$. Thus the current is

$$\frac{1}{c}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\mathbf{P} = \frac{1}{c}\,(\mathbf{V},\,\operatorname{grad})\,\mathbf{P}.$$

The force on this current is consequently

$$\frac{1}{c}\iiint[(\mathbf{V},\,\operatorname{grad})\,\mathbf{P},\,\mathbf{H}]\,dv$$

for the whole element.

Hence the total force on an element of the stream at any point is proportional to the volume, and is of amount

$$\mathbf{F} = \frac{1}{c}\,\{(\mathbf{P},\,\operatorname{grad})\,[\mathbf{V},\,\mathbf{H}] + [(\mathbf{V},\,\operatorname{grad})\,\mathbf{P},\,\mathbf{H}]\}$$

per unit volume. Since $\mathbf{P} = (3/4\pi c)\,[\mathbf{V},\,\mathbf{H}]$, the force $\mathbf{F}$ is proportional to V^2, in the sense that if the velocity everywhere were increased in a certain ratio k, the force $\mathbf{F}$ would be changed only by the numerical factor k^2; and similarly it is proportional to H^2.

(5) The equation of motion of an element, at a point where the density of the stream is ρ, is

$$\rho\mathbf{V} = \mathbf{F},$$

and since the time-dimension of the left-hand side is of degree -2, it is clear that for a given magnetic field, a change in the velocity of the stream at infinity will simply result in a proportional change of velocity at every point of the field, without altering the geometry of the stream-lines or the distribution of density throughout the field. Hence a knowledge of the geometry of the paths (e.g. the radius of the auroral zone, should such a zone exist also in the present case) could give no indication of the velocity of the stream; the geometry of the paths is determined solely by the charge-density of the stream; on the ordinary auroral theory, as has been seen, the scale of the paths depends on $V^{-\frac{1}{2}}$, and the auroral zone varies in radius with V.

(6) Another important difference from the force on a charged particle or stream is that the latter force depends on H, and therefore diminishes outwards (*ceteris paribus*) as the inverse third power of the radial distance r; in the case of the neutral stream, the force depends on the space derivative of H^2, which varies as the inverse *seventh* power of r. Hence the deflection of the neutral ionised stream is confined much more closely within the immediate neighbourhood of the earth than in the case of the charged stream.

Motion in the equatorial plane.

(7) Since the force $\mathbf{F}$ per unit volume of the stream depends on the space derivatives of the velocity as well as on the magnetic field, it seems impracticable to obtain a general solution of the equations of motion. The problem is somewhat simpler when consideration is restricted to the two possible cases of plane motion, viz. in the equatorial plane and in the meridian plane containing the sun. These fortunately suffice to indicate the character of the paths in the more general case.

The components of $\mathbf{F}$ along any three rectangular axes Ox, Oy, Oz are given by the following and two similar expressions:

$$\frac{1}{c}\left\{\left(P_x\frac{\partial}{\partial x} + P_y\frac{\partial}{\partial y} + P_z\frac{\partial}{\partial z}\right)(V_yH_z - V_zH_y)\right.$$
$$\left. + H_z\left(V_x\frac{\partial}{\partial x} + V_y\frac{\partial}{\partial y} + V_z\frac{\partial}{\partial z}\right)P_y - H_y\left(V_x\frac{\partial}{\partial x} + V_y\frac{\partial}{\partial y} + V_z\frac{\partial}{\partial z}\right)P_z\right\}$$
$$\ldots\ldots(1),$$

where

$$P_x = \frac{3}{4\pi c}(V_yH_z - V_zH_y), \quad P_y = \frac{3}{4\pi c}(V_zH_x - V_xH_z)$$
$$P_z = \frac{3}{4\pi c}(V_xH_y - V_yH_x) \qquad\qquad \Bigg\}\ldots(2).$$

First consider the motion in the equatorial plane, and take the z axis parallel to the earth's axis of rotation, so that H_x, H_y and

their derivatives are zero at all points in the plane, as also are P_z and its derivatives. In this case the z component of $\mathbf{F}$ vanishes, showing that a stream initially projected in the equatorial plane would continue to move in the plane. The x and y components of $\mathbf{F}$ reduce to

$$-\frac{3}{8\pi c^2}\left[\frac{\partial}{\partial x}\{H_z^2(V_x^2 - V_y^2)\} + \frac{\partial}{\partial y}(2H_z^2 V_x V_y)\right],\Bigg\}$$
$$-\frac{3}{8\pi c^2}\left[\frac{\partial}{\partial y}\{H_z^2(V_y^2 - V_x^2)\} + \frac{\partial}{\partial x}(2H_z^2 V_x V_y)\right]\Bigg\} \quad ...(3).$$

It is convenient to take the x axis along the direction of the stream at the point considered, and the y axis normal to it, in the direction $[\mathbf{z}, \mathbf{x}]$; then $V_x = V$, $V_y = 0$. This choice is better remembered if s and n are used in place of x and y; this notation will be adopted, and also the suffix z in H_z will be dropped, as unnecessary in the present instance. The components F_s and F_n of $\mathbf{F}$ now become

$$-\frac{3}{8\pi c^2}\left[\frac{\partial}{\partial s}(H^2 V^2) + 2H^2 V\frac{\partial V_n}{\partial n}, \quad -\frac{\partial}{\partial n}(H^2 V^2) + 2H^2 V\frac{\partial V_n}{\partial s}\right]$$
$$......(4).$$

Hence the equation of motion along the stream is

$$\tfrac{1}{2}\rho\frac{\partial V^2}{\partial s} = -\frac{3}{8\pi c^2}\left\{\frac{\partial}{\partial s}(H^2 V^2) + 2H^2 V\frac{\partial V_n}{\partial n}\right\} \quad ......(5).$$

The last term on the right depends on the convergence or divergence of the stream-line in the equatorial plane, and is negative if, as will appear, the stream converges in the plane. The convergence seems to be so small, however, that this term can safely be neglected; if it were taken into account, the estimated bending and convergence of the stream would be slightly reduced.

Even when this term is omitted, however, no simple exact integral of the resulting equation is obtainable, because ρ is an unknown function of position in the field. Since, however, the equation may be written in the form

$$\left(\frac{4\pi c^2\rho}{3} + H^2\right)\frac{\partial V^2}{\partial s} = -V^2\frac{\partial H^2}{\partial s} \quad ....(6),$$

it is clear that so long as the stream is approaching the earth, and therefore (in the equatorial plane) entering into a stronger magnetic field, the velocity along it is decreasing. The diminution of kinetic energy supplies the gain of energy of electric polarisation.

The equation of continuity may be written in the form

$$\frac{\partial(\rho V)}{\partial s} = -\rho\left(\frac{\partial V_n}{\partial n} + \frac{\partial V_z}{\partial z}\right) \quad ....(7),$$

taking the axes as already described and omitting the term $\partial\rho/\partial t$, since the state of motion is supposed steady. The last two terms represent the effect of convergence of the stream in and normal to the equatorial plane. The convergence in the plane seems to be outweighed by a greater divergence normal to the plane, so that the right-hand side is negative as a whole, i.e. ρV decreases along the stream; but the change of section is so small that it is sufficient for our purpose to neglect it, and to take ρV as constant and equal to its initial value $\rho_0 V_0$. Using the equation

$$\rho V = \rho_0 V_0 \quad \dots\dots\dots\dots\dots\dots(8),$$

the equation of motion can now be integrated, with the result

$$V = V_0 \frac{\left(1 + \dfrac{3H^2}{2\pi c^2 \rho_0}\right)^{\frac{1}{2}} - 1}{3H^2/2\pi c^2 \rho_0} \quad \dots\dots\dots\dots(9).$$

The retardation thus depends on the quantity $3H^2/2\pi c^2 \rho_0$, the maximum value of which is attained when the stream reaches the earth, where

$$H = H_0 = 0\cdot35 \text{ c.g.s.} \quad \dots\dots\dots\dots\dots(10)$$

approximately. It is convenient, for later work, to introduce the symbol

$$\lambda \equiv \frac{4\pi c^2 \rho_0}{3H_0{}^2} \quad \dots\dots\dots\dots\dots\dots(11).$$

The following table gives the value of the retardation V/V_0, on reaching the earth, for various values of ρ_0 and λ. The value of $3H_0{}^2/4\pi c^2$, which equals ρ_0/λ, is approximately $3\cdot2 . 10^{-23}$; the table shows that if ρ_0 much exceeds this, i.e. if λ much exceeds unity, the retardation will be very slight.

ρ_0	$3\cdot2 . 10^{-25}$	$3\cdot2 . 10^{-21}$	$6\cdot4 . 10^{-24}$	$3\cdot2 . 10^{-23}$
λ	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{5}$	1
V/V_0	$0\cdot132$	$0\cdot358$	$0\cdot463$	$0\cdot732$

ρ_0	$6\cdot4 . 10^{-23}$	$3\cdot2 . 10^{-22}$	$3\cdot2 . 10^{-21}$
λ	2	10	100
V/V_0	$0\cdot828$	$0\cdot955$	$0\cdot995$

Since the mass of a hydrogen atom is $1\cdot66 . 10^{-24}$ gm., the density $3\cdot2 . 10^{-23}$ corresponds to a neutral stream in which the mass per c.c. is equivalent to that of about 20 hydrogen atoms.

It may be noted that the density of the stream considered in

Prof. Lindemann's paper* is approximately $8 . 10^{-24}$ on nearing the earth, corresponding in mass to about 5 hydrogen atoms per c.c.; such a stream would suffer a retardation of about one-half its speed. Since V/V_0 varies as H^2, the retardation, and corresponding increase of density of the stream, affects the stream only in the near neighbourhood of the earth.

Consider next the deflection of the stream, and let R denote the radius of curvature in the plane of motion. The equation of transverse acceleration may be written

$$\rho \frac{\partial V_n}{\partial t} = \frac{3}{8\pi c^2} \left\{ \frac{\partial}{\partial n} (H^2 V^2) - 2H^2 V \frac{\partial V_n}{\partial s} \right\} \quad \ldots\ldots(12).$$

Since $\partial V_n/\partial t = V (\partial V_n/\partial s)$, we have

$$V \frac{\partial V_n}{\partial s} \left\{ \rho + \frac{3H^2}{4\pi c^2} \right\} = \frac{3}{8\pi c^2} \frac{\partial}{\partial n} (H^2 V^2) \quad \ldots\ldots(13);$$

and, since $\partial V_n/\partial s = V/R$, this gives

$$\frac{1}{R} = \frac{3H^2}{4\pi c^2} \left\{ \rho + \frac{3H^2}{4\pi c^2} \right\}^{-1} \frac{\partial \log HV}{\partial n} \quad \ldots\ldots\ldots(14).$$

The term in F_n depending on the space-variation of the normal velocity V_n has not been neglected here, as in the former case, since it is of the same type as the term $\rho (\partial V_n/\partial t)$, and more important than the latter when ρ is small.

The values of ρ and V already obtained in terms of H are now inserted in the last equation; using the symbol λ already defined, and remembering that $H/H_0 = (a/r)^3$, after a little reduction it appears that

$$\frac{1}{R} = \frac{\left(1 + \frac{2}{\lambda} \frac{a^6}{r^6} \right)^{\frac{1}{2}} - 1}{1 + \frac{2}{\lambda} \frac{a^6}{r^6}} \cdot \frac{\partial \log H}{\partial n} \quad \ldots\ldots\ldots(15),$$

and

$$\frac{\partial \log H}{\partial n} = - \frac{3}{r} \cdot \frac{\partial r}{\partial n} = - \frac{3 \sin \psi}{r} \quad \ldots\ldots\ldots(16),$$

where ψ is the angle between the direction of motion of the stream at a point P and the radius OP from the earth's centre. Moreover

$$\frac{1}{R} = \frac{\sin \psi}{r} + \frac{d \sin \psi}{dr} \quad \ldots\ldots\ldots\ldots(17).$$

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* The mass striking the earth in a storm of 20 hours' duration is given as $6 . 10^7$ grammes; assuming a nearly parallel beam, the volume occupied by this mass, taking the velocity of the stream as $8 . 10^7$ cm. sec., as assumed by Prof. Lindemann, is $\pi (6\cdot 4 . 10^8)^2 20.60.60.8 . 10^7$ c.c., $6\cdot 4 . 10^8$ cm. being the earth's radius. This leads to the above estimate of the density.

Using these formulae, the last equation reduces to

$$\frac{d \sin \psi}{\sin \psi} = \left\{ \frac{3 \left(1 + \frac{2}{\lambda}\frac{a^6}{r^6}\right)^{\frac{1}{2}} - 1}{1 + \frac{2}{\lambda}\frac{a^6}{r^6}} - 1 \right\} \frac{dr}{r} \quad \text{.........(18)},$$

which has the integral

$$r \sin \psi = \tfrac{1}{2} p_0 \left\{ 1 + \left(1 + \frac{2}{\lambda}\frac{a^6}{r^6}\right)^{-\frac{1}{2}} \right\} \quad \text{............(19)},$$

where p_0 is a constant of integration. Clearly p_0 is the value of $r \sin \psi$ at infinity. In place of $r \sin \psi$ we may write p, which, of course, denotes the distance from O of the tangent to the path at P.

As foretold on general grounds (§ 5), this equation of the path of the stream does not depend upon the initial velocity of the stream, but only on ρ (through λ).

As r diminishes, p also steadily diminishes, but always remains greater than $\tfrac{1}{2} p_0$. Hence no part of the stream not initially directed towards O can ever reach O, but must always remain beyond half the original perpendicular distance of its path from O. Again, as r diminishes, $\sin \psi$ increases, i.e. the path becomes more and more inclined to the radius OP; it is easily seen that $\sin \psi$ increases from zero at infinity to its maximum value unity, which corresponds to the minimum value of r and p (when, in fact, r and p are equal). Beyond this apse-point the stream recedes from the earth along a path symmetrical with respect to the apsidal radius. This minimum value of r and p will be denoted by r_m or p_m, and is obtained by writing $\sin \psi = 1$ in the last formula. Thus

$$r_m = \tfrac{1}{2} p_0 \left\{ 1 + \left(1 + \frac{2}{\lambda}\frac{a^6}{r_m^6}\right)^{-\frac{1}{2}} \right\} \quad \text{............(20)},$$

giving r_m in terms of p_0, and *vice versa*. It is easily proved that p_0 steadily increases as r_m increases, and conversely.

Impact upon the earth.

(8) If a path intersects the earth, that part of the stream will, of course, be absorbed and conclude its path in the atmosphere. The semi-breadth p_0' of the part of the stream thus absorbed is obtained by writing $r_m = a$ in the equation for p_0, so that

$$p_0' = 2a \left\{ 1 + \left(1 + \frac{2}{\lambda}\right)^{-\frac{1}{2}} \right\}^{-1}.$$

Just as p_0 is a monotonic function of r_m, so p_0' is a monotonic function of λ. The following table shows how p_0'/a varies with λ and ρ_0.

λ	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{2}$
ρ_0	0	$3\cdot2.10^{-25}$	$3\cdot2.10^{-24}$	8.10^{-24}	$1\cdot6.10^{-23}$
p_0'/a	2	$1\cdot87$	$1\cdot64$	$1\cdot50$	$1\cdot38$
$\lambda p_0'/a$	0	$0\cdot019$	$0\cdot164$	$0\cdot375$	$0\cdot69$

λ	1	2	10	∞
ρ_0	$3\cdot2.10^{-23}$	$6\cdot4.10^{-23}$	$3\cdot2.10^{-22}$	∞
p_0'/a	$1\cdot27$	$1\cdot17$	$1\cdot04$	$1\cdot00$
$\lambda p_0'/a$	$1\cdot27$	$2\cdot34$	$10\cdot4$	∞

The amount of matter impinging on the earth in the various cases is proportional to $\lambda p_0'/a$, which is tabulated in the last row. It diminishes considerably with λ, though not quite in the same ratio as λ itself.

Bending of the stream round the earth.

(9) It is of interest to determine how far the limiting path (corresponding to p_0') bends round the earth, and this may be done as follows: let r, θ be the coordinates of a point on the limiting stream-line, θ being measured from the direction of this stream-line at infinity, so that the initial value is 0. Then

$$r \cdot \frac{d\theta}{dr} = \tan\psi \quad \text{or} \quad d\theta = \frac{dr}{r} \cdot \frac{\sin\psi}{\sqrt{1 - \sin^2\psi}}.$$

Let $x = a/r$, so that for the limiting path x varies from 0 to 1, while

$$\sin\psi = \frac{a}{r} \cdot \frac{1 + \left(1 + \frac{2}{\lambda}\frac{a^6}{r^6}\right)^{-\frac{1}{2}}}{1 + \left(1 + \frac{2}{\lambda}\right)^{-\frac{1}{2}}} = x \, \frac{1 + \left(1 + \frac{2}{\lambda}x^6\right)^{-\frac{1}{2}}}{1 + \left(1 + \frac{2}{\lambda}\right)^{-\frac{1}{2}}}.$$

Then the value of θ when $r = a$ or $x = 1$, i.e. the angular distance, on either side of the central meridian of the earth, within which the neutral stream will impinge on the earth, is given by

$$\theta_0 = \int_0^1 \left\{\frac{1 + \left(1 + \frac{2}{\lambda}x^6\right)^{-\frac{1}{2}}}{1 + \left(1 + \frac{2}{\lambda}\right)^{-\frac{1}{2}}}\right\} \left[1 - x^2 \left\{\frac{1 + \left(1 + \frac{2}{\lambda}x^6\right)^{-\frac{1}{2}}}{1 + \left(1 + \frac{2}{\lambda}\right)^{-\frac{1}{2}}}\right\}\right]^{-\frac{1}{2}} dx$$

$$= \phi(\lambda).$$

The limiting value of this integral when $\lambda \to 0$ and also when $\lambda \to \infty$ is $\pi/2$, so that for very dense and also for very rare streams the matter impinges only on the front half of the earth; this is obviously the case for very dense streams, which are scarcely

deflected by the earth's magnetic field; for very rarefied streams, again, as our formula for $1/r$ indicates, the curvature is proportional to $\lambda^{\frac{1}{2}}$, i.e. to $\rho^{\frac{1}{2}}$, and is consequently very small.

The values of θ_0 have been calculated from the above integral for a number of values of λ, and are as follows:

λ	0	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
ρ	0	$3 \cdot 2 \cdot 10^{-25}$	$3 \cdot 2 \cdot 10^{-24}$	$5 \cdot 3 \cdot 10^{-24}$	$8 \cdot 0 \cdot 10^{-24}$	$1 \cdot 6 \cdot 10^{-23}$
θ_0	$90°$	$132°$	$158°$	$165°$	$172°$	$166°$

λ	1	2	4	8	∞
ρ	$3 \cdot 2 \cdot 10^{-23}$	$6 \cdot 4 \cdot 10^{-22}$	$1 \cdot 3 \cdot 10^{-22}$	$2 \cdot 6 \cdot 10^{-22}$	0
θ_0	$148°$	$126°$	$110°$	$101°$	$90°$

It is of interest to observe that with the density considered by Prof. Lindemann ($8 \cdot 10^{-24}$) the stream would impinge on the earth almost all round the equator; and that the same is true for a considerable range of density on either side of this value.

The utmost bending, it will be noticed, does not extend to a turning through a right angle. This is markedly different from the case of the corpuscular paths discussed by Prof. Störmer, which can twist to an unlimited extent.

Paths other than the limiting paths can easily be calculated by an extension of the same method. A drawing of several of the paths, including the limiting one, has been made for the case of $\lambda = 1/4$, $\rho_0 = 8 \cdot 10^{-24}$, which is approximately the case of maximum bending
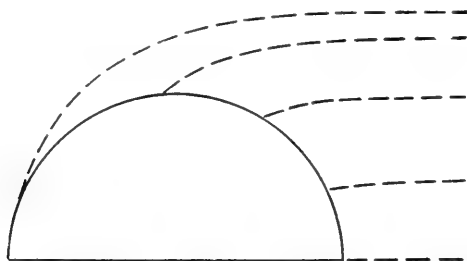
Fig. 1. Stream-lines in a neutral ionised stream (of density $8 \cdot 10^{-24}$), lying in the earth's equatorial plane. The stream-lines beyond the uppermost here drawn do not impinge upon the earth. Only half the earth's equatorial section is shown.

round the earth. It appears that while the path which has grazing incidence gets nearly to the midnight meridian, the fraction of the stream which gets to the back of the earth is only a small one, about $1/8$ of that which impinges on the front of the earth. The obliqueness of incidence increases with increasing angular distance

from the midday meridian, though, of course, far less rapidly*
than in the case of a beam of light. But the incidence of the
stream on the dark half of the earth is sufficiently oblique to
impair the penetrative power of the stream, and near the mid-
night meridian the stream must always remain high up in the
atmosphere.

As regards the radius of curvature R for the streams, from the
equation (7·17) it is easy to see that for paths outside the earth,
R is a minimum when $\psi = 90°$ and $r = a$, while the value of λ
which then makes R a minimum is 2/3. The value thus found for
R is $4a/3$, so that the curvature of the stream in the equatorial
plane is always less than that of the equatorial circle.

The paths in a meridian plane.

(10) The only other family of plane paths is that of the streams
moving in a meridian plane. Take any point P on such a path,
and a right-handed set of axes, P_s along the path, P_n normal to
the path and in the meridian plane, and P_x normal to the meridian
plane. Then V_x, H_x are both zero, so that the polarisation is entirely
along the direction of P_x. From the general expressions (7·1) for
the components of force per unit volume on the stream, it readily
appears that $F_x = 0$, i.e. there is no force tending to divert the path
from the meridian plane; and that

$$F_s = \frac{3}{4\pi c^2} \left\{ -V_s H_n{}^2 V_{xx} - \tfrac{1}{2} \frac{\partial}{\partial s} (H_n V_s)^2 + H_n H_s V_s \frac{\partial V_n}{\partial s} \right\},$$

$$F_n = \frac{3}{4\pi c^2} \left\{ V_s H_n H_s V_{xx} - V_s{}^2 H_n H_{xx} + H_s V_s \frac{\partial}{\partial s} (V_s H_n) - V_s H_s{}^2 \frac{\partial V_n}{\partial s} \right\}.$$

These expressions are much more complicated than those for
the equatorial case, because in a meridian plane the magnetic force
varies in direction at different points in the plane. The paths in the
meridian plane are consequently more complicated than those
hitherto discussed; in low latitudes they diverge from the equator,
in higher latitudes they converge towards it; their geometrical form
is still fairly simple, however, though the analytical expressions
representing the paths are unworkably complex. It therefore
seemed best to determine the paths by a combination of analysis,
calculation and drawing. The equatorial case had indicated that,
though a considerable degree of bending might occur near the limit

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* The angle of impact on reaching the earth, for an element of the stream with
p_0 as its initial value of p, is obtained by writing $r = a$ in the formula for $r \sin \psi$ in
§ 7. We thus have $\sin \psi = p_0/p_0'$, so that as p_0 increases from 0 to p_0', ψ increases
from 0 (corresponding to direct incidence) to 90° (grazing incidence). Since $\sin \psi$
increases very slowly as ψ approaches 90°, the obliquity of incidence for the element
which meets the earth on the twilight circle ($\theta = 90°$) may be 10° or 20° short of
grazing incidence.

of that part of a stream which impinges on the earth, the actual convergence or change of section of any element of the stream is small. While taking the longitudinal retardation and the curvature of the stream into account, therefore, the convergence was neglected, as before.

The equation of motion along the stream thus becomes

$$\rho V_s \frac{\partial V_s}{\partial s} = - \frac{3}{8\pi c^2} \frac{\partial}{\partial s} (H_n V_s)^2,$$

or, using the relation $\rho V_s = \rho_0 V_0$,

$$\frac{\partial \log \rho}{\partial s} = - \frac{\partial \log V_s}{\partial s} = \left\{ 1 + \frac{4\pi\rho c^2}{H_n^2} \right\}^{-1} \frac{\partial \log H_n}{\partial s},$$

which has the integral

$$\frac{\rho}{\rho_0} = \frac{V_0}{V} = \tfrac{1}{2} \left\{ 1 + \left(1 + \frac{3H_n^2}{2\pi\rho_0 c^2} \right)^{\frac{1}{2}} \right\} = \tfrac{1}{2} \left\{ 1 + \left(1 + \frac{2}{\lambda} \frac{H_n^2}{H_0^2} \right)^{\frac{1}{2}} \right\}.$$

The equation of normal motion likewise becomes

$$\left(\rho + \frac{3H_s^2}{4\pi c^2} \right) \frac{\partial V_n}{\partial s} = \frac{3}{4\pi c^2} \left\{ H_n H_s \frac{\partial V_s}{\partial s} + V_s \left(H_s \frac{\partial H_n}{\partial s} - H_n H_{xx} \right) \right\}.$$

Substituting for $\frac{\partial V_s}{\partial s}$ from the former equation, we get, after a little reduction,

$$\frac{1}{R} = \frac{1}{V_s} \frac{\partial V_n}{\partial s} = \frac{1}{1 + \frac{4\pi\rho c^2}{3H_s^2}} \left\{ \frac{1}{1 + \frac{4\pi\rho c^2}{3H_n^2}} \cdot \frac{1}{H_s} \cdot \frac{\partial H_n}{\partial s} - \frac{H_n H_{xx}}{H_s^2} \right\},$$

where R is the radius of curvature in the meridian plane.

Now at a point P distant r from the earth's centre O, and in latitude θ,

$$H_{xx} = - \frac{3H_0}{r^4} \sin \theta,$$

while if the directions of the stream and of the magnetic force at the point make angles β, ξ (as shown in Fig. 2) with OP,
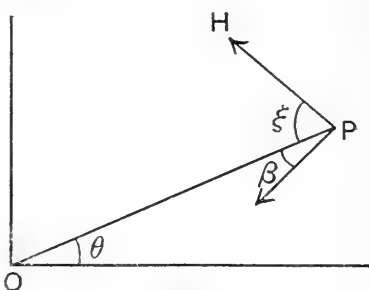
Fig. 2

$$H_s = \tfrac{1}{2} \{ 3 \sin (\theta - \beta) + \sin (\theta + \beta) \} H_0 / r^3,$$

$$H_n = \tfrac{1}{2} \{ 3 \cos (\theta - \beta) - \cos (\theta + \beta) \} H_0 / r^3,$$

$$\frac{\partial H_n}{\partial s} = 3 \{ 5 \cos (\theta - 2\beta) - \cos (\theta + 2\beta) \} H_0 / 4r^4,$$

$$\tan \xi = \tfrac{1}{2} \cot \theta.$$

Substituting these values, after some reduction we obtain the following formula, from which R was actually calculated:

$$\frac{r}{R} = \left\{1 + \lambda \frac{\rho}{\rho_0} \cdot \frac{H_0{}^2}{H_s{}^2}\right\}^{-1} \frac{\sin\theta \cdot \tan(\xi+\beta) + \left\{1 + \lambda \frac{\rho}{\rho_0} \cdot \frac{H_0{}^2}{H_n{}^2}\right\}^{-1} \left\{\frac{5}{4}\cos(\theta-2\beta) - \frac{1}{4}\cos(\theta+2\beta)\right\}}{3\sin(\theta-\beta) + \sin(\theta+\beta)}.$$

When the expression on the right is positive, the paths are convex to the equatorial plane; when negative, they are concave. It may be seen from the above formula that the curvature is zero when either H_n or H_s is zero, i.e. when the stream becomes either normal to, or along, the lines of force. The above formula was used in the following way.

A stream initially moving parallel to the equatorial plane was considered, and its path was regarded as straight up to a certain distance from the earth. Its radius of curvature at this distance was calculated by the above formula, the values of H_n, H_s and ρ/ρ_0 being first calculated from the previous formulae. The path was then continued with this curvature for a small distance, the new values of θ, ξ, β being either computed or found graphically, and R was again calculated, enabling a further extension of the path to be made. This procedure, being rather laborious, was carried only so far as would make it clear whether or not the amount of bending in the polar regions was sufficiently in excess of, and different from, that in the equatorial plane, to be likely to aid in explaining the main features of auroral incidence. The important parts of three paths were worked out, for one value of λ or of the initial density ρ_0; this value of λ was 1/4, being the one for which most bending was found to occur in the equatorial plane, and also corresponding to the value 8.10^{-24} for ρ_0, which was considered by Prof. Lindemann. The results indicate rather less convergence of the stream in the meridian plane than in the equatorial plane, and there is no indication of any tendency for the streams to enter along any special "auroral" zone.

The first path considered was the one which, if undeflected, would reach the earth's pole; the curvature of this path is very slight; at a distance $1.5a$ from the earth, the value of R is $57.5a$ approximately; the values of R at the radial distances 1.4, 1.3, 1.2, 1.1 and 1.0 times the earth's radius are approximately $20a$, $8.4a$, $4.1a$, $2.5a$ and $1.6a$. The total deflection is $12°$, and the path meets the earth in latitude $72°$ approximately. The equatorial path at the same initial distance from the earth's centre is deflected through $17°$ and meets the earth at $59°$ from the central meridian.

The second path considered was one at an initial perpendicular distance $1.2a$ from the equatorial plane. This is more curved than the former one, the values calculated for R at the radial distances $1.4a$, $1.3a$, $1.2a$, $1.1a$, being $6.0a$, $3.8a$, $2.3a$, $1.3a$, diminishing

approximately to a on reaching the earth. The total deflection of the path is roughly $37°$, and it reaches the earth at a distance of $116°$ from the centre of the meridian, i.e. at $26°$ beyond the pole; it falls short of grazing incidence by only about $12°$, and it is very likely that this path is not far from the limiting path. It may be compared with the equatorial path which, though starting at the rather greater initial distance of $1.25a$ from the earth's centre, is deflected through $27°$ and reaches the earth at a longitude $83°$ from the central meridian.

The third path considered was the one at initial distance $1.5a$ from the equatorial plane. It was found that this path does not reach the earth, and, in fact, seems not to approach it within a distance $1.45a$, though it is deflected by a few degrees out of its original line. The corresponding equatorial path reaches the earth after a deflection through $82°$, in longitude $172°$.

General conclusions as to the motion of a neutral ionised stream.

(11)　Slightly different results might be got for other values of λ or of the initial density ρ_0 of the stream, but the above seem sufficient to show that the deflection of a strictly neutral ionised stream within the earth's magnetic field is not such as can parallel the observed facts about the incidence of aurorae. The part of the stream which gets round to the back of the earth is only a small fraction of the whole (this fraction being smaller near the poles than near the equator), and it impinges at a very oblique angle of incidence. Auroral streamers, on the contrary, seem to lie nearly along the earth's magnetic lines of force, which, in the polar regions, are not far from being normal to the earth. Moreover, it seems doubtful whether a neutral ionised stream could reach the earth, along the continuation of the central meridian round the back of the earth as viewed from the sun, at such great polar distances as are observed during great magnetic storms. There seems also to be no likelihood that the paths which are neither in the equatorial nor in the central meridian plane would show any markedly different characteristics, though their double curvature would make them far more difficult to calculate.

Thus, whether or not a strictly neutral ionised stream from the sun can explain the magnetic phenomena, it is unable to account for the auroral phenomena associated with magnetic storms. The facts about aurorae seem explicable only on the assumption that the stream from the sun has a resultant charge, though it may be ionised and contain charges of both signs.

The residual charge on the stream.

(12) So far it has been assumed that the solar stream would remain strictly neutral during its journey from the sun, but as Prof. Lindemann has indicated, there would be a slight loss of the more mobile negative charges (electrons) from the stream, until sufficient residual positive charge accumulated to prevent further escape of electrons. The order of magnitude of this charge, per unit of mass, was estimated by Prof. Lindemann at $2 \cdot 5 . 10^3$ E.S.U. The corresponding charge per unit of volume, taking $\rho = 8 . 10^{-24}$, is $2 . 10^{-20}$ ($\equiv E$, say). It is of interest to compare the additional force on the stream, due to this residual charge, with the polarisation forces which have alone been considered in this paper up to the present point.

Taking the most favourable case, when the directions of V and H are mutually perpendicular, the additional force per unit volume, in the equatorial plane, is EVH/c. The order of magnitude of the polarisation force in the equatorial plane is $(3/4\pi c^2) \, V^2 H^2 \, (6/r)$, the last factor arising on differentiating H^2 with respect to r. The ratio of the former to the latter force is $2\pi \, Ecn/9VH$, which in the present case, when $r = a$, is approximately 10^{-8}; this estimate might, however, be increased to 10^{-7}, on account of the increase in E/V as the stream approaches the earth, when V decreases and, on account of the increase in ρ, E is increased.

The ratio varies with distance in the ratio $(r/a)^6$, if the change in E/V is ignored. Thus, even when $r = 50a$, the additional force is still less than the polarisation force. Since the latter is itself negligible except within about $1\cdot5$ radii from the earth, it is clear that the neglect of the additional force, in the preceding investigation, is justifiable.

(13) It has been seen in § 3 that the geometry of the motion of a corpuscle on Prof. Störmer's theory depends on the value of e/mV, since the magnetic moment (M) of the earth is a definite constant ($8\cdot3 . 10^{25}$ C.G.S.). This parameter is independent of the *number* of corpuscles per unit volume; the intensity of aurorae must depend on this number, but the theory described in § 3 does not deal with this point. The difficulties confronting theories of aurorae and magnetic storms have arisen mainly through the excessive number of particles required to account for the energy and production of magnetic storms; it seems imperative for the success of any such theory that it should demand less in the way of charge density than past theories have done. At the same time there is at least some reason for thinking that the auroral theory of § 3 would be simplified if e/mV had a smaller value than those appropriate to cathode-rays, β particles or α particles. Prof. Lindemann's theory, though itself not tenable as it was propounded,

perhaps indicates the direction in which there is hope of success-fully modifying the theories which depend upon a charged stream. By supposing the stream to be composed to some extent of neutral ionised gas the ratio of the residual charge to the total mass (e/m) is diminished. If, as Prof. Lindemann suggests, a neutral ionised stream can produce the same magnetic effects as in my theory are attributed to a charged stream, the reconciliation of the theories of magnetic storms and of aurorae with each other, and with the facts, might thus be brought nearer. The discussion of magnetic storms is, however, beyond the scope of the present paper.

Summary.

(14)   The motion of a neutral ionised stream directed towards the earth is investigated, in connection with Prof. Lindemann's hypothesis that such a stream from the sun, of suitable dimensions, speed, and density, could produce terrestrial magnetic storms and aurorae. The stream would become polarised in the earth's magnetic field, and the slight redistribution of charge would cause the resultant electromagnetic forces on the two sets of positive and negative charges to differ slightly; the stream as a whole would be subject to a force having components both along and normal to the stream. The force varies approximately as the inverse seventh power of the distance from the earth's centre, and becomes appreciable only within a distance from the earth comparable with the earth's radius. The analysis of the motion is complicated, and resort to numerical computation has to be made, even after introducing simplifying assumptions. There are two families of plane paths, in the equatorial plane, and in the meridian plane through the sun. In both cases part of the stream is bent towards the earth and may impinge, nearly at grazing incidence, on the dark half of the earth. The bending is greatest when the density has a value of the same order as that considered by Prof. Lindemann; the speed of the stream does not affect the geometry of its motion. The bending, however, is not of such a character as suffices to account for auroral phenomena. Hence if the stream is to produce both aurorae and magnetic storms it cannot be neutral, and the resultant charge must exceed that which would arise in an originally neutral stream by mere leaking away of negative electrons.

Dougall's Theorem on Hypergeometric Functions. By C. T. PREECE. (Communicated by Dr G. N. WATSON.)

The theorem which is the subject of this paper may be expressed, in a form free from Gamma-Functions, by means of the equation

$$\frac{t^{(u)}\,\Pi\,(y+z+t)^{(u)}}{(\Sigma+t)^{(u)}\,\Pi\,(x+t)^{(u)}} = 1 - \frac{(t+1)\,u\,(\Sigma+u+2t-1)\,\Pi x}{1!\,(u+t)\,(\Sigma+u+t-1)\,\Pi\,(x+t)}$$
$$+ (t+3)\,\frac{tu_{(2)}\,(\Sigma+u+2t-1)^{(2)}\,\Pi x_{(2)}}{2!\,(u+t)^{(2)}\,(\Sigma+u+t-1)_{(2)}\,\Pi\,(x+t)^{(2)}}$$
$$- (t+5)\,\frac{t^{(2)}u_{(3)}\,(\Sigma+u+2t-1)^{(3)}\,\Pi x_{(3)}}{3!\,(u+t)^{(3)}\,(\Sigma+u+t-1)_{(3)}\,\Pi\,(x+t)^{(3)}}$$
$$\dots\dots\dots\dots\dots\dots,$$

in which
$$x^{(r)} = x\,(x+1)\,(x+2)\dots(x+r-1),$$
$$x_{(r)} = x\,(x-1)\,(x-2)\dots(x-r+1),$$
$$\Pi x^{(r)} = x^{(r)}y^{(r)}z^{(r)},\quad \Pi x_{(r)} = x_{(r)}y_{(r)}z_{(r)},$$
$$\Sigma = x+y+z,$$

and u is any positive integer.

Dougall's proof[*] is an ingenious application of the algebraical theorem that if a polynomial of degree n vanishes for more than n different values of the variable it vanishes identically. No proof could be more rigorous, but a theorem like Dougall's will probably stir in most mathematicians who encounter it some desire for a direct proof which builds up the series from the factorial product, and such a direct proof is given in this paper.

The identity

$$t\,(y+z+t)\,(z+x+t)\,(x+y+t)$$
$$= (x+y+z+t)\,(x+t)\,(y+t)\,(z+t) - (x+y+z+2t)\,xyz$$

establishes Dougall's Theorem for the case in which u is equal to 1. If we replace x, y, z by $x+a$, $y+a$, $z+a$, and $a+t$ by c, we obtain the identity

$$(c-a)\,\Pi\,(y+z+c+a)$$
$$= (x+y+z+c+2a)\,\Pi\,(x+c) - (x+y+z+2c+a)\,\Pi\,(x+a).$$
$$\dots\dots(\mathrm{A})$$

* *Proc. Edinburgh Math. Soc.*, vol. xxv (1907), pp. 114–132.

From this follow the special cases

$$t\Pi(y+z+t) = (\Sigma+t)\Pi(x+t) - (\Sigma+2t)\Pi x, \quad\ldots\ldots\ldots\ldots(1)$$

$$(t+2)\Pi(y+z+t) = (\Sigma+t-1)\Pi(x+t+1) - (\Sigma+2t+1)\Pi(x-1),$$
$$\ldots\ldots(2)$$

$$(t+1)\Pi(y+z+t+1) = (\Sigma+t+1)\Pi(x+t+1) - (\Sigma+2t+2)\Pi x. \ldots(1)'$$

To evaluate $\Pi(y+z+t)^{(2)}$ multiply the first term on the right of (1)′ by the expression on the right of (1) and the second term on the right of (1)′ by the expression on the right of (2). We obtain

$$t(t+1)\Pi(y+z+t)^{(2)}$$

$$= (\Sigma+t+1)\Pi(x+t+1)\left[(\Sigma+t)\Pi(x+t) - (\Sigma+2t)\Pi x\right]$$

$$-(\Sigma+2t+2)\Pi x[(\Sigma+t-1)\Pi(x+t+1) - (\Sigma+2t+1)\Pi(x-1)]t/(t+2).$$

Hence

$$\frac{t^{(2)}\Pi(y+z+t)^{(2)}}{(\Sigma+t)^{(2)}\Pi(x+t)^{(2)}}$$

$$= 1 - \frac{\Pi x}{(\Sigma+t)\Pi(x+t)}\left[\Sigma+2t+\frac{t(\Sigma+2t+2)(\Sigma+t-1)}{(t+2)(\Sigma+t+1)}\right]$$

$$+\frac{t(\Sigma+2t+1)^{(2)}\Pi x_{(2)}}{(t+2)(\Sigma+t+1)_{(2)}\Pi(x+t)^{(2)}}$$

$$= 1 - \frac{(t+1)\,2\,(\Sigma+2t+1)\Pi x}{(t+2)(\Sigma+t+1)\Pi(x+t)} + \frac{t(\Sigma+2t+1)^{(2)}\Pi x_{(2)}}{(t+2)(\Sigma+t+1)_{(2)}\Pi(x+t)^{(2)}},$$

and the theorem is proved when $u = 2$.

To establish the theorem generally assume that it is true for $u-1$, so that

$$\frac{(t+1)^{(u-1)}\Pi(y+z+t+1)^{(u-1)}}{(\Sigma+t+1)^{(u-1)}\Pi(x+t+1)^{(u-1)}}$$

$$= 1 - \frac{(t+2)(u-1)(\Sigma+u+2t)\Pi x}{1!\,(u+t)(\Sigma+u+t-1)\Pi(x+t+1)}$$

$$+ (t+4)\frac{(t+1)(u-1)_{(2)}(\Sigma+u+2t)^{(2)}\Pi x_{(2)}}{2!\,(u+t)^{(2)}(\Sigma+u+t-1)_{(2)}\Pi(x+t+1)^{(2)}} - \cdots$$

$$+ (-)^r(t-2r)\frac{(t+1)^{(r-1)}(u-1)_{(r)}(\Sigma+u+2t)^{(r)}\Pi x_{(r)}}{r!\,(u+t)^{(r)}(\Sigma+u+t-1)_{(r)}\Pi(x+t+1)^{(r)}}\cdots$$

$$\ldots\ldots(B)$$

For $\Pi(y+z+t)$ we use expressions given in the following set of equations:

$$t\Pi(y+z+t) = (\Sigma+t)\Pi(x+t) - (\Sigma+2t)\Pi x, \quad\ldots\ldots\ldots\ldots(1)$$

$$(t+2)\Pi(y+z+t) = (\Sigma+t-1)\Pi(x+t+1) - (\Sigma+2t+1)\Pi(x-1),\ldots(2)$$
$$\ldots\ldots\ldots\ldots;$$

$$(t + 2r - 2)\,\Pi\,(y + z + t)$$
$$= (\Sigma + t - r + 1)\,\Pi\,(x + t + r - 1) - (\Sigma + 2t + r - 1)\,\Pi\,(x - r + 1), \dots (r)$$
$$(t + 2r)\,\Pi\,(y + z + t)$$
$$= (\Sigma + t - r)\,\Pi\,(x + t + r) - (\Sigma + 2t + r)\,\Pi\,(x - r), \quad \dots\dots(r+1)$$
$$\dots\dots\dots\dots$$

all special cases of (A).

To obtain an expression for $\dfrac{t^{(u)}\,\Pi\,(y + z + t)^{(u)}}{(\Sigma + t)^{(u)}\,\Pi\,(x + t)^{(u)}}$ multiply successive terms of the expansion (B) by the successive expressions given by (1), (2), ..., (r), ..., for $\dfrac{t\,\Pi\,(y + z + t)}{(\Sigma + t)\,\Pi\,(x + t)}$. The coefficient of $(-)^r\,\Pi x_{(r)}$ in the resulting expansion is

$$\frac{t\,(u - 1)_{(r)}\,(\Sigma + t - r)\,\Pi\,(x + t + r)\,(t + 1)^{(r-1)}\,(\Sigma + u + 2t)^{(r)}}{(\Sigma + t)\,\Pi\,(x + t)\,r!\,(u + t)^{(r)}\,(\Sigma + u + t - 1)_{(r)}\,\Pi\,(x + t + 1)^{(r)}}$$

$$+ \frac{t\,(u - 1)_{(r-1)}\,(\Sigma + 2t + r - 1)\,(t + 1)^{(r-2)}\,(\Sigma + u + 2t)^{(r-1)}}{(\Sigma + t)\,\Pi\,(x + t)\,(r-1)!\,(u + t)^{(r-1)}\,(\Sigma + u + t - 1)_{(r-1)}\,\Pi\,(x + t + 1)^{(r-1)}}$$

$$= \frac{t^{(r-1)}\,(u - 1)_{r-1}\,(\Sigma + u + 2t)^{(r-1)}}{r!\,(u + t)^{(r)}\,(\Sigma + t)\,(\Sigma + u + t - 1)_{(r)}\,\Pi\,(x + t)^{(r)}}$$

$$\times \begin{bmatrix}(t + r - 1)\,(u - r)\,(\Sigma + t - r)\,(\Sigma + u + 2t + r - 1)\\ + r\,(\Sigma + 2t + r - 1)\,(u + t + r - 1)\,(\Sigma + u + t - r)\end{bmatrix}.$$

The term in square brackets reduces to

$$[(\Sigma + t)^{(2)} + (\Sigma + t)\,(u + t - 1)]\,u\,(t + 2r - 1).$$

Hence the term of the expansion we are seeking which contains $\Pi x_{(r)}$ is

$$\frac{(-)^r\,(t + 2r - 1)\,t^{(r-1)}\,u_{(r)}\,(\Sigma + u + 2t - 1)^{(r)}\,\Pi x_{(r)}}{r!\,(u + t)^{(r)}\,(\Sigma + u + t - 1)_{(r)}\,\Pi\,(x + t)^{(r)}},$$

and the theorem is proved.

If, instead of using equations (1), (2), ..., (r), ..., we use

$$(c - a)\,\Pi\,(y + z + t) = (\Sigma + c + 2a)\,\Pi\,(x + c) - (\Sigma + 2c + a)\,\Pi\,(x + a),$$
$$(c - a + 2)\,\Pi\,(y + z + t)$$
$$= (\Sigma + c + 2a - 1)\,\Pi\,(x + c + 1) - (\Sigma + 2c + a + 1)\,\Pi\,(x + a - 1),$$

etc., where $c + a = t$, we obtain for $\Pi\,(y + z + t)^{(u)}$ an expansion whose successive terms contain $\Pi\,(x + a)$, $\Pi\,(x + a)^{(2)}$, etc., in place of Πx, $\Pi x^{(2)}$, etc., but the extension of the theorem obtained in this way is only the trivial one which is more easily obtained by replacing x, y, z by $x + a, y + a, z + a$ and t by $-2a$.

A new expansion is obtained when we use the following equations:

$$\Pi (y + z + t) = \frac{(\Sigma + t) \, \Pi \, (x + t) - (\Sigma + 2t) \, \Pi x}{t},$$

$$\Pi (y + z + t + 1) = \frac{(\Sigma + t + 1) \, \Pi \, (x + t + 1) - (\Sigma + 2t + 2) \, \Pi x}{t + 1}$$

$$= \frac{(\Sigma + t + 2) \, \Pi \, (x + t) - (\Sigma + 2t + 1) \, \Pi \, (x + 1)}{t - 1},$$

$$\Pi (y + z + t + 2) = \frac{(\Sigma + t + 2) \, \Pi \, (x + t + 2) - (\Sigma + 2t + 4) \, \Pi x}{t + 2}$$

$$= \frac{(\Sigma + t + 3) \, \Pi \, (x + t + 1) - (\Sigma + 2t + 3) \, \Pi \, (x + 1)}{t}$$

$$= \frac{(\Sigma + t + 4) \, \Pi \, (x + t) - (\Sigma + 2t + 2) \, \Pi \, (x + 2)}{t - 2},$$

etc.

The first and second terms in the expression for $\Pi (y + z + t)$ are multiplied by the first and second expressions for $\Pi (y + z + t + 1)$ respectively, and a series is obtained for $\Pi (y + z + t)^{(2)}$ with terms containing 1, Πx, $\Pi x^{(2)}$. These three terms are multiplied respectively by the three expressions for $\Pi (y + z + t + 2)$ to give a series equal to the product $\Pi (y + z + t)^{(3)}$. The general result, established without difficulty by induction, is

$$\frac{t^{(u)} \, \Pi \, (y + z + t)^{(u)}}{(\Sigma + t)^{(u)} \, \Pi \, (x + t)^{(u)}} = 1 - \frac{(t + u - 2) \, u \, (\Sigma + u + 2t - 1) \, \Pi x}{(t - 1) \, (\Sigma + t) \, \Pi \, (x + u + t - 1)}$$

$$+ (t + u - 4) \frac{(t + 2) \, u_{(2)} \, (\Sigma + u + 2t - 1)_{(2)} \, \Pi x^{(2)}}{(t - 1)_{(2)} \, 2! \, (\Sigma + t)^{(2)} \, \Pi \, (x + u + t - 1)_{(2)}}$$

$$- (t + u - 6) \frac{(t + 2)_{(2)} \, u_{(3)} \, (\Sigma + u + 2t - 1)_{(3)} \, \Pi x^{(3)}}{(t - 2)_{(3)} \, 3! \, (\Sigma + t)^{(3)} \, \Pi \, (x + u + t - 1)_{(3)}}$$

$$+ \text{etc.}$$

On a Quintic Locus defined by five points in a plane. By WILLIAM L. MARR, M.A., D.Sc. (Aberdeen). (Communicated by Mr W. P. MILNE, Clare College.)

Consider the linear system of cubic curves, passing through the five given points A, B, C, D, E in a plane, viz.:

$$S \equiv \lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 + \lambda_4 S_4 + \lambda_5 S_5 = 0 \qquad \ldots\ldots(1).$$

The locus of a point moving so that the tangents at A, B, C, D, E to the cubic S meet in a point P is a quintic curve Γ. For, if $P \equiv (x', y', z')$, and we express the conditions that the polar conic of P with respect to the cubic curve S is a given conic, we obtain six conditions linear in both x', y', z' and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ separately.

Hence eliminating the λ's, we obtain a determinant of the sixth order, one of whose rows (or columns) does not contain x', y', z'. The curve Γ is therefore of the fifth degree.

Plainly the curve Γ passes through the 15 points of intersection of the joins of any two pairs of the given points. Let, for example, L be the point of intersection of AB and CD. Then LAB, LCD, LE may be considered a cubic curve of the system (1), whose tangents at A, B, C, D, E are concurrent. Hence L lies on Γ. Also Γ touches the conic through the five given points at the five given points themselves. For, isolating A, we can draw a unique cubic through A so as to touch AB, AC, AD, AE at B, C, D, E respectively. Plainly the tangent at A to this cubic passes through A. Hence A lies on Γ and similarly with regard to the other four given points.

Let now P be the point on Γ adjacent to A. It is evident from the fundamental property of Γ that a cubic curve touches PA at A as well as PB, PC, PD, PE at B, C, D, E respectively.

Proceeding to the limit, when P moves up to coincidence with A, we see that this cubic curve touches Γ at A. But the conic $ABCDE$ is the polar conic of A with respect to this cubic curve and hence touches this cubic at A. Thus Γ touches the conic $ABCDE$ at A, and similarly at B, C, D, E.

We have thus the following result: *Given five points in a plane, the locus of a point* P *moving so that the tangents at the five points to a cubic curve passing through them meet in* P *is a quintic curve touching the conic through the five points at these points, and passing through the intersections of the lines joining the given points in pairs.*

On the Possible Mechanics of the Hydrogen Atom. By Mr W. M. H. GREAVES, St John's College.

In the quantum theory of the hydrogen atom as developed by Bohr, it is assumed that there are certain orbits in which the electron may move without emitting radiation. Such orbits are called "stationary states." It is further assumed that the electron may undergo a transition from any stationary state to any other, and that the total energy lost in such a transition will be emitted in the form of monochromatic radiation, the frequency being given by Planck's formula $\delta E = h\nu$.

In the present paper it will be assumed that the electron, both in its motion in the stationary states and in its passage from one such state to another, behaves in accordance with the ordinary principles of classical dynamics. More precisely we shall assume that in addition to the electrostatic force e^2/r^2 towards the nucleus, the electron is subjected to radial and transverse forces S and T which for the moment may be regarded as arising from its interaction with a medium, and we shall seek to determine S and T so as to account for the phenomena postulated by the quantum theory.

It must be emphasised that these assumptions are made purely for the sake of investigating their consequences. We shall find that the analytical expressions obtained for S and T are rather complicated, and if the existence of such forces be regarded as physically unplausible, then it will follow that the assumptions must be rejected.

Denoting as usual the nuclear charge by e, the mass of the electron by m (the mass of the nucleus is large in comparison with m) and the distance between the electron and the nucleus by r, the electrostatic force on the electron is e^2/r^2 towards the nucleus and this gives rise to a central acceleration e^2/mr^2.

We are supposing that in addition to the electrostatic force e^2/r^2 which gives rise to this acceleration, the electron is subjected to the action of a radial force S tending to increase r and a force T perpendicular to the radius vector in the plane of the instantaneous orbit and acting in the direction of increasing azimuth.

At any time t let a, ϵ, and p be the respective values of the semi-major-axis, the eccentricity and the semi-latus-rectum of the instantaneous ellipse, and let ω, w, u, and ζ be the respective values of the mean angular motion, the true anomaly, the eccentric anomaly, and the mean anomaly in the instantaneous ellipse.

Then we have*

$$\frac{da}{dt} = \frac{2}{e^2} \frac{\omega a^3}{\sqrt{1-\epsilon^2}} \left[S\epsilon \sin w + T \frac{p}{r} \right]$$

$$\frac{d\epsilon}{dt} = \frac{1}{e^2} \omega a^2 \sqrt{1-\epsilon^2} \left[S \sin w + T (\cos u + \cos w) \right] \Bigg\} \quad \dots(1).$$

We also have the relations,

$$\omega^2 a^3 = e^2/m \quad \dots\dots\dots\dots\dots\dots\dots(2),$$

$$p = a (1 - \epsilon^2) \dots\dots\dots\dots\dots\dots\dots(3),$$

and

$$r = a (1 - \epsilon \cos u) = \frac{p}{1 + \epsilon \cos w} \quad \dots\dots\dots(4).$$

We then get from (1) after an easy reduction,

$$\frac{d\sqrt{p}}{dt} = \frac{1}{e^2} \omega a^{\frac{3}{2}} r T$$

$$\frac{d\sqrt{a}}{dt} = \frac{1}{e^2} \frac{\omega a^{\frac{5}{2}}}{\sqrt{1-\epsilon^2}} \left[\epsilon S \sin w + T \frac{p}{r} \right] \Bigg\} \quad \dots\dots\dots(5).$$

Now write

$$\left.\begin{array}{l} x = \sqrt{e^2 m a} \\ y = \sqrt{e^2 m p} \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots\dots(6),$$

and we get on using (2),

$$\frac{dx}{dt} = \frac{a}{\sqrt{1-\epsilon^2}} \left[\epsilon S \sin w + T \frac{p}{r} \right]$$

$$\frac{dy}{dt} = r T \Bigg\} \quad \dots\dots\dots\dots(7).$$

We shall assume that the disturbing forces S and T are small compared with the electrostatic force e^2/r^2. Then the general nature of the motion will be obtained by retaining only the non-periodic parts of the right-hand sides of equations (7), as in the astronomical theory of the secular perturbations of the major planets.

In order to proceed further it is necessary to make some more assumptions as regards S and T.

Let us write

$$rT = T_1$$

and

$$S = 2S_1 \sin \zeta \Bigg\} \quad \dots\dots\dots\dots\dots\dots(8),$$

ζ being the mean anomaly in the instantaneous ellipse.

We shall assume that T_1 is a function of y only and that S_1 is function of x only.

* See Tisserand, *Mécanique Céleste*, vol. I, ch. XXVII, p. 433.

Now $T\dfrac{p}{r} = T_1\dfrac{p}{a^2}\dfrac{a^2}{r^2}$, and when we expand in Fourier series in sines and cosines of multiples of ζ the non-periodic part of this is

$$T_1\frac{p}{a^2}\cdot\frac{1}{\sqrt{1-\epsilon^2}}\quad {}^* .$$

Again we have†,

$$\sin w = 2\sqrt{1-\epsilon^2}\sum_{i=1}^{i=\infty}\frac{dJ_i(i\epsilon)}{d\epsilon}\frac{\sin i\zeta}{i},$$

J_i denoting as usual a Bessel function of order i. And so the non-periodic part of $S\sin w$ is $2\sqrt{1-\epsilon^2}\,J_1'(\epsilon)\,S_1$.

Hence neglecting periodic terms the equations (7) become,

$$\left.\begin{aligned}\frac{dx}{dt} &= 2a\epsilon\,J_1'(\epsilon)\,S_1 + T_1\\[2mm]\frac{dy}{dt} &= T_1\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(9).$$

Now according to the quantum theory the stationary states of the system are given by $x = nh/2\pi$ and $y = kh/2\pi$ where h is Planck's constant and n and k are any two positive integers subject to the condition that $n \geqslant k$.

[In the simple theory of the hydrogen atom it is not usual to quantise y, as the energy of an electron in an orbit is a function of x only, and we should not be able to recognise the quantisation of y from the spectral lines. There seems however to be no very serious objection to the quantisation of y, if necessary, and we shall see that although it is not observable directly it will play an important part in our analysis.]

The assumptions of the present paper demand that in the stationary orbits the disturbing forces S and T (and therefore S_1 and T_1) should vanish. The simplest trial solution thus suggested would be to take:

$$S_1 = \lambda f(x)$$

and $$T_1 = \mu f(y),$$

where λ and μ are constants and $f(z)$ is a function of z with zeros at the points $z = h/2\pi$, $z = 2h/2\pi$, $z = 3h/2\pi$, etc.

This simple assumption would certainly satisfy the condition that the disturbing forces should vanish in the stationary states, but when we come to consider the question of transfers between stationary states it becomes untenable. For it would mean that some of the states are stable and others unstable, and in order to

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* Tisserand, *Mécanique Céleste*, vol. I, p. 242. † *Ibid.* p. 225.

produce a transition from a stable state to any other state, a definite external force would be necessary. Such an assumption seems to be foreign to the ideas of the quantum theory and in what follows we shall assume that a transition is started by an infinitesimal displacement of the electron from a stationary state. Again the ideas of the quantum theory demand that in such a transition the quantity x is diminished and this leads us to suppose that when the electron is not moving in one of its stationary orbits $\dfrac{dx}{dt}$ is negative. Further, it is necessary to postulate that one stationary state is stable, namely the state defined by $n = k = 1$ in which the electron is finally bound.

The simplest trial solution which satisfies these conditions is to take

$$\left. \begin{aligned} S_1 &= \lambda f(x) \\ T_1 &= \mu f(y) \end{aligned} \right\} \quad \dots\dots\dots\dots\dots\dots(10),$$

where λ and μ are positive constants and $f(z)$ is a function of z with a simple zero at $z = h/2\pi$ and double zeros at the points $z = 2h/2\pi$, $z = 3h/2\pi$, $z = 4h/2\pi$, etc., $f(z)$ being also negative when $\dfrac{2\pi z}{h} > 1$ and is not an integer and positive when $\dfrac{2\pi z}{h} < 1$. We shall assume, moreover, that $f(z)$ is a bounded function of z.

Adopting these assumptions let us proceed to discuss the possibilities of transitions between the stationary states. We are assuming such transitions to be started by an infinitesimal displacement of the electron from a stationary orbit.

We shall first of all examine the general behaviour of y. Suppose that initially $\alpha > y > \beta$ where α and β are any two consecutive double zeros of $f(y)$.

We may write

$$T_1 = -(\alpha - y)^2 (y - \beta)^2 \phi(y),$$

where $\phi(y)$ does not vanish for values of y between α and β and is everywhere positive in that interval.

The equation $\dfrac{dy}{dt} = T_1$ may be replaced* by the equations

$$\left. \begin{aligned} \frac{dy}{d\chi} &= -(\alpha - y)^2 (y - \beta)^2 \\ t &= \int_0^\chi \frac{d\chi}{\phi(y)} \end{aligned} \right\} \quad \dots\dots\dots\dots\dots(11).$$

and

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* Cf. Charlier, *Mechanik des Himmels*, Bd I, p. 89; or the original source, Weierstrass, *Ges. Werke*, Bd II, p. 1.

The first of these equations gives

$$-\frac{1}{(\alpha-\beta)^2}\left[\frac{1}{\alpha-y}-\frac{1}{y-\beta}\right]+\frac{2}{(\alpha-\beta)^3}\log\frac{\alpha-y}{y-\beta}=\chi+\epsilon,$$

where ϵ is an arbitrary constant.

We then see that as $\chi \to +\infty$, $y \to \beta$. And $\phi(y)$ being bounded and positive the second of equations (11) shows that as $t \to +\infty$, $\chi \to +\infty$ and conversely. It then follows that as $t \to +\infty$, $y \to \beta$. It can be easily verified in the same way that this result also holds when $\alpha = 2h/2\pi$, and $\beta = h/2\pi$.

We thus see that if we start in a stationary orbit for which $y = kh/2\pi$, and if, owing to a slight disturbance y becomes $\frac{kh}{2\pi} + \delta'$, then as t increases indefinitely, y will tend to $kh/2\pi$ or $(k-1)h/2\pi$ according as δ' is positive or negative.

Now suppose that we start in the stationary orbit given by $x = nh/2\pi$ and $y = kh/2\pi$, and that owing to a slight disturbance we have initially $x = \frac{nh}{2\pi} + \delta$, $y = \frac{kh}{2\pi} + \delta'$, where δ and δ' are positive. We have seen that $y \to \frac{kh}{2\pi}$ as $t \to \infty$; we proceed to enquire what happens to x.

Write
$$2a\epsilon J_1'(\epsilon)S_1 = -H\left(x-\frac{nh}{2\pi}\right)^2$$
$$T_1 \quad\quad = -K\left(y-\frac{kh}{2\pi}\right)^2 \quad\quad \ldots\ldots\ldots\ldots(12).$$

Then as long as x and y remain in the neighbourhood of the values $x = nh/2\pi$, $y = kh/2\pi$, H and K will be approximately constant (and positive), and the motion is given approximately by the equations

$$\frac{dx}{dt} = -H\left(x-\frac{nh}{2\pi}\right)^2 - K\left(y-\frac{kh}{2\pi}\right)^2$$
$$\frac{dy}{dt} = -K\left(y-\frac{kh}{2\pi}\right)^2 \quad\quad \ldots\ldots\ldots(13),$$

where H and K are to be regarded as positive constants.

[This approximation could be avoided by considering the upper and lower bounds of H and K in small intervals including the values $x = nh/2\pi$, $y = kh/2\pi$, thus obtaining bounds to the motion. Such a procedure would only complicate the analysis and would lead to the same results as those obtained by the method actually adopted.]

The initial conditions are $x = \frac{nh}{2\pi} + \delta$, $y = \frac{kh}{2\pi} + \delta'$ for $t = 0$.

The second of equations (13) gives

$$y - \frac{kh}{2\pi} = \frac{1}{Kt + 1/\delta'},$$

and we then get from the first of equations (13), writing x_1 for $x - \frac{nh}{2\pi}$:

$$\frac{dx_1}{dt} + Hx_1{}^2 + \frac{K}{(Kt + 1/\delta')^2} = 0.$$

Writing $Kt + 1/\delta' = \tau$ this becomes

$$\frac{dx_1}{d\tau} + \frac{H}{K} x_1{}^2 + \frac{1}{\tau^2} = 0,$$

and then putting $x_1 = v/\tau$ we get

$$r \frac{dv}{d\tau} + \frac{H}{K} v^2 - v + 1 = 0,$$

which gives
$$\int \frac{dv}{\dfrac{H}{K} v^2 - v + 1} + \log A\tau = 0 \quad \dots\dots\dots\dots(14),$$

where A is an arbitrary constant of integration.

Write
$$v - \frac{K}{2H} = v_1$$

and
$$\frac{K^2 - 4HK}{4H^2} = q^2.$$

We shall assume that $K > 4H$ so that q is real. This condition can always be satisfied by adjusting the constants λ and μ suitably. We shall take the positive value of q.

(14) then gives

$$\int \frac{dv_1}{v_1{}^2 - q^2} + \frac{H}{K} \log A\tau = 0 \quad \dots\dots\dots\dots\dots(14').$$

We now have to consider three cases:

Case (a):

Suppose $v_1 > q$ initially.
$(14')$ then gives on integration

$$\frac{1}{2q} \log \frac{v_1 - q}{v_1 + q} = -\frac{H}{K} \log A\tau,$$

and therefore
$$\frac{v_1 - q}{v_1 + q} = \left(\frac{1}{A\tau}\right)^{2qH/K} \quad \dots\dots\dots\dots\dots(15\,a).$$

Case (b):

Suppose that initially $-q < v_1 < q$.

We then get
$$\frac{1}{2q} \log \frac{q-v_1}{v_1+q} = -\frac{H}{K} \log A\tau$$

and so
$$\frac{q-v_1}{v_1+q} = \left(\frac{1}{A\tau}\right)^{2qH/K} \quad \ldots\ldots\ldots\ldots\ldots(15\,b).$$

Case (c):

Suppose that initially $v_1 < -q$.
In this case we have
$$\frac{1}{2q} \log \frac{q-v_1}{-q-v_1} = -\frac{H}{K} \log A\tau,$$

giving
$$\frac{q-v_1}{-q-v_1} = \left(\frac{1}{A\tau}\right)^{2qH/K} \quad \ldots\ldots\ldots\ldots\ldots(15\,c).$$

The constant A is to be determined by the initial conditions. Now we have initially $t = 0$ and so $\tau = 1/\delta'$ and is positive. It follows that in each of the three cases A is positive.

In case (a) we have initially $A\tau > 1$ and as τ increases indefinitely v_1 decreases steadily to the limit q.

In case (b) as τ increases indefinitely v_1 increases steadily to the limit q.

In each of these cases x_1 tends to zero as t increases to infinity.

But in case (c) we have initially $A\tau < 1$, and as τ increases to the value $1/A$, v_1 decreases to $-\infty$.

The above formulae then indicate that x_1, and therefore x, decreases to $-\infty$ as τ increases to the value $1/A$. But this does not mean that x really becomes infinite, as the analysis is only applicable to the neighbourhood of $x = nh/2\pi$. We have proved, however, that x passes through the value $nh/2\pi$, and it will then go on decreasing.

We thus see that x settles down into the state $x = nh/2\pi$ or passes through this state according as $v_1 \gtrless -q$ initially.

Now initially we have $\tau = 1/\delta'$,

and
$$\delta = x_1 = v/\tau = v\delta',$$

so that initially
$$v_1 = v - K/2H = \delta/\delta' - K/2H,$$

and the above condition becomes:
$$\frac{\delta}{\delta'} \gtrless \frac{K - \sqrt{K^2 - 4HK}}{2H} \quad \ldots\ldots\ldots\ldots\ldots(16).$$

We now see that if δ/δ' be sufficiently large x will settle down to the value $nh/2\pi$, but that if δ/δ' be sufficiently small x passes through this value and goes on decreasing.

In the latter case after a certain interval of time, x will have arrived in the neighbourhood of the value $x = (n - 1) h/2\pi$. Suppose that after a time t_1 the values of x and y are

$$x = \frac{(n - 1) h}{2\pi} + \epsilon \text{ and } y = \frac{kh}{2\pi} + \epsilon'.$$

Then the above analysis shows that x will settle down to the value $(n - 1) h/2\pi$ or pass through that value according as ϵ/ϵ' is sufficiently large or sufficiently small. But clearly as x approaches the value $(n - 1) h/2\pi$, the ratio ϵ/ϵ' begins by being very large. So that in this case (at any rate if δ' is sufficiently small) x will settle down to the value $(n - 1) h/2\pi$.

For the sake of brevity we shall denote the stationary state $x = nh/2\pi$, $y = kh/2\pi$ by an "n_k orbit." We have now proved that when δ and δ' are both positive and are both sufficiently small the electron will return to the n_k orbit after its small initial disturbance if δ/δ' is sufficiently large and will settle down into an $(n - 1)_k$ orbit if δ/δ' is sufficiently small. In the latter case a transition has taken place from the original n_k orbit to an $(n - 1)_k$ orbit.

When δ and δ' are not both positive the problem is more complicated, but it is possible to see from general considerations what kind of phenomena are to be expected.

Suppose for instance that δ and δ' are both negative. Then from the previous work we see that as t increases without limit, y will tend to the value $(k - 1) h/2\pi$, and that in the meanwhile x is steadily decreasing. Suppose that after a time t_1 we have $x = \frac{n'h}{2\pi} + \gamma$ and $y = \frac{(k - 1) h}{2\pi} + \gamma'$, where γ and γ' are both very small. Then from the foregoing we see that the electron will settle down into an $n'_{(k-1)}$ orbit or an $(n' - 1)_{(k-1)}$ orbit according as the ratio γ/γ' is sufficiently large or sufficiently small. It is also clear since the geometrical conditions demand that we must always have $x > y$, that in a transition of this kind the principal quantum number n cannot diminish by any number of units, but that the minimum value to which it can attain is $k - 1$.

Of course a general argument of this kind does not prove that a transition may occur in which the principal quantum number n decreases by any assigned number of units consistent with the geometrical conditions. So far the only kind of transition which we have definitely proved to be possible is a transition from an n_k orbit to an $(n - 1)_k$ orbit. To prove or disprove the more general result would be difficult, and we shall confine ourselves to proving the possibility of transitions in which the second quantum number k diminishes by one unit and the principal quantum number n diminishes by at least two units.

To prove this we first of all observe that during any transition

$\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are both negative and that $\dfrac{dx}{dt}$ is numerically greater than $\dfrac{dy}{dt}$. It follows that in any finite interval of time the total diminution of x is greater than the total diminution of y.

Now consider the case in which δ and δ' are both negative. We can obviously choose δ' so small that after some time t_1 we have $x = \dfrac{(n-1)\,h}{2\pi} + \epsilon$ and $y = \dfrac{kh}{2\pi} - \epsilon'$, where ϵ and ϵ' are both small. During the subsequent motion y will tend to the limit $(k-1)\,h/2\pi$ and as the total diminution of x exceeds the total diminution of y in any finite interval of time, it follows at once that x will pass through the value $(n-1)\,h/2\pi$, so that in this transition the principal quantum number n diminishes by more than unity.

We have obtained

$$S = 2S_1 \sin \zeta = 2\lambda f(x) \sin \zeta$$

and

$$T = T_1/r = \frac{\mu}{r} f(y),$$

where $f(z)$ is any function of z satisfying the conditions set forth above. The simplest form of $f(z)$ satisfying these conditions would be to take

$$f(z) = \frac{-\sin^2\left(\dfrac{2\pi^2}{h} z\right)}{z - h/2\pi}$$

and we should then have

$$\left. \begin{aligned}
S &= \frac{-2\lambda \sin^2\left(\dfrac{2\pi^2}{h} e \sqrt{ma}\right) \sin \zeta}{e\sqrt{ma} - h/2\pi} \\[2mm]
T &= -\frac{\mu}{r} \cdot \frac{\sin^2\left(\dfrac{2\pi^2}{h} e \sqrt{mp}\right)}{e\sqrt{mp} - h/2\pi}
\end{aligned} \right\} \quad \ldots\ldots\ldots\ldots(17).$$

In these formulae for the radial and transverse disturbing forces S and T, λ and μ are positive constants, e is the electrostatic charge of the electron and m its mass, r is the distance between the electron and the nucleus, h is Planck's constant, a and p are the semi-major-axis and semi-latus-rectum of the instantaneous elliptic orbit, and ζ is the mean anomaly of the electron in the instantaneous ellipse.

We have been concerned only with the behaviour of the electron and we have not attempted to develop any mechanical explanation of the assumption made in the quantum theory that the emitted

radiation is monochromatic, its frequency being given by the well-known formula $\delta E = h\nu$. If we assume that the disturbing forces S and T arise from the interaction of the electron with a medium, then the energy lost in a transition will be transmitted to the medium, and its subsequent behaviour will depend on the mechanics of the medium, a question which is beyond the scope of this paper.

An alternative hypothesis would be to suppose that the forces S and T arise from the interaction between the electron and the nucleus, in which case the energy lost in a transition would be transmitted to the nucleus in the first place and we should then be compelled to postulate some mechanism in the nucleus which would radiate the energy in the required way.

We have seen that in the above analysis the second quantum number k cannot change by more than unity in any transition. The same would have been true for the principal quantum number n if the term T_1 had been absent from the right-hand side of the equation (9) for $\dfrac{dx}{dt}$. In fact it is the variation of the eccentricity ϵ which is capable of, so to speak, carrying the major axis through one of its stationary values.

The expressions (17) for the radial and transverse disturbing forces are very complicated, and to many it will probably appear almost inconceivable that they should have a real physical existence. If we deny their existence it seems as if we shall have to deny the original assumption that the motion of the electron in the stationary states and in its transitions between these states is in accordance with the ordinary principles of Dynamics. It is true that the determination of S and T has not been unique, but the various assumptions introduced have been of such a nature throughout as to render it very probable that the solution (17) is one of the simplest possible solutions. It may be noted in this connection that the results obtained above would not be invalidated if the expressions (17) for the radial and transverse disturbing forces S and T were multiplied by any bounded functions of a and ϵ, which have no zeros and are always positive when $0 < \epsilon < 1$.

On Complexes of Cubic Curves in Ordinary Space. By C. G. F. JAMES, Trinity College.

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Up to the present, to the best of our knowledge, no detailed discussion has been given of any case of a complex, or system ∞^3, of space curves. In the present paper we consider a few cases of complexes of space cubics, which appear to be worth notice.

§ 1. For the present we do not restrict ourselves to curves of any particular type. A curve of the system will not in general pass through two arbitrary points, but there are ∞^5 pairs of points so connected. Let q be the number of curves which pass through such a pair in general. This number will be the most fundamental characteristic the complex possesses, but will be unity in most cases which naturally present themselves, and in particular with all systems with which we shall deal. It is, however, possible to devise classes of complexes in which this is not so. In future, then, we assume that two points, connected by a curve of the system, are in general connected by that one alone.

The curves through an arbitrary point of space P fill a surface T_P, on which the curves form a linear system, or pencil. P is in general a singular point on the surface. The character of this surface appears to determine the general nature of the complex. We therefore propose to define as *the order* of the complex the order of multiplicity of P on the surface; and as the *class*, the order of the surface itself. Thus *the class equals the order of the congruence of curves meeting a line**, while the difference between the class and the order determines the order of multiplicity of a line on the surface of those curves, which have it as chord†. Twice this number gives *the number of curves which touch a line* in general position. If no curve passes through an arbitrary point, the order is taken to be zero.

Those points through which ∞^2 curves pass are said to be *singular*. Unless all the curves of the complex lie on a fixed surface there is at most a curve of such points. They are divided into two types, according as the curves in question do not lie, or lie on a surface, the former being the general case. We may also have *fundamental points*, through which all the curves pass. These are at most finite in number.

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* Or the number of curves of the congruence through a point. In general it equals this order divided by q.

† In general equal to this number divided by q.

Singular points of a different type are given by points, at which the associated surface of curves through such has a higher singularity than in general. Thus, if the order is two we shall have a surface of points P for which T_P has a binode at P. This surface corresponds in a sense to the singular surface of a line complex*, but is not of the same importance. In the same way we may have *singular chords* of the complex, or lines which ∞^2 of the curves have as chord, or *fundamental chords*, which are chord to every curve of the system.

There are other systems of lines having special properties. Thus, in general, we have a complex formed by the quadrisecants of the curves; and similarly, in general, an arbitrary line will be trisecant to a finite number of curves, which will be triple curves on the surface, generated by those curves having the line as chord. These can, however, be set aside, as not arising in the case of cubic curves.

§ 2. We shall now confine our attention to systems of space cubics. We shall denote by a_x^n an homogeneous polynomial in $(x_1 \ldots x_4)$ of order n, a_x^1 being replaced by a_x. We shall employ the symbolic notation for polar forms, etc. Then the matrix equation

$$\left\| \begin{array}{ccc} a_x & b_x & c_x \\ a_x' & b_x' & c_x' \end{array} \right\| = 0$$

represents such a cubic curve. If we allow the elements to be simultaneously homogeneous functions of $\alpha_1 \ldots \alpha_4$, we thereby determine such a complex. We restrict ourselves to the case when the α enter linearly; and more particularly to the cases when the complex contains ∞^g linear congruences of the Stuyvaert Group†. This is the group of those congruences in which the elements are linear functions of three parameters, and which are themselves linear. We get the complexes in question by adding α_4 to one or more of the terms in which the α occur for such a congruence. The index g is at most equal to three, and attains this limit in those cases when any linear relation in (α) gives such a congruence. This occurs for the first two Stuyvaert types; and, in general, for these only ‡.

There is a similar group of complexes associated with the three row matrix for a cubic

$$\left\| \begin{array}{cccc} a_x & b_x & c_x & d_x \\ A_x & B_x & C_x & D_x \\ a & b & c & d \end{array} \right\| = 0,$$

* Jessop, *A Treatise on the Line Complex*, Cambridge, 1903, p. 89.

† Stuyvaert, "Sur les congruences de triangles, cubiques gauches...," *Crelle*, vol. 132, 1907, p. 216; and for later developments *Congruences de cubiques gauches*, Ghent, 1920.

‡ Stuyvaert, *loc. cit.* These two types are explained in §§ 7, 21 respectively. The remaining cases will not be used.

but, for this to be advantageous, it is necessary that the constants $a, \ldots, d$ shall all involve the parameters $\alpha_1 \ldots \alpha_4$.

§ 3. We may alternatively regard the matrix of our complex as representing a curve in (a), with the x as parameters. This gives in a space $S_{(a)}$ a complex of curves*, such that the points of any one curve represent the curves of the original complex through a point (x). There will be similarly systems of curves representing the cubics having as chord the single lines of $S_{(x)}$, etc. We shall suppose this representation of the complex on $S_{(a)}$ to be birational, since this is the case in all the systems we discuss. A large number of enumerative results may be obtained from this representation. Some of these are given by a consideration of the above complex in $S_{(a)}$, and of these we give a few examples in parallel column.

<table>
<tr><td>(a)-space</td><td>(x)-space</td></tr>
<tr><td>(a) Equation of the surface of curves through a fixed point (β).</td><td>Condition that two curves (a), (β) may have a point in common.</td></tr>
<tr><td>(b) Congruence of curves (x) meeting a line g.</td><td>Points (x) through which pass curves of the series ∞^1 given by assigning two linear relations in (a).</td></tr>
<tr><td>(c) Its order m.</td><td>Number of curves of this ∞^1, meeting a fixed curve of the complex.</td></tr>
<tr><td>(d) Its class n.</td><td>Number of curves of this ∞^1, through which pass two curves of a second ∞^1 of like nature.</td></tr>
<tr><td>(e) Number of curves meeting three lines.</td><td>Points on curves of three such series.</td></tr>
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These results may be interpreted arithmetically. They may also be duplicated by interchanging the spaces, and in this form the last four cases are presented in a more useful form.

§ 4. The first complex we shall consider is given by

$$(G_0) \qquad \begin{Vmatrix} a_x & b_x & c_x & d_x \\ a_x' & b_x' & c_x' & d_x' \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{Vmatrix} = 0 \qquad \ldots\ldots(1),$$

the last row being essentially a row of arbitrary linear forms. It contains ∞^3 linear congruences with a fundamental chord, and the same four fundamental points

$$\begin{Vmatrix} a_x & b_x & c_x & d_x \\ a_x' & b_x' & c_x' & d_x' \end{Vmatrix} = 0,$$

which are fundamental points of the complex. If the linear relation giving the congruence is $\mu_a = 0$, then the fundamental chord in question is

$$\mu_1 a_x + \ldots + \mu_4 d_x = \mu_1 a_x' + \ldots + \mu_4 d_x' = 0 \qquad \ldots\ldots(2).$$

* Cf. Stuyvaert, *loc. cit.* § 3.

Taking the four fundamental points as the vertices $A_1, \ldots, A_4$ of the tetrahedron of reference the matrix is reducible to

$$| \; x_i \quad \kappa_i x_i \quad \alpha_i \; | = 0, \qquad i = 1, \ldots 4 \qquad \ldots\ldots(3),$$

and *the complex of fundamental chords is the tetrahedral complex, π,* given by

$$\Sigma \, p_{12} p_{34} \, (\kappa_1 - \kappa_3)(\kappa_1 - \kappa_4)(\kappa_2 - \kappa_3)(\kappa_2 - \kappa_4) = 0 \ \ldots(4);$$

associated with the cross-ratio $J = \{\kappa_1 \kappa_2 \kappa_3 \kappa_4\}$. We can show

(1) *that the entire system of chords of the complex is itself a complex,* namely π,

(2) *that the cubics through a point (y) lie on the complex cone of (y) for π, and form a pencil on it, base points $A_1, \ldots, A_4$. Thus G_0 is of order two, and class two.*

They generate, in fact, the surface $T_y{}^2$,

$$| \; x_i \quad \kappa_i x_i \quad y_i \quad \kappa_i y_i \; | \equiv \Sigma \, x_1 x_2 y_3 y_4 \, (\kappa_1 - \kappa_2)(\kappa_3 - \kappa_4) = 0 \ \ldots(5).$$

From this the first result follows.

(3) *that G_0 and π are transformed into one another by the transformation*

$$x_i' = 1/x_i, \qquad i = 1, \ldots 4.$$

In fact G_0 becomes

$$| \; 1 \quad \kappa_i \quad \kappa_i x_i \; | = 0, \qquad i = 1, \ldots 4.$$

Incidentally we have deduced an alternative generation of the complex G_0, as the intersection of projective systems of quadrinodal cubic surfaces

$$\sum_1^4 \frac{\mu_i}{x_i} = \sum_1^4 \frac{\mu_i \kappa_i}{x_i} = 0 \qquad\qquad \ldots\ldots(6).$$

From (3) we deduce

(4) *that the cast of $(A_1 \ldots A_4)$ on a cubic of G_0 is constant.*

Thus the main interest of G_0 lies in its connection with π. This complex is thus a complex of singular chords of G_0, and has the maximum dimensionality such a system can have.

§ 5. A metrically special case of G_0 is given by

$$\| \; x - x' \quad y - y' \quad z - z' \; \| = 0 \qquad \ldots\ldots(7),$$
$$\| \; x/a^2 \quad y/b^2 \quad z/c^2 \; \|$$

namely the system of cubics which pass through the feet of the normals from the various points (x', y', z') of space to the quadric $\Sigma \, x^2/a^2 = 1$. The corresponding π is Reye's complex of axes*.

§ 6. *Degenerate curves of the complex.* Projecting, from a point (z) on a generator h_z of $T_y{}^2$, the system of curves through y, we obtain on a general plane a pencil of cubic curves having A as

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* Jessop, *Treatise on the Line Complex*, Cambridge, 1903, p. 125.

double base point, with one fixed tangent, and four simple base points, projection of $A_1 \ldots A_4$. Using this we see that:

The lines of degenerate conics form the four stars $(A_1) \ldots (A_4)$; *the associated conics four nets defined respectively by* A_1, A_2, A_3; *etc.* With $A_4 y$ is associated the conic $T_y^2(x) = x_4 = 0$, etc. In a complex of cubics there are, in general, ∞^1 composed of three lines, of which one (the cross line) meets the other two, the wing lines. In our case *there are six groups of such doubly degenerate cubics*, the cross line is formed by a line $A_i A_j$, the wing lines pass through the remaining fundamental points, and the nodes N_k, N_l are such that

$$\{A_i A_j N_k N_l\} \barwedge \{\kappa_i \kappa_j \kappa_k \kappa_l\}.$$

Each set of three concurrent edges of the tetrahedron $A_1 \ldots A_4$ forms such a cubic.

The congruence of cubics of G_0 *meeting a line* l is of the second order and class* zero. Its focal surface is the envelope of the T_y^2 for points on the line, and is therefore the Plücker Surface† of l for π. The congruence is linear for a line through a point such as A_4, the condition of meeting with l outside A_4 being a double one, since the cones for points of such a line have a common tangent plane along l.

For a line of π the congruence becomes the congruence of cubics having the line as chord.

§ 7. We pass on to the complex we shall consider in greatest detail, namely

$$(G_1) \qquad \left\| \begin{array}{ccc} \sum\limits_1^4 a_i a_{ix} & \sum\limits_1^4 a_i b_{ix} & \sum\limits_1^4 a_i c_{ix} \\ a_x{}' & b_x{}' & c_x{}' \end{array} \right\| = 0 \quad \ldots\ldots(8),$$

containing ∞^3 Stuyvaert congruences of Type I‡. These are given by assigning a relation $\mu_a = 0$, the typical singular $c_6{}^3$ being

$$\left\| \begin{array}{cccc} a_{ix} & b_{ix} & c_{ix} & \mu_i \\ a_x{}' & b_x{}' & c_x{}' & 0 \end{array} \right\| = 0, \qquad i = 1, \ldots 4 \quad \ldots\ldots(9).$$

The common fundamental point of the congruences

$$a_x{}' = b_x{}' = c_x{}' = 0$$

is a fundamental point F of the complex.

The cubics through a fixed point (y) lie on a cubic surface $T_y{}^3$ with a node at (y). The equation of $T_y{}^3$ and the tangent cone at (y) can be written down, but this will not be necessary for our purpose. We can similarly write down the cubic through (y) and (z), when such a cubic exists.

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* Number of cubics having a line as chord. † Jessop, *loc. cit.* p. 105.
‡ Stuyvaert, "Une congruence linéaire de cubiques gauches," *Bull. Acad. Roy. de Belge*, 1907, p. 470.

Thus G_1 *is of the second order, and third class.* Two cubics touch a given line, and the cubics having a line as chord cut on it an ordinary involution.

§ 8. *The associated complex* in the (α)-space (§ 3) is a quartic complex of lines, each of which represents the cubics through some point of $S_{(x)}$. Thus the cubics which meet one of their number (β) are represented by points of a quartic cone, the complex cone of (β) in $S_{(\alpha)}$.

The representation is indeterminate for points which satisfy

$$\| \ a_{1x} \quad a_{2x} \quad a_{3x} \quad a_{4x} \quad a_x{}' \ \| = 0 \qquad \ldots\ldots(10)$$

giving 10 *singular points* Q_i through which pass ∞^2 cubics, represented by points of a plane of the (α)-space. We shall refrain throughout from stating explicitly the enumerative results deducible from these and other representative constructs.

§ 9. It is important to observe that the tangent cone to $T_y{}^3$ at (y) passes, in general, neither through F nor through the Q_i. As an equation in (y) it gives the surface locus of points of contact of tangents from (x) to curves of the complex, $R_x{}^4 \equiv (x)^2, Q_i, F*$. Its tangent cone at (x) is the tangent cone of $T_x{}^3$. Further, if in the same original equation we write $y = \alpha + k_\beta$, determine the envelope with respect to k, and then regard (x) as fixed, the result can be expressed in terms of the line coordinates p_{ij} of $(\alpha\beta)$, and gives the line equation of $R_x{}^4$, of order 12 as it should be.

Corollary. One cubic of G_1 issues in every direction from a point Q_i.

§ 10. There is a surface $R^9 - Q_i{}^3, F^3$ of points for which $T_y{}^3$ has a binode at (y). On it there lies a curve of points, for which the binode has its edge contained in the cubic surface, and a finite number of points for which the edge is oscular†. These loci have special properties in connection with the distribution of cubics of G, but they are not apparently of great importance (cf. the succeeding paragraphs).

§ 11. *On the degenerate cubics of* G_1. We may represent the $T_y{}^3$ of an arbitrary point birationally on a plane‡ by projection from its node (y). We have six base points $H_1, H_2, H_3, G_1, G_2, G_3$ lying on a conic, to which correspond lines $h_1, \ldots, g_3$ on $T_y{}^3$, through (y). Our system of cubic curves on $T_y{}^3$ is represented by a family of conics, base points F_0, G_1, G_2, G_3, where F_0 is the projection of F.

* In this paper R^n, etc. denote surfaces of order n, etc. The notation $\equiv$ implies that the surface passes through (x) etc. with the multiplicity of the index.
† See Salmon, *Geometry of Three Dimensions*, 5th Edn., 1915, vol. II, § 522, p. 167, for the character of these singular binodes.
‡ Clebsch, "Die Geometrie auf den Flächen dritter Ordnung," *Crelle's Journal*, LXV, 1866.

So also the lines H_iH_j, G_iG_j, G_iH_j are images of as many lines which we shall call h_{ij}, g_{ij} and g_{ij}' respectively. These various groups of lines play essentially distinct rôles. In fact we may divide the whole 21 lines into the following sets:

(a) The three lines h_i *form part of degenerate cubics in which the line passes through* (y). The residual conics are represented by conics $\varpi_i^2 \equiv F_0$, G_1, G_2, G_3, H_i.

(b) The three lines g_{ij} *form part of degenerate cubics in which the conic*, represented by G_kF_0, *passes through* (y).

Hence the lines (a) and (b) describe the same congruence of the third order formed by lines of degenerate cubics of G_1.

(c) *The lines* g_i are lines which meet every cubic through (y) again. They describe a complex, since, on each, (y) is a selected point.

(d) *The nine lines* g_{ij}' are lines which meet every cubic through (y) once, and the three lines h_{ij} are chord to all cubics through (y), and each set describes a certain complex. This division breaks down in special cases, giving sets of lines common to the various complexes.

§ 12. We thus see that *the congruence of lines, and that of conics, of degenerate cubics are both of the third order*. The points Q are singular for each; while F is fundamental for the congruence of conics, since it lies, in general, on no line of T_y^3. In fact the conics associated with a line h_i, or g_{jk}, are the residual sections by the planes Fh_{jk}, Fg_i respectively.

F is indeed the vertex of a quartic cone of lines of degenerate cubics, but these are cross lines of doubly degenerate cubics, and do not all belong to the congruence. If we express that the line joining (x) (η) lies integrally on the quadrics given by two determinants from (8), and eliminate the (α) from the resulting equations we obtain the equation of the cone in question, (η) being taken as F. This equation is

$$\sum_{a,\,b,\,c} \mid b_{ix} \quad c_{ix} \quad a_{i\eta}b_x' - b_{i\eta}a_x' \quad a_{i\eta}c_x' - c_{i\eta}a_x' \mid = 0 \quad …(11).$$

This cone contains the lines FQ_i, and exhausts the cross lines of the cubics in question. The wing lines describe a scroll which I have not determined. There exist, in each case in finite number,

(a) curves whose wing lines coincide, and

(b) curves for which the two intersections coincide, and thus formed by three concurrent lines.

In exactly the same way we can show that the points Q_i are vertices of quartic cones of lines of the congruence of lines of degenerate cubics. These cones pass through Q_iF respectively, but not through the lines Q_iQ_j.

We may consider these double degenerations from the point of view of the specialities which occur on the T_y^3. If h_1 be a line for

which the corresponding conic degenerates, then F must be either on a line g_{i1}' (say g_{11}) or a line g_{ij} (say g_{23}). In the latter case, from the plane representation every cubic on the surface degenerates into a line through F (g_{23}) and a conic in a plane through F. This case occurs for at most a finite number of lines, which certainly includes the lines of the congruence through F. To all such lines then are associated ∞^1 conics to form cubics of G_1.

Taking, secondly, a line such as g_{23}, if the associated conic degenerates we must have either $g_1 F$ passing through g_2, with F either on g_2 or g_{12}, or $g_1 F$ passing through h_1, with F on h_1 or g_{11}'. Examining these in turn we find that a line of the quartic cone of cross lines is always a line g for points on it, and a line g_{ij}' for such other cubic surfaces as it lies on. These cases correspond in the plane representation to the falling of F_0 into a point G_i or on to a line $G_i H_j$.

It is possible to find the surface locus of double points on the simply degenerate cubics by expressing that two of the quadrics in (8) touch at (ξ), but the calculation is troublesome, and hardly appears worth recording. We will close this discussion by remarking that for a point on the singular surface (§ 10) and for the singular points of higher type, the pencil of cubics through it, and, in particular, the degenerate cubics, acquire specialities corresponding to specializations of the plane representation. The same is true for the 16 points for which T_y^3 acquires a node at F. It would be tedious to enumerate these in detail.

§ 13. We pass on to consider some loci connected with the lines of space, and leading up to the consideration of *singular lines in association with* G_1.

In the first place there are ∞^1 lines which meet a pair of arbitrary lines. Now the cubics through a point are represented by points of a line (in the a-space, § 8), and three such meet l. We see, then, that the cubics meeting l are represented by points of a cubic surface, which as locus of the lines representing the cubics through its single points must be ruled. *Its double directrix must therefore represent the cubics having l as chord* (we shall verify, in fact, that it is a line), and the simple directrix gives a sub-system of ∞^1 cubics which have a set of six other associated lines as chords (cf. § 15). The two singular generators correspond to the two cubics which touch l.

So also the cubics meeting two skew lines are represented by points on a 9-tic curve, of genus 18, *and* 27 *cubics meet three lines*[*] *in ar-*bitrary position. The surface of cubics meeting two lines is therefore an $R^{27} \equiv l^3,\ l^{13}$.

If the two lines meet, the representative curve is an 8-tic of

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[*] It is seen that the cubic surfaces in (d) have no common curves.

genus 14, and the curves fill an R^{24}. Remembering that a curve of order and rank R may degenerate into a set of lines with $\frac{1}{2}R$ intersections we see that the order of the surface filled by cubics bisecant to such a curve is*

$$M = 27 \binom{n}{2} - \tfrac{3}{2}R + 3n,$$

while they are represented by points on a curve of order

$$9 \binom{n}{2} - \tfrac{1}{2}R + n.$$

The surface has c_n as a $(3n-2)$-fold curve; while, if c_n be rational, $2(3n-2)$ curves meet it in coincident points.

§ 14. Consider the involution cut on l by cubics having it as chord. Its two united points give the cubics touching l. If these points coincide, we have, from the theory of degenerate involutions, a line for which all the cubics in question have one point in common. In fact *we have a line of the complex* (g), *and this point is the selected point on it* (§ 11). The equation of the complex is found by taking the line as joining (X) (Y) and expressing that (Y) lies on the tangent cone of T_ξ^3 at $(\xi) = (X + \lambda Y)$. This gives the points where a cubic touches the line, and the condition that its roots may coincide gives the sextic equation of the complex, when expressed in terms of $p_{ij} = X_i Y_j - X_j Y_i$.

§ 15. We pass on to determine *the surface filled by cubics, which have a line l as chord*, and to associated questions. A chord of the cubic (a) is given by

$$\left. \begin{array}{r} \Sigma\, a_i\,(la_{ix} + mb_{ix} + nc_{ix}) = 0 \\ la_x' + mb_x' + nc_x' = 0 \end{array} \right\} \qquad \ldots\ldots(12).$$

If this line is the same as $l_x = l_x' = 0$ we must have

$$l_i + \lambda l_i' = \underset{j}{\Sigma}\, a_j\,(la_{ji} + mb_{ji} + nc_{ji}) \qquad \ldots\ldots(13),$$

$$\sigma_1 l_i + \sigma_2 l_i' = la_i' + mb_i' + nc_i' \qquad \ldots\ldots(14),$$

for $i = 1, \ldots, 4$. Solving (14) we have, to a factor

$$l = |\, b_i'\, c_i'\, l_i\, l_i'\,|_1^4, \quad m = |\, c_i'\, a_i'\, l_i\, l_i'\,|_1^4, \quad n = |\, a_i'\, b_i'\, l_i\, l_i'\,|_1^4 \quad (15).$$

Substituting back in (13),

$$l_i + \lambda l_i' = \underset{j}{\Sigma} a_j \begin{vmatrix} a_{ji} & b_{ji} & c_{ji} & & \\ a_k' & b_k' & c_k' & l_k & l_k' \end{vmatrix}, \quad k = 1, \ldots, 4 \quad \ldots\ldots(16)$$

$$= \underset{j}{\Sigma} A_j{}^i\, a_j,$$

where $A_j{}^i \equiv \underset{r,s}{\Sigma} p_{rs} \{ \underset{a,b,c}{\Sigma} a_{ji}\,(b_u'c_v' - b_v'c_u') \}, \quad r \neq s \neq u \neq v \ldots(17).$

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* Assuming provisionally that the cubics having a line as chord fill a cubic surface (§ 15).

Here the p_{rs} denote the Plücker Coordinates $l_r l_s' - l_s l_r'$ of l.

Solving this system

$$\alpha_1 = |\ l_i + \lambda l_i' \quad A_2{}^i \quad A_3{}^i \quad A_4{}^i\ | \equiv \kappa_1 + \lambda \kappa_1', \text{ etc. }\ldots(18).$$

Hence the cubics having l as chord are represented by a line in the (α)-space, as we have already seen. This line may be any line. If, however, it belongs to the quartic complex of lines representing systems of cubics through the single points of space, then l lies on the $T_y{}^3$ of the corresponding (y). Now the representative line in (α) has its coordinates cubic functions of those of l (Eqn. 18). From this we might conclude that the sum of the orders of the complexes of lines g_i and h_{ij} is 12, but actually this is redundant. We shall see that inversely a line of the (α)-space determines six lines in the x-space.

To obtain finally the equation of our surface we substitute from (18) in

$$\Sigma\, a_j a_{jx} + w a_x' = 0 \qquad \ldots\ldots(8\,a),$$

and eliminate w and λ. We obtain a cubic surface

$$|\ \Sigma\, \kappa_j\, a_{jx} \quad \Sigma\, \kappa_j'\, a_{jr} \quad a_{r}'\ | = 0 \qquad \ldots\ldots(19).$$

The surfaces pass through l, F, and Q, the last conclusion being given by the fact that the cubics through such a point are represented by points of a plane. This cubic surface is, in general, of general type.

Deductions from the plane representation[*]. It is known that the cubic surface may be represented on a plane with six fundamental points $A_1 \ldots A_6$, not on the same conic, which represent as many lines of a Double-Six, the conjugate set being given by conics $A_2 \ldots A_6$, etc., and the remaining 15 lines by lines $A_i A_j$. Let us take the conic $\varpi_2 \equiv A_1 \ldots A_5$ to represent l. If F_0 is the image of F, then our cubic curves are represented either by the pencil of lines (F_0) or by that of quintics $F_0 A_1{}^2 \ldots A_6{}^2$, and it is easily seen that the latter is not possible.

The section of the cubic surface by lF consists of l and a conic, which can only form part of a degenerate cubic, for no cubic can meet it outside l and F. Hence *the congruence of conics of degenerate cubics is of class unity.* This conic is represented by $A_6 F_0$, and the completing line by A_6, so that it is opposite to l in the double-six. The remaining lines are distributed with respect to l as follows.

(a) $A_1 \ldots A_5$ representing lines forming part of degenerate cubics, of which the line and conic both meet l. The line a_1, of image A_1, is completed by the conic, represented by $A_1 F_0$, residual section by the plane $l_1 F$, l_1 being the line represented by $(A_2 \ldots A_6)$. This exhausts the degenerate cubics of G_1 on $L_l{}^3$.

* Clebsch, *loc. cit.* §§ 1, 2, etc.

(b) The lines l, l_i, forming half a double-six, form a group of six associated lines, chord to the same group of ∞^1 cubics. There are ∞^4 such sets of six lines.

(c) The lines represented by (A_iA_j) meet every cubic in question once.

§ 16. *Singular chords of* G_1. We pass on to consider under what conditions the methods of the last section are indeterminate. In the first place, if

$$\| \ l_i \quad l_i{}' \quad a_i \quad b_i \quad c_i \ \| = 0,$$

then l, m, n are not determined by (15). This equation expresses that F lies on l, so that all lines through F are chord to ∞^2 cubics, as can be seen *à priori*.

Secondly the (α) are infinite, but with definite ratios for lines of the quartic complex π^4 given by $| A_j{}^i | = 0$; so that, for a value of λ, and a line l, for which

$$\| \ l_i + \lambda l_i{}' \quad A_1{}^i \quad A_2{}^i \quad A_3{}^i \quad A_4{}^i \ \| = 0,$$

we have ∞^1 solutions for the (α). Thus *for each line* l *of* π^4 *we have a value* λ *associated with* ∞^1 *cubics*.

Thirdly, the lines for which

$$\| \ l_i \quad l_i{}' \quad A_1{}^i \quad A_2{}^i \quad A_3{}^i \quad A_4{}^i \ \| = 0 \qquad \ldots\ldots(20)$$

are chord to ∞^2 cubics of G_1, ∞^1 for each value of λ. Using the representation of the lines of S_3 on the quadric form in S_5 we can show that these lines form a scroll of order 40.

We can identify π^4 *with the complex of lines* h_{ij} *which are chord to* ∞^1 *cubics through some point outside themselves*. To see this, consider the (1, 1) correspondence between the cubics having l as chord, and the planes of the pencil (l), which is expressed by (16) and (17). The plane in question is the first plane in (12)*, and meets the cubic expressed by

$$\sum_j \alpha_j \, a_{jx} + w a_x{}' = 0, \text{ etc.,}$$

in the point $w = 0$ or

$$x_i = \left| \ \sum_{j=1}^{4} \alpha_j \, a_{j\sigma} \quad \sum_1^4 \alpha_j \, b_{j\sigma} \quad \sum_1^4 \alpha_j \, c_{j\sigma} \ \right| \quad \sigma \neq i, \ i = 1, \ldots 4 \quad \ldots(21)$$

$$\equiv \left| \ \sum_1^4 \kappa_j \, a_{j\sigma} + \lambda \sum_1^4 \kappa_j{}' \, a_{j\sigma} \ \ldots\ldots \ \right|.$$

This point describes a fixed cubic $\varpi_3 \equiv F\,(l)^2$, its points being in projective correspondence with these cubics†, each of which it meets once. The above identification is now immediate.

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* The second being lF.

† In the representative plane we have a range (g) perspective to the pencil (F_0).

§ 17. By a slightly different procedure we may obtain a simpler form for the equation of $L_l{}^3$, but one which is not so useful in other connections. Returning to the equations (12) let (y) (z) be points on this line, (x) a point on the cubic. Solving

$$l : m : n = \left\| \begin{array}{ccc} a_y' & b_y' & c_y' \\ a_z' & b_z' & c_z' \end{array} \right\|.$$

Substituting in the two equations, which arise by expressing that (y) (z) lie on the first plane in (12), we may eliminate (α) and w between these and (8 a) of § 15, obtaining finally*

$$\left| \begin{array}{ccc} a_{ix} & |\; a_{iy}\, a_y'\, a_z' \;| & |\; a_{iz}\, a_y'\, a_z' \;| \\ a_x & 0 & 0 \end{array} \right| = 0, \quad i = 1, \dots 4 \quad (22).$$

This equation may be expressed in terms of p_{ij}. The solution is indeterminate for lines through F, and lines satisfying

$$\left\| \begin{array}{c} |\; a_{iy}\, a_y'\, a_z' \;| \\ |\; a_{iz}\, a_y'\, a_z' \;| \end{array} \right\| = 0, \qquad i = 1, \dots 4 \qquad \dots\dots(23),$$

giving the scroll of singular chords again.

§ 18. *The congruence* Γ_l *of cubics of* G_1 *meeting a line* l is of the third order and class. l is a singular line of the third order, the cubics through any point of it filling a cubic surface. The point F is fundamental, and the 10 points Q_i singular. It acquires another singular point if l belongs to π^1. The congruence is represented, as we have seen, by a cubic scroll in the α-space, and its simple directrix gives a set of six associated singular chords of Γ_l, all chord to the same ∞^1 curves. l is included in a second set of six such lines. To obtain the focal surface we substitute $y = \alpha - k\beta$ in the equation of $T_y{}^3$, thus obtaining the envelope of all these $T_y{}^3$ for points on l. We find an $R^{12} \equiv F^4 Q_i{}^4 (l)^4$, touched seven times by the curves of Γ_l.

§ 19. *The cubics which touch a plane* ϖ are represented by points on the focal surface of the congruence of lines, representing the cubics through the single points of ϖ. It must be met in six points by a line, this being the number of coincidences† of the linear series $g_3{}^1$ cut on the section of $T_y{}^3$ and ϖ by cubics through (y), or the number of cubics through (y) which touch ϖ. Hence the surface is of the sixth order. This may be verified directly. Many enumerative results follow. Thus the congruence of cubics touching ϖ is of the sixth order and class, and six pass through two singular points $Q_i Q_j$, etc.

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* The remaining rows of the inner determinants are given by writing b and c for a.

† Severi, *Lezione di Geometria Algebrica*, § 68, p. 234.

Two cubics touch ϖ at a given point on it, two cubics through a point P on it touch it elsewhere. These facts are shown by a correspondence argument on the section of ϖ by $T_y{}^3$.

§ 20. A special case of G_1 deserves mention on account of special features in the distribution of the singular points. It is given by

$$\left\| \begin{array}{ccc} a_1\,a_x + a_4\,a_x{}'' & a_2\,b_x + a_4\,b_x{}'' & a_3\,c_x + a_4\,c_x{}'' \\ a_x{}' & b_x{}' & c_x{}' \end{array} \right\| = 0 \qquad \ldots\ldots(24).$$

The surface $T_y{}^3$ is given by

$$\mid\ a_x{}''\,a_y - a_y{}''\,a_x \quad a_x{}'\,a_y \quad a_y{}'\,a_x\ \mid = 0,$$

and the singular points fall into the following sets:

(1) A single point Q_0: $\qquad\qquad a_y = b_y = c_y = 0,$

(2) Three pairs Q_{11}, Q_{12}, etc.: $\qquad a_y = b_y = a_y{}'\,b_y{}'' - b_y{}'\,a_y{}'' = 0$, etc.

(3) Three single points Q_1, etc.: $\qquad a_y = a_y{}' = a_y{}'' = 0$, etc.

(4) Three triads $Q_{11}{}'$, $Q_{12}{}'$, $Q_{13}{}'$, etc.: $\quad a_y = \left\| \begin{array}{ccc} a_y{}' & b_y{}' & c_y{}' \\ a_y{}'' & b_y{}'' & c_y{}'' \end{array} \right\| = 0$, etc.

Of these the first ten correspond to the singular points in the general case, and they only lie on the $T_y{}^3$ of an arbitrary point (y). The remaining nine are of the second type (§ 1). Thus the cubics through each point $Q_{1i}{}'$ lie on

$$b_y{}'\,c_y{}'' - b_y{}''\,c_y{}' = 0.$$

The cubic curve

$$\left| \begin{array}{ccc} a_x{}' & b_x{}' & c_x{}' \\ a_x{}'' & b_x{}'' & c_x{}'' \end{array} \right| = 0 \qquad\qquad \ldots\ldots(25)$$

also plays a special part. It lies in fact on the $T_y{}^3$ of each of its points. Hence a cubic of the system passes through any pair of its points, so that it is bisecant to ∞^2 curves.

§ 21. The last complex we shall consider is that represented by

$$(G_2) \qquad \left\| \begin{array}{ccc} \sum\limits_1^4 a_i\,a_{ix} & \sum\limits_1^4 a_i\,b_{ix} & A_x \\ \sum\limits_1^4 a_i\,a_{ix}{}' & \sum\limits_1^4 a_i\,b_{ix}{}' & A_x{}' \end{array} \right\| = 0 \qquad \ldots\ldots(26),$$

containing ∞^3 Stuyvaert congruences[*] of Type II, with a common fundamental chord l, or

$$A_x = A_x{}' = 0,$$

which is a fundamental chord of the complex. The cubics of G_2 are given by the intersection of projective systems ∞^4 of quadrics, having a common base line:

$$\sum\limits_1^4 a_i\,(a_{ix}\,A_x{}' - a_{ix}{}'\,A_x) = 0, \quad \sum\limits_1^4 a_i\,(b_{ix}\,A_x{}' - b_{ix}{}'\,A_x) = 0 \ (27).$$

The surface of cubics through a point (y) is a quartic surface

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[*] "Deuxième Congruence linéaire de cubiques gauches," *Rend. Circ. Mat. Palermo*, t. xxvi, 1907.

$T_y^4 \equiv (y)^2$, l^2, $16Q_i$, so that the complex *is of the second order and fourth class.* The 16 singular points* Q_i are:

$$\left\| \begin{array}{l} a_{ix} A_x' - a_{ix}' A_x \\ b_{ix} A_x' - b_{ix}' A_x \end{array} \right\| = 0, \quad i = 1, \ldots 4,$$

through each such point, as through every point of l, ∞^2 cubics of G_2 pass. The complex is represented on a space $S_{(a)}$ without singularities. Those cubics through a point are represented by points of a line, and those through a singular point Q by points of a plane. To obtain the representative surface for those through a point (τ) of l, we have to express that the surfaces (27) touch at (τ). This gives the quadric surface,

$$\left| \begin{array}{ll} \Sigma a_i a_{i\tau} & \Sigma a_i a_{i\tau}' \\ \Sigma a_i b_{i\tau} & \Sigma a_i b_{i\tau}' \end{array} \right| = 0 \qquad \ldots\ldots(28).$$

§ 22. The quartic surface T_y^4 can be represented on a plane†, in such a way that the plane sections are represented by quartic curves $\varpi_4 = A^2 B B' C_1 \ldots C_6$, where B' is a point consecutive to B along a straight line. The nodal line has as image a curve

$$\rho_3' \equiv A B B' C_1 \ldots C_6,$$

and the node y the base point B. Our family of cubics, passing through (y), and meeting l twice, can only be represented by the system of cubics‡ of base $A^2 B C_1 C_2 C_3 C_4$. We can again apply this to the study of the degenerate cubics. Two lines pass through y, represented respectively by BA. and B'. Both form part of degenerate cubics, the residual conics being represented by a conic $\varpi_2 \equiv A C_1 C_2 C_3 C_4$, and a cubic $\varpi_3 \equiv A^2 B B' C_1 C_2 C_3 C_4$. The lines represented by C_5, C_6, and by $A C_i$ ($i = 1, \ldots 4$), also form part of degenerate conics, but for these the conic passes through (y). The latter are represented by cubics $A^2 B C_1 C_2 C_3 C_4 C_5$, $A^2 B C_1 C_2 C_3 C_4 C_6$, and conics $A B C_2 C_3 C_4$, $A B C_1 C_3 C_4$, etc., respectively.

Hence *the congruence of lines of degenerate cubics is of the second order and sixth class.* The class is deduced from the fact that each point of l is the vertex of a sextic cone $C^6 \equiv (l)^4$, Q_i of lines of the congruence. Its equation is obtained, as for (11), by taking (a) on l, and expressing that the line (a) (x) lies on the surfaces (27), afterwards eliminating the (a). Writing $a = \xi + k\eta$ in this equation, and taking its envelope with respect to k (in which it is quadratic) we obtain *the focal surface of the congruence,* a surface $F^{12} \equiv (l)^{10}$, $16Q_i^2$.

Thus *the lines in the planes through l envelope conics. The points Q are vertices of pencils of lines of the congruence, and must lie by*

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* Their number is determined by formulae in my paper "On the intersection of Constructs...," *Proc. Cambridge Phil. Soc.* vol. XXI, 1923.

† Jessop, *Quartic Surfaces,* Cambridge, 1916, chap. v, p. 119.

‡ Jessop, *op. cit.* Type V, p. 124.

pairs in planes through l. The congruence may be considered completely known.

The congruence of conics of degenerate cubics is of the sixth order.

§ 23. *The cubics of G_2 which have a line g as chord* are represented by points of a cubic curve in the (α)-space, as we proceed to show. If we express that $\xi + k\eta$ lies integrally on the surfaces in (27), we get equations which solved for (α) give them as cubic functions of k. If now g, or (ξ) (η), does not meet l this gives the cubics of G_2 in question. For g lies on a quadric through such a cubic and l, and must belong to the same regulus as l.

Hence also these cubics fill a surface $\kappa^{12} \equiv (l)^3, (g)^2, 16Q_i$. If g meets l one of the equations obtained is evanescent, when (ξ) is chosen to be on l. The cubics meeting such a line are represented by points on a quadric.

§ 24. *The congruence of cubics of G_2 through a point (τ) of l* is of order 2 and class 3. Its focal surface is given by the condition that (27), regarded as a line in $S_{(\alpha)}$, touches the quadric (28), and is therefore an $R^8 \equiv (l)^4, 16Q_i^2$.

The cubics of G_2 meeting a line g are represented by points of a quartic surface, which must be ruled, having as double curve the cubic of § 23. The congruence of curves in question is of the fourth order, and twelfth class. Its focal surface, given by the envelope of T_y^4, as y describes g, is an $R^{24} \equiv (l)^{12}, (g)^6, 16Q_i^2$.

The congruence of cubics which touch a plane is, exactly as for G_1, of order 8 and class 24, its curves being represented by points of the focal surface of the congruence of lines, representing the systems of curves through the various points of the plane.

A large number of other results are deducible from the representation in $S_{(\alpha)}$, but we shall not enumerate them in detail.

§ 25. *Conclusion.* The complexes G_0, G_1. G_2 are all of the second order, and respectively of class 2, 3, 4. They are, if we confine ourselves to the case of matrices whose elements are general (of their order) in the α, the only cases in which the cubics through a point are represented by a line in the α-space. It will be observed that the cubics which meet a line are represented by points on the three simplest ruled surfaces with a single nodal curve at most, namely, a quadric, a ruled cubic, and a ruled quartic with a nodal space cubic, respectively. In other words they are the only cases of such complexes in which the representation on the (α)-space has no singular elements. The next simplest case is that in which we have a singular line, to each point of which correspond ∞^1 curves, and cases of which are constructed without difficulty.

On Some Approximate Numerical Applications of Bohr's Theory of Spectra. By D. R. HARTREE.

[*Read* 21 May 1923.]

§ 1. *Introduction.*

In the past two years a theory of spectra and atomic constitution has been developed in considerable detail by Bohr[*] in a more or less qualitative form. This theory is accepted completely in the present paper, which deals with an attempt to make a more quantitative application of it to the investigation of the spectra and structure of certain atoms. The general idea of this attempt is the determination of an electric field such that the orbits specified by the application of the quantum conditions to the motion of an electron in this field shall have the energies assigned to the different terms in the optical and X-ray spectra. The dimensions of the orbits and variation of time along them can then be calculated.

For various reasons a type of field is assumed much simpler than the actual atomic field must be, and exact agreement between all calculated and observed terms is not to be expected and cannot in fact be obtained, but good enough agreement is obtained to make the quantitative results interesting; and both for the dimensions of the orbits and for the field they probably form a fairly good first approximation. The orbits of the electrons normally present in the atom having been calculated, the field due to them could be determined and compared with the field deduced from the spectral terms. This final stage of the work depends on the orientation of the orbits and possibly on the relative phases of the electrons in them, and in the present paper is only considered very briefly.

From the calculated dimensions of and variation of time along the orbits, the X-ray scattering by an atom can be calculated on certain assumptions, and the results compared with Bragg's experimental work[†] on the subject. Some results for sodium are quoted here; more detailed consideration is deferred for the present.

Work somewhat similar to that described in the present paper has been published by Fues in three recent papers[‡]. The work here considered was undertaken independently; it is in some ways more general and has led to several further points of interest.

[*] N. Bohr, *Zeit. für Phys.* 9, p. 1 (1922); *The Theory of Spectra and Atomic Constitution* (Camb. Univ. Press, 1922). See also N. Bohr and D. Coster, *Zeit. für Phys.* 12, p. 342 (1923).

[†] W. L. Bragg, R. W. James and C. H. Bosanquet, *Phil. Mag.* 41, p. 309; 42, p. 1 (1921); 43, p. 439 (1922).

[‡] E. Fues, *Zeit. für Phys.* 11, p. 369; 12, p. 1 (1922); 13, p. 211 (1923). It is possible that similar calculations have been made by Bohr himself, but if so they have not to my knowledge been published.

§ 2. *General Theory.*

We will consider at present only atoms which in the normal state have one electron only in the most lightly bound type of orbit present, such as neutral atoms of the first group of the periodic table and singly ionised atoms of the second group. The optical spectra of such atoms are conveniently called one-electron spectra.

It will be convenient to speak of this electron as the 'outer' or 'valency' electron, and the others present in more closely bound orbits as the 'inner' or 'X-ray' electrons; also the inner electrons and the nucleus together are conveniently referred to as the 'kernel' and the surface and radius of the smallest sphere enclosing all the inner orbits are taken as the 'boundary' and 'radius' of the atom (strictly of the ion).

The possible orbits of the outer and inner electrons, determined by the application of the quantum conditions to their motion in the atomic field, correspond to the different terms of the optical and X-ray spectra respectively. If the field of force is given and the equations of motion are such that we know how to apply the quantum conditions, the terms can be calculated; in this paper a solution of the opposite problem is attempted, namely, the determination of the field of force from the observed term values.

For a neutral atom the field may be thought of as a perturbing field superposed on the field of an H nucleus, and it is convenient to speak of this superposed field as the 'added field,' but it obviously cannot be assumed that the added field is small compared to the original field, even for the valency electron, as on Bohr's theory this electron does not stay outside the kernel the whole time, but may penetrate right into it. It is assumed in this paper that at a given point the field is the same for an X-ray electron and a valency electron; consideration of this is deferred till § 4.

For a singly ionised atom the field may be considered similarly as that of a He nucleus with an 'added field' superposed.

Both for simplicity and in view of the lack of detailed knowledge of the arrangement of the electrons in the atom, the potential of the field is assumed to be a function of the distance r from the nucleus only. That is to say, the field is assumed to be radial, and each orbit is then plane.

Taking polar coordinates r, θ in the plane of the orbit, the variables in the Hamilton-Jacobi equation separate, and the solution is

$$S = \alpha_2 \theta + \int [-2m\alpha_1 + 2me^2 V - \alpha_2{}^2 r^{-2}]^{\frac{1}{2}} dr \quad ...(2 \cdot 0),$$

where

$-\alpha_1 =$ negative energy of the orbit;

$\quad \alpha_2 =$ angular momentum about an axis perpendicular to plane of orbit (α_2 is constant in a radial field);

eV = potential of the field, a function of r only;

m, e = mass and charge of the electron.

The variables being separable, there is no doubt of the form in which to apply the quantum conditions, and they give

$$2\pi\alpha_2 = kh \qquad \ldots\ldots(2\cdot1),$$

$$\oint \left[- 2m\alpha_1 + 2me^2 V - \alpha_2{}^2 r^{-2}\right]^{\frac{1}{2}} dr = n'h = (n - k) h \ldots(2\cdot2),$$

where n', k, and n are integers, usually called the radial, angular, and total or principal quantum numbers. The orbit for which n and k have specified values is usually referred to as the n_k orbit (for circular orbits $n = k$).

The field at any distance r, being radial, can be thought of as due to a charge of $+ Ze$ at the nucleus; this quantity Z is introduced for various reasons mentioned later (§ 6), it will be called the *effective nuclear charge**. If Ke is the net charge on the kernel (K is one greater than the degree of ionisation of the atom), then Z obviously tends to K for large r, and to the atomic number N for small r. It is convenient to speak of $Z - K$ as the *added charge*, as it is the charge to which the added field is due.

Z is connected to V by the relations

$$\frac{\partial V}{\partial r} = \frac{Z}{r^2}, \quad V = \int^\infty \frac{Z}{r^2} dr = \int_0 Z d\left(\frac{1}{r}\right) \quad \ldots\ldots(2\cdot3).$$

On substituting $(2\cdot1)$ in $(2\cdot2)$ and dividing through by h we obtain the expression

$$(n - k) = n' = \oint \left[- \frac{2m}{h^2} \alpha_1 + \frac{2me^2}{h^2} V - \frac{k^2}{4\pi^2 r^2}\right]^{\frac{1}{2}} dr \quad (2\cdot4).$$

From this formula, if V is given, the energy of the orbit of given n and k can be found, and the wave number of the corresponding series term is then given by the usual relation

$$\nu = \alpha_1/hc \qquad \ldots\ldots(2\cdot5).$$

A significant as well as more convenient form for the integral is obtained if we use the new variable

$$\rho = r/a \qquad \ldots\ldots(2\cdot6),$$

where a is the radius of the 1_1 hydrogen orbit and has the value

$$a = h^2/4\pi^2 me^2 = 0\cdot532 \times 10^{-8} \text{ cm.}$$

Then, using $(2\cdot5)$ and the value for the Rydberg number,

$$R = 2\pi^2 me^4/ch^3 = 109737 \text{ cm.}^{-1},$$

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* See § 4, p. 630, footnote *.

the expression (2·4) for the radial quantum integral can be reduced to*

$$n - k = n' = \frac{1}{2\pi} \oint \left[-\frac{\nu}{R} + 2v - \frac{k^2}{\rho^2} \right]^{\frac{1}{2}} d\rho \quad \ldots\ldots(2·7),$$

where, corresponding to (2·3),

$$v = \int_\rho^\infty \frac{Z}{\rho^2} d\rho = \int_0 Z d\left(\frac{1}{\rho}\right) \quad \ldots\ldots(2·8).$$

The interest of formulae (2·7), (2·8) is that all electronic constants e, m, h have disappeared, and if Z is given as a function of ρ, i.e. the structure of the atom given on the scale of the hydrogen atom, the spectrum is determined on the scale of the hydrogen spectrum.

It should be emphasised that formula (2·7) holds whatever the degree of ionisation of the atom considered; for a singly ionised atom the first term is still $-\nu/R$, not $-\nu/4R$.

The problem is then the determination of Z as a function of ρ such that for given n and k the values of ν/R calculated from (2·7) and (2·8) become as nearly as possible those of the terms of the optical and X-ray spectra; or such that when the actual values of ν/R are inserted in (2·7) with the appropriate values of k, the values of n obtained by evaluation of the integral shall be as nearly as possible the whole numbers assigned to the terms by Bohr's theory.

If Z is left as an empirical function of ρ specified by a graph or table, the integration has to be carried out numerically; this is not difficult if the work is suitably arranged, and it gives valuable extra freedom compared to Fues' method† which consists essentially in dividing up the range of integration into sections, and making V quadratic in r^{-1} in each range, in which special case the integral can be evaluated in terms of known functions.

When the change of mass with velocity is taken into account, the radial quantum integral becomes

$$n - k = \frac{1}{2\pi} \oint \left[-\frac{\nu}{R} + 2v - \frac{k^2}{\rho^2} + b\left(2v - \frac{\nu}{R}\right)^2 \right]^{\frac{1}{2}} d\rho \quad (2·9),$$

where

$$b = \frac{1}{4}\left(\frac{2\pi e^2}{ch}\right)^2 = \frac{1}{75180}.$$

In this form the relativity correction is exact and quite easy to take into account; it has been applied where appreciable.

The actual form of the orbit and course of time along it can be obtained in the usual way from the solution (2·0) of the Hamilton-

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* The finite mass of the nucleus has been neglected. To take it into account, R should presumably have its appropriate value for each element, but except for H, He and possibly Li the difference is not appreciable.

† E. Fues, *loc. cit.* (second and third papers).

Jacobi equation. In terms of ρ and without the relativity correction the relations are

$$\theta - \beta_2 = k \int \rho^{-2} \left[-\frac{\nu}{R} + 2v - \frac{k^2}{\rho^2} \right]^{-\frac{1}{2}} d\rho \quad ......(2\cdot91),$$

$$t - \beta_1 = \frac{2\pi m a^2}{h} \int \left[-\frac{\nu}{R} + 2v - \frac{k^2}{\rho^2} \right]^{-\frac{1}{2}} d\rho \quad ...(2\cdot92).$$

With the very simple type of field assumed, complete agreement with all actual values of ν/R is not of course to be expected, and the calculated orbits will be only approximations to the real orbits, but the results are sufficiently good to be interesting.

§ 3. *Circular Orbits.*

The integral $(2\cdot7)$ or $(2\cdot9)$ has to be taken round a cycle, that is from one root of the quantity inside the square bracket to the other and back (there are usually if not always only two real roots). For a circular orbit the roots must coincide, hence, omitting the relativity correction, we must have

$$-\frac{\nu}{R} + 2v - \frac{k^2}{\rho^2} = 0 \qquad ......(3\cdot1),$$

$$\frac{d}{d\rho} \left[-\frac{\nu}{R} + 2v - \frac{k^2}{\rho^2} \right] = 0 \qquad ......(3\cdot2).$$

On differentiating and using $(2\cdot8)$, $(3\cdot2)$ gives

$$Z\rho = k^2 \qquad ......(3\cdot3).$$

Hence, if Z is plotted as a function of ρ, the radii of the circular orbits are given by the points where the (Z, ρ) curve is cut by the hyperbolae $Z\rho = k^2$, and the values of ν/R are then given by $(3\cdot1)$.

Equations similar to $(3\cdot1)$, $(3\cdot3)$ can be obtained retaining the relativity correction, but are not of much interest.

§ 4. *Term Values and Assignment of Quantum Numbers.*

For the optical terms the determination of the values of ν/R presents little difficulty, nor does there seem much doubt that these values really represent very nearly the orbital energies of the valency electron. The values of ν are taken in general from Fowler's tables[*], values otherwise obtained are specially noted. The values to be taken for doublet terms are considered later (see § 5). The assignment of quantum numbers to the different terms is taken from Bohr's theory.

For the X-ray terms the values of ν/R are taken from Bohr and Coster's recent paper[†], except for Na for which they are not given (see § 7·3), and the quantum numbers there assigned have been used.

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[*] A. Fowler, "Report on Series in Line Spectra" (*Phys. Soc. Lond.* 1922).
[†] N. Bohr and D. Coster, *Zeit. für Phys.* 12, p. 342 (1923).

It seems doubtful whether the values of ν/R for the X-ray terms really represent the energies of the electrons in the corresponding orbits, on account of the alteration of the orbits of the outer X-ray electrons when an electron is removed from an inner orbit; and it also seems doubtful whether the field can reasonably be taken to be the same for an X-ray electron and for the valency electron at the same point. On consideration it appears that the errors involved in neglecting these difficulties are in opposite directions, and that further compensation probably occurs also, so that the use of the same field for X-ray and optical orbits, and of the term values as representing orbital energies of the X-ray electrons, is not so violent an approximation as it appears at first sight.

It is interesting to see that formula (3·1) for circular orbits can be reduced approximately to Moseley's form for X-ray orbits.

If there is a circular orbit radius ρ_0, and Z_0, v_0 are the values of Z and v at this radius, we have

$$\nu/R = 2v_0 - (k^2/\rho_0^2) \ \ \ldots\ldots(3\cdot1); \quad Z_0\rho_0 = k^2 \ \ \ldots\ldots(3\cdot3).$$

Put
$$\rho_0 v_0 = \rho_0 \int_{\rho=\infty}^{\rho_0} Z\, d(1/\rho) = Z_0 - s \quad \ldots\ldots(4\cdot1),$$

where
$$s = \int_{\rho=\infty}^{\rho_0} (Z_0 - Z)\, d(\rho_0/\rho) \quad \ldots\ldots(4\cdot2),$$

and represents roughly the effect of the screening of the outer electrons in diminishing the potential inside the atom.

Then substituting (4·1) in (3·1) and eliminating ρ_0 by (3·3), we get

$$\nu/R = [2Z_0 (Z_0 - s) - Z_0^2]/n^2 = [Z_0^2 - 2sZ_0]/n^2,$$

since $n = k$ for a circular orbit. Now s must be smaller than Z and actually is usually a good deal smaller, so this can be written approximately in Moseley's form*

$$\nu/R = (Z_0 - s)^2/n^2. \qquad \ldots\ldots(4\cdot3).$$

It appears that $N - Z_0$ corresponds roughly to the 'inner screening number' of Bohr and Coster's paper†, and s to the 'outer screening number.'

For various reasons (fine structure of absorption edge, effect of chemical combination on X-ray levels, etc.) there is an uncertainty in the value of ν/R for the X-ray levels, probably of the same order

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* The quantity $(Z_0 - s)$ appearing in this formula has sometimes been called the 'effective nuclear charge' for the X-ray orbit; in the present paper the name is applied to Z_0 only. Z_0 is the nuclear charge which gives the same *field* at radius ρ_0, $Z_0 - s$ is the nuclear charge which gives the same *potential* there, while $n\,(\nu/R)^{\frac{1}{2}}$ is nearly $Z_0 - s$, as shown here, but not exactly. In some early work on the calculation of X-ray levels from assumed atomic structures these quantities seem to have been rather confused.

† N. Bohr and D. Coster, *loc. cit.* p. 359.

of magnitude as the value of ν/R for the greatest optical term, i.e. about 0·4; this is unimportant for the deeper X-ray levels but it is serious in the case of the most lightly bound ones. On account of the importance of the circular orbits in the course of the work, it is not really satisfactory if there is a lightly bound circular X-ray orbit present. For example, the analysis for Na is not so satisfactory as that for K, the lowest values of ν/R for circular X-ray orbits being about 2·4 and 21 respectively.

A small further point which might be noted here is that in the case of elements of small atomic number, the L spectrum has not been observed, and the values of ν/R for the X-ray levels for which $k = 1$ (other than the K level) are not available, since in conformity with the combination rules transitions from these levels to the K level do not occur.

§ 5. *Divided Levels.*

Both in X-ray and optical spectra we meet the phenomenon of divided levels, or levels for the specification of which a third quantum number appears to be necessary. For X-ray and one-electron optical spectra the division appears to be into a doublet, except for the $k = 1$ levels, which are always single (the k_1 of Bohr and Coster's paper* is here identified with the angular quantum number k, as implied by the curves shown in that paper). Bohr and Coster* class the levels for which $k_1 = k_2$ in their notation as 'normal' levels, and these are the levels for which the values of ν/R have been taken here.

Similarly for optical levels where the doublet separation is appreciable the value of ν which has been taken is that for the level for which $k = k'$, where k' is Sommerfeld's third quantum number (k' is equal to k for the level of the pair concerned in the strongest lines). For both X-ray and optical terms it is the level of the doublet with the smaller wave number which is taken.

The selection may appear arbitrary, but if the third quantum number specifies the angular momentum about the axis of symmetry of the atom, the orbits for which it is equal to that specifying the angular momentum of the individual electron are those lying in the equatorial plane of the atom, and the assumption of a radial field is likely to be least in error for them†.

§ 6. *Sketch of Practical Work.*

It is not proposed to give here an account of the details of the numerical work, but a few general remarks may be made.

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* N. Bohr and D. Coster, *loc. cit.* See particularly pp. 348, 365.
† I am indebted to Prof. Ehrenfest for this justification of a procedure I had decided to adopt for less convincing reasons.

The work depends on the two integrals

$$n - k = \frac{1}{2\pi} \oint \left[2v - \frac{\nu}{R} - \frac{k^2}{\rho^2} \right]^{\frac{1}{2}} d\rho \qquad \ldots\ldots(2\cdot7),$$

$$v = \int_{\rho\,=\,\infty} Z\, d\!\left(\frac{1}{\rho}\right) \qquad \ldots\ldots(2\cdot8).$$

[The relativity correction is very easily taken into account where necessary, see formula (2·9).] The integral (2·7) is replaced by

$$n - k = \frac{1}{\pi} \int_{\rho_0}^{\rho_1} \left[2v - \frac{\nu}{R} - \frac{k^2}{\rho^2} \right]^{\frac{1}{2}} d\rho \qquad \ldots\ldots(6\cdot1),$$

where ρ_0, ρ_1 are the two roots of the quantity in the square bracket, and are the extreme radii of the orbit. In most cases the complete integration can be avoided for most orbits by various artifices, and of several different forms of the integral which have been tried, this has been found the most satisfactory.

It has been possible to arrange the computational part of the work in such a way that no apparatus is required beyond a 10-inch slide rule and a table of squares.

The circular orbits are very important steps in the analysis, depending as they do on one value of v only and not on an integral containing values of v over a range of ρ. The orbits with the same value of k are conveniently grouped together; the $(n + 1)_k$ orbit penetrates some way inside the circular k-quantum orbit, and the others slightly further, but the minimum radius for an orbit of given k tends to a limit very rapidly with increasing n; this limit is roughly half the radius of the k-quantum circular orbit in most cases.

The work proceeds from the outside of the atom inwards, the (Z, ρ) curve being successively extrapolated and adjusted to fit (as nearly as possible) first the circular orbit with some given value of k, then the other orbits of the same k, then the circular orbit with next smaller value of k, and so on. [This sketch of the procedure is very rough, actually the calculations for the different values of k cannot be kept so independent as suggested here.]

It might seem that in using the quantity Z rather than v as the basis of the work, an extra integration is introduced unnecessarily. But the practical advantages of using Z are very appreciable and quite outweigh the trouble of the extra integration, which is anyhow very simple.

To start with Z is an easier quantity to think about than v, it is easier to consider how it should or should not behave. This is important in the adjustments of the field which are necessary unless a very fortunate first estimate has been made. Experience has shown that it is difficult to adjust v satisfactorily, but using Z the adjustments are simple.

Also v varies over a range of 0 to ∞ while Z varies from K to N only, and this makes Z easier to work with, especially for both very large and very small values of ρ. This is really connected to the previous objection to v; to put it very crudely, v varies faster than Z, and hence it is easier to detect deviations from smoothness in Z than in v; for the same reason Z is an easier quantity to extrapolate satisfactorily.

The following restrictions are placed on Z:

(1) (a) It shall tend to K for large values of ρ (K = net charge on kernel).

    (b) It shall tend to N for small values of ρ (N = atomic number).

(2) The (Z, ρ) curve shall be continuous and have no discontinuity in gradient.

(3) Z shall be monotonic considered as a function of ρ.

With a perfectly arbitrary function Z it would probably be possible to obtain complete agreement with the observed optical and X-ray terms; it might even be possible to do so if only conditions (1) and (2) were imposed, but the results would have no physical significance. The third restriction is really very severe, but it seems almost necessary as well as the other two in order to give significance to the result, and it has been applied strictly*.

§ 7. *Results.*

For the purpose of comparing the calculated with the actual term values it is convenient to introduce for the optical terms the quantity q defined by the relation

$$(v/R) = K^2/(n - q)^2 \qquad \ldots\ldots(7 \cdot 1),$$

where K is the net kernel charge and n the total quantum number assigned to the term. This quantity q which has been called the 'quantum defect' is a suitable measure of the deviation of the orbit from the orbit of the same quantum number in the field of a point charge K. If q_c is the value of q calculated in order to give n and k integral for the given field, then the smallness of $q_c - q$ compared to q seems the fairest test of the accuracy with which this deviation has been reproduced by the added field specified by the added charge $Z - K$.

The value of $q_c - q$ is very nearly equal to another quantity which is more easily calculated. The value n_c of n calculated from the *observed* value of v/R by means of the radial quantum integral

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* Fues (*loc. cit.*) appears to take a formula for V which makes Z have a maximum and then tend to $-\infty$ as ρ tends to zero. By this means he gets better agreement for the $k = 1$ orbits than I have been able to do, but the significance of this result appears doubtful.
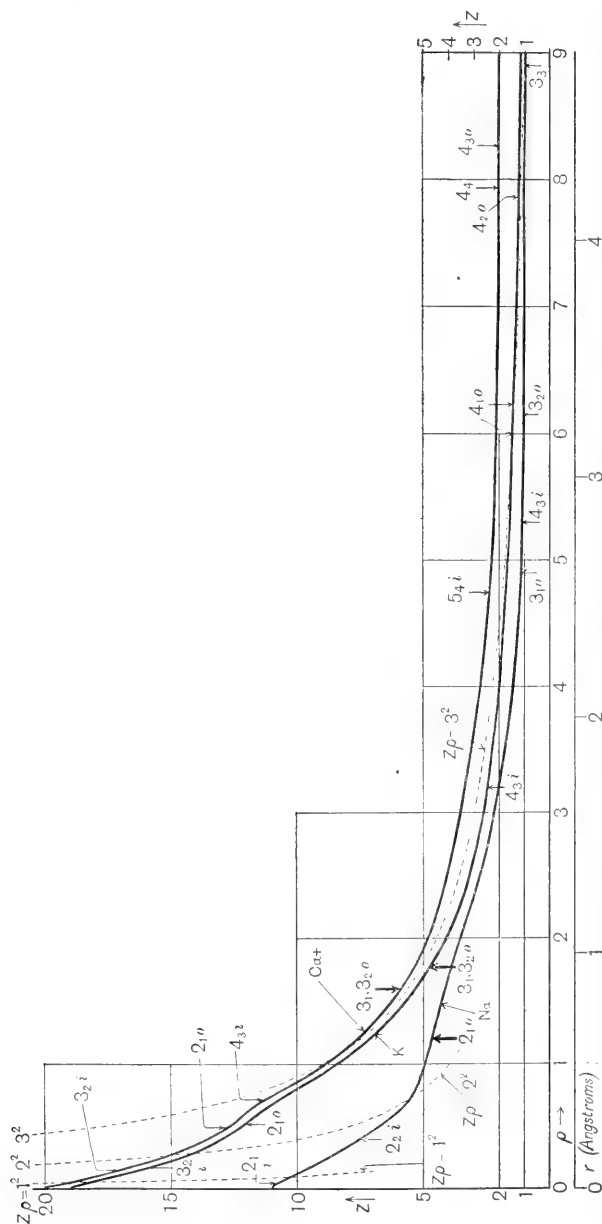
Fig. 1. (Z, ρ) curves for Na, K and Ca$^+$, determined by analysis of optical and X-ray spectrum terms. The full line curves are the curves of Z as a function of ρ. The broken curves are the hyperbolae $Z\rho = k^2$ ($k = 1, 2, 3$). The vertical arrows indicate the extreme values of ρ for various orbits of the different atoms. The quantum specification of each orbit is written at the end of the arrow, with the letter i or o to show whether the arrow indicates the inner or outer limit of the orbit. The positions of the circular orbits are given by the points of intersection of the (Z, ρ) curve and the hyperbolae and are not usually otherwise marked. The heavy vertical arrows each show the position of the boundary of the corresponding atom kernel.

will not generally be exactly equal to the value of n assigned to the orbit. It can be shown that

$$n_c - n = q_c - q$$

very nearly. Now n_c can be calculated directly, while an interpolation is necessary to obtain q_c, so a more convenient and sufficiently accurate test of the work is furnished by a comparison of $\delta = n_c - n$ with q. For the X-ray orbits a similar test is probably best given by the smallness of δ compared to unity.

The results for each atom considered are given in the form of a graph of Z as a function of ρ to specify the field, and a table comparing calculated and observed term values in the manner just indicated. Each table gives for a number of orbits the values of ν/R (observed)*, $q = n - (\nu/K^2 R)^{-\frac{1}{2}}$ for the optical orbits only, and $\delta = n_c - n$. In the calculation of n_c, k has been assumed correct and n' calculated from the observed ν/R using the integral (2·9), except for the circular orbits for which n' has been assumed zero and k calculated from the relativity modification of (3·1). The results are given to three places of decimals when the accuracy of the observed values of ν/R justifies it; the third may be in error by two or three units. The maximum and minimum radii of the orbits are also given in the table.

In the case of levels for which reliable values of ν/R were not available, the value of δ has been estimated† and the values of ν/R and the extreme radii calculated. Values for such levels entered in italics.

The (Z, ρ) curves are all given on the same diagram for comparison, the hyperbolae $Z\rho = k^2$ are also shown; the intersections of these with the (Z, ρ) curve give the positions of the circular orbits. Greatest and least radii of certain other orbits are also shown.

§ 7·1. *Potassium* ($N = 19$, $K = 1$).

This element seemed the most suitable to begin work on; the doublet separation is small and it is unimportant what way of treating it is used; for higher elements it cannot be dismissed so easily, on the other hand for lower elements in the periodic table there are fewer inner electrons and so, perhaps, more chance of their net effect not averaging out so closely to a radial field. Also, the values of ν/R for all the circular orbits are well determined for potassium; the 3_3 orbit is still an optical orbit, the 2_2 orbit has a

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* The values of ν/R are given in the table to 3 significant figures; for optical orbits they are known more accurately, and it is sometimes necessary to make use of the fourth and even the fifth significant figure in the numerical work.

† These values are at present pure guesses. It is possible that later more satisfactory means of making the estimates may be available. It seemed more satisfactory to guess what value of δ the actual value of ν/R would give than to guess ν/R directly.

value of ν/R of about 21 and the uncertainty already mentioned (§ 4) is unimportant.

Also, the comparison of K and Ca$^+$ promised to show very strikingly the process of transition of the 3_3 orbit from the optical to the X-ray region in conformity with Bohr's theory of atomic structure, and is hence of greater interest than the comparison of Na and Mg$^+$, and further it seemed that it might be possible to extrapolate from K and Ca to Cl, and connect up with Bragg's results on X-ray reflection for Cl as well as Na.

Since only the K X-ray spectrum has been observed, no values of ν/R are available for the 2_1 or 3_1 orbits; that for the 3_2 orbit is seriously affected by the uncertainty already mentioned.

TABLE I. *Results for Potassium.*

Orbit	ν/R	$q = n - (\nu/R)^{-\frac{1}{2}}$	$\delta = n_c - n$	Extreme values of ρ
1_1	265	—	$-0\cdot040$	$0\cdot055$
2_1	$23\cdot5$	—	$0\cdot07$	$0\cdot03,\ 0\cdot5$
2_2	$21\cdot3$	—	$0\cdot002$	$0\cdot3$
3_1	$2\cdot4$	—	$0\cdot15$	$0\cdot03,\ 1\cdot8$
3_2	$1\cdot5$	—	$0\cdot07$	$0\cdot17,\ 1\cdot75$
3_3	$\cdot123$	$\cdot146$	$0\cdot000$	6
4_1	$\cdot319$	$2\cdot230$	$0\cdot246$	$0\cdot03,\quad 6$
4_2	$\cdot200$	$1\cdot765$	$0\cdot078$	$0\cdot16,\quad 8$
4_3	$\cdot0693$	$0\cdot202$	$0\cdot006$	$3\cdot1,\quad 23$
4_4	$\cdot0627$	$0\cdot006$	$0\cdot000$	16
5_1	$\cdot127$	$2\cdot199$	$0\cdot171$	$0\cdot03,\ 15$
5_2	$\cdot0938$	$1\cdot734$	$0\cdot003$	$0\cdot16,\ 19$
5_3	$\cdot0439$	$0\cdot229$	$-0\cdot002$	$2\cdot9,\quad 40$
5_4	$\cdot0401$	$0\cdot008$	$0\cdot001$	$10,\quad 40$
6_1	$\cdot0688$	$2\cdot189$	$0\cdot156$	$0\cdot03,\ 28\frac{1}{2}$
6_2	$\cdot0547$	$1\cdot724$	$-0\cdot009$	$0\cdot16,\ 34$
6_3	$\cdot0302$	$0\cdot240$	$-0\cdot005$	$2\cdot7,\quad 61$
6_4	$\cdot0279$	$0\cdot008$	$0\cdot003$	$9,\quad 63$

§ 7·2. *Ionised Calcium* ($N = 20$, $K = 2$).

Not many terms of any of the optical series are observed, so the series limits are not very well determined, all the values of ν/R may therefore be slightly in error by the same amount.

Of the principal series only the first line is observed.

TABLE II. *Results for Ionised Calcium.*

Orbit	ν/R	q	δ	Extreme values of ρ
1_1	297	—	-0.036	0.05
2_1	*30*	—	*0.06*	*0.03, 0.5*
2_2	25.6	—	0.005	0.25
3_1	*4.0*	—	*0.13*	*0.03, 1.6*
3_2	*2.9*	—	*0.05*	*0.15, 1.6*
3_3	0.748	0.687	0.046	1.3
4_1	0.872	1.859	0.173	0.03, 4
4_2	0.641	1.504	0.061	0.15, 5
4_3	0.354	0.640	-0.047	0.7, 8
4_4	0.252	0.018	0.000	8
5_1	0.397	1.826	0.136	0.03, 10
5_2	—	—	—	—
5_3	0.210	0.633	-0.044	0.7, 16
5_4	0.161	0.023	0.006	5, 20
6_1	0.228	1.815	0.126	0.03, 17
6_2	—	—	—	—
6_3	0.139	0.632	-0.047	0.7, 26
6_4	0.112	0.029	0.004	4, 31

TABLE III. *Results for Sodium.*

Orbit	ν/R	q	δ	Extreme radii
1_1	79.0	—	-0.045	0.09
2_1	*4.0*	—	*0.12*	*0.06, 1.2*
2_2	2.45	—	0.022	0.75
3_1	0.378	1.373	0.212	0.05, 5
3_2	0.223	0.881	0.007	0.4, 6
3_3	0.112	0.010	0.000	8.9
4_1	0.143	1.357	0.144	0.05, $13\frac{1}{2}$
4_2	0.102	0.867	-0.024	0.4, 17
4_3	0.0629	0.012	-0.001	5.3, 26
4_4	0.0625	0.0004	0.000	16
5_1	0.0752	1.353	0.139	0.05, 26
5_2	0.0584	0.863	-0.029	0.4, 32
5_3	0.0402	0.013	0.001	4.9, 45
5_4	0.0400	0.0005	0.001	10, 40

§ 7·3. *Sodium* ($N = 11$, $K = 1$).

The work for sodium is not entirely satisfactory on account of the small value of ν/R for the 2-quantum circular orbit, and the consequent large effect of the uncertainty already mentioned (§ 4). Also no X-ray levels have been directly observed, so values have to be extrapolated from higher elements or obtained indirectly, which adds to the uncertainty.

The chief interest of the results for sodium is in the application to X-ray reflection. This will be considered more fully in a later paper, but a preliminary statement will be made here.

If the electron distribution is averaged out into a spherically symmetrical distribution of charge which is supposed to scatter according to classical electromagnetic laws, then the quantity F of Bragg's papers*, which may roughly be called the scattering power of a single atom, is given by†

$$F = \Sigma \int \frac{\sin \phi}{\phi} \, df \qquad \ldots\ldots(7\cdot2),$$

where df is the fraction of a radial period which an electron spends between radii r and $r + dr$, and

$$\phi = \frac{4\pi r \sin \theta}{\lambda} \qquad \ldots\ldots(7\cdot3),$$

where θ is the angle of reflection and λ the wave-length of the X-rays used, and the summation is over all orbits.

Table IV gives the values of F

(*a*) Observed by Bragg*.

(*b*) Calculated from atom model, with dimensions of orbits evaluated from the field deduced by analysis of optical and X-ray spectral terms.

(*c*) Calculated from atom model, with dimensions of orbits estimated by Bragg.

TABLE IV. *X-ray scattering.*

	$\sin \chi = 0\cdot1$	0·2	0·3	0·4	0·5
$F \begin{cases} (a) \\ (b) \\ (c) \end{cases}$	8·32	5·40	3·37	2·02	0·76
	8·87	6·18	3·41	1·66	1·18
	8·73	6·04‡	3·76	2·53	1·80

* W. L. Bragg, R. W. James and C. H. Bosanquet, *loc. cit.*

† This formula is a simple extension of Bragg's (*Phil. Mag.* vol. 43, p. 440, formula (4)).

‡ In Bragg's paper this value is given as 5·04. This is presumably a misprint, it has been recalculated.

The values (c) are entered for comparison. For both (b) and (c) the electron arrangement is supposed to be: 2 electrons in 1_1 orbits, 4 each in 2_1 and 2_2 orbits.

The agreement seems satisfactory in view of the difficulties and uncertainties in both the theoretical and the experimental side.

§ 8. *Discussion of Results.*

It must be emphasised first that no details of atomic structure are postulated; on the single assumption of a radial field, the same for all orbits, the strength of the field at different distances is determined by the analysis of the experimental data on the energy levels in the atom. The field so deduced should be compared afterwards with those due to various arrangements of the X-ray electrons; the latter field, however, will not in general be radial and a satisfactory comparison is difficult to make (see end of this section).

It will be noticed on examining the (Z, ρ) curves that the added field becomes appreciable quite a long way outside the boundary of the atom. This alone precludes the assumption of central symmetry, for if Laplace's equation is to hold outside the boundary of a spherically symmetrical atom, it is easy to see that the added field must be zero at all points outside.

Another general characteristic of the results, as far as they go, is that it seems impossible to obtain good agreement for the orbits for which $k = 1$, if the restrictions on Z specified at the end of § 6 are adhered to. This is scarcely surprising, as the optical orbits for which $k = 1$ penetrate right inside the K ring, where the absolute magnitude of the deviation from a radial field is probably much the greatest*. Also the interaction of the two K electrons themselves is probably very close, and the assumption of the same field for the X-ray and the valency electrons is probably most in error when applied to them. Using a field which tends to N in a similar way for all elements as ρ tends to zero, the values of $n_c - n$ for the K level have very similar values, and those for the optical orbits for which $k = 1$ have roughly similar values and behave in the same sort of way with increasing n.

The values† of δ are rarely greater than 10 per cent. of q (or of 1 for X-ray orbits), even for the $k = 1$ terms, and are in many cases much less; the agreement is probably as good as could reasonably be expected so long as the field is assumed radial.

An interesting result is the very considerable difference between

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* Since there are only two electrons in the K group and at least four in any other group, the deviation from radial symmetry may be expected to be most pronounced in the region of the K ring.

† For small values of q, it must be remembered that δ may be in error by 3 in the third decimal place due to errors in the approximate integration formulae used, etc.

the added field for K and Ca$^+$ at distances of the order of 5 times
the atomic radius, that for Ca$^+$ being very much smaller. (This is
independent of the type of added field used and can be illustrated
from the observed values of ν/R alone; the same appears to be true
for Na and Mg$^+$.) The whole scale of the added field must shrink
slightly with the contraction of the kernel under the attraction of
the increased nuclear charge, but this effect does not appear to be
large enough to account for more than a half, at most, of the
observed difference. Attention may also be called to the very
similar values of the added charge $Z-K$ at the boundary of the
atom; they are approximately 3·7 for Na, 3·8 for K, 3·9 for Ca$^+$.

The most interesting phenomenon shown by these results is
the first stage of the process of the transition of the 3_3 orbit from
an optical to an X-ray orbit.

It will be remembered that the position of the 3_3 orbit is given
by the point where the (Z, ρ) curve cuts the hyperbola $Z\rho = 3^2$.
For potassium the intersection is at a very small angle, and the
(Z, ρ) curve remains near the hyperbola for a long way further in.
On going to Ca$^+$, the added field shrinks slightly, and superposed
on this Z is increased by 1 for all ρ. The shrinkage does not counter-
act more than half the uniform increase of Z, and the point of
intersection is shifted a long way in, actually inside the boundary
of the atom. By extrapolating roughly from the curves for K
and Ca$^+$, it is evident that on going to the succeeding elements of
the periodic table the 3_3 orbit is going to continue its progress in-
wards, though not so rapidly (as far as its radius is concerned,
though its ν/R may increase considerably as it is within the atom
boundary).

Another phenomenon also connected with this is the large
variation of q between the different diffuse ($k = 3$) terms of the
optical spectrum of potassium. The value of α in the Hicks'
formula for the sequence term

$$\nu/R = K^2/[m + a + (\alpha/m)]^2$$

is exceptionally large for the d sequence of potassium, and no ex-
planation of this is apparent till we come to Bohr's theory of atomic
structure and then only when we consider numerical dimensions
in some detail. This phenomenon then appears as a consequence
of the fact that the 3_3 orbit is situated just where the added field
is beginning to increase rapidly, so that the inner parts of the
'elliptic' orbits with $k = 3$ go into very appreciably stronger added
fields than that in which the circular orbit lies; moreover, this
effect is cumulative, as in the stronger field the electron gets pulled
in further, into yet stronger fields. Now Bohr has shown that if the
inner parts of orbits of the same k are very nearly the same, then
a very small value of α is to be expected; a peculiarly large value of

α may in a similar way be expected when the inner parts are more than usually different, as is the case here.

So far, the comparison of the (Z, ρ) curve determined from the analysis of spectral terms with that to be expected from various arrangements of the X-ray electrons has hardly been considered other than in a rough qualitative way. The contribution to Z by m electrons averaged out into a uniform spherical shell, and m units of positive electricity at the nucleus, is given by curve A in Fig. 2. With an actual arrangement of point electrons the field will not in general be radial, but neglecting the deviations from radial symmetry the contribution to Z may be expected to be something like curve A with the corners smoothed off, say curve B. The whole (Z, ρ) curve may be considered as compounded of a series of such curves, one for each electron group in the atom.

The little quantitative work which has, so far, been done on these lines is not sufficient to lead to any definite results.

In conclusion I wish to express my thanks to Mr R. H. Fowler for his interest in this work, and for his advice in connection with it.
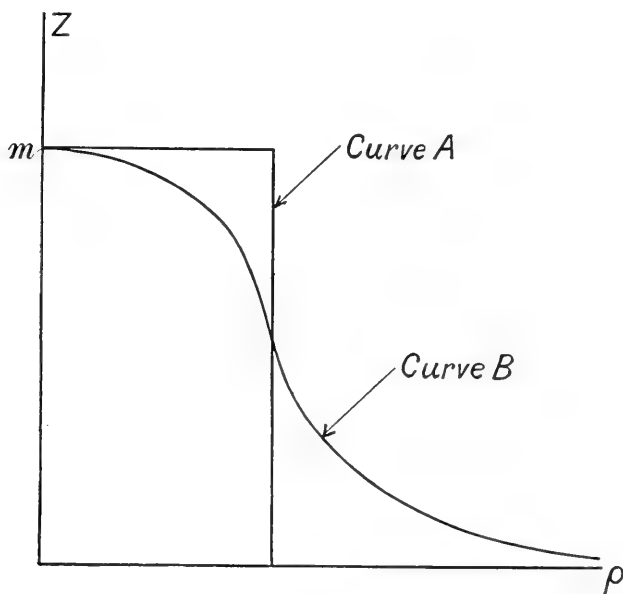
Fig. 2

The Partitions of Infinity with some Arithmetic and Algebraic consequences. By Major P. A. MacMahon.

1. The representation of numbers by means of a systematic scale of notation depends upon the well-known theorem:

"Let r_1, r_2, r_3, ..., r_n, r_{n+1}, ... denote an infinite series of integers, restricted in no way except that each is to be greater than 1; then any integer N may be expressed in the finite form

$$N = p_0 + p_1 r_1 + p_2 r_1 r_2 + p_3 r_1 r_2 r_3 + ... + p_n r_1 r_2 ... r_n,$$

where $p_3 < r_{s+1}$. When r_1, r_2, r_3, ... are given, this can be done in one way only."

The place of this theorem in the theory of the Partition of Numbers has not, I believe, hitherto been explicitly stated.

In the year 1886* I discussed certain special partitions of numbers which I provisionally termed "Perfect Partitions" with the definition:

"A perfect partition of a number is one which contains one, and only one, partition of every lower number."

I recurred again to the subject† in "The Theory of the Perfect Partitions of Numbers and the Compositions of Multipartite Numbers" and subsequently‡ I made applications to various branches of Combinatory Analysis. The theory was an important auxiliary in the final solution of Euler's problem of the "Latin Square" and of its wide generalisation.

2. In the paper of 1890 (*l.c.*), p. 119, *foot-note*, I gave what is virtually the most general expression of the perfect partition of an infinite number.

This was

$$1^{\alpha_1} (1 + \alpha_1)^{\alpha_2} \{(1 + \alpha_1)(1 + \alpha_2)\}^{\alpha_3} \{(1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3)\}^{\alpha_4} ... \text{ } ad \text{ } inf.,$$

where (i) α_1, α_2, α_3, α_4, ... are positive integers.

(ii) α_1, α_2, ..., when used as exponents, denote repetitions of the numbers which they affect, so that the expression denotes an infinite succession of integers in such wise that

1^{α_1} denotes a succession of α_1 units,

$(1 + \alpha_1)^{\alpha_2}$ denotes a succession of α_2 integers each equal to $1 + \alpha_1$, etc.

(iii) $\alpha_1, \alpha_2, \alpha_3, \ldots$ may each be given any one of the values $1, 2, 3, \ldots$ subject to

(iv) one number α_s may be an infinite integer and this implies that α with a suffix, $> s$, is zero.

Every number, from 1 to ∞, may be composed in one way (and one only) by means of integers selected from the infinite series whatever values be assigned to $\alpha_1, \alpha_2, \alpha_3, \ldots$ in agreement with the specified conditions.

This perfect partition of infinity is derived from the general theorem given at the commencement of the paper by writing

$$p_s = \alpha_{s+1}, \quad r_s = 1 + \alpha_s.$$

It is important to notice that the partition expression, which is in accord with the universally recognised notation, involves a *single* system of integers

$$\alpha_1, \alpha_2, \alpha_3, \ldots,$$

because the notation causes the system

$$p_0, p_1, p_2, \ldots$$

to disappear automatically.

3. The most interesting particular cases in the partition notation are:

I. $\alpha_1 = \infty$, $\alpha_s \, (s > 1) = 0$, an infinite succession of units

$$(1111\ldots);$$

II. $\alpha_s = r - 1$ for all values of s

$$\{1^{r-1} . r^{r-1} . (r^2)^{r-1} . (r^3)^{r-1} \ldots\},$$

involving the scale of numeration for radix r;

III. $\alpha_s = s$ for all values of s

$$(1! \, 2!^2 \, 3!^3 \, 4!^4 \ldots);$$

IV. $\alpha_1, \alpha_2, \alpha_3, \ldots$ in succession are, each of them, one less than the uneven primes $3, 5, 7, \ldots$

$$\{1^2 . 3^4 . (3.5)^6 . (3.5.7)^{10} . (3.5.7.11)^{12} \ldots\}.$$

Every partition of infinity corresponds to a scale of numeration, and has an algebraic formula connected with it.

Thus I above give

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + \ldots.$$

II with $r = 2$,

$$\frac{1}{1-q} = (1+q)(1+q^2)(1+q^{2^2})(1+q^{2^3}) \ldots ad \; inf.$$

II in general,

$$\frac{1}{1-q} = (1 + q + \dots + q^{r-1})(1 + q^r + q^{2r} + \dots + q^{(r-1)\cdot r})(1 + q^{r^2} + q^{2r^2} + \dots + q^{(r-1)\cdot r^2}) \dots ad\ inf.$$

III,

$$\frac{1}{1-q} = (1 + q^{1!})(1 + q^{2!} + q^{2\cdot 2!})(1 + q^{3!} + q^{2\cdot 3!} + q^{3\cdot 3!}) \dots ad\ inf.$$

etc.

Application to the Generating Function which enumerates the Partitions of Numbers.

4. The function is $\quad \prod\limits_{1}^{\infty}(1 - q^s)^{-1},$

if the partitions be unrestricted.

In the first place transform each factor by application of the binary scale formula

$$\frac{1}{1-q} = (1 + q)(1 + q^2)(1 + q^4)(1 + q^8) \dots.$$

We find

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\dots} = (1+q)(1+q^2)^2(1+q^3)(1+q^4)^3 \dots (1+q^n)^{n_2}\dots,$$

where $n_2 = 1 + \nu$, and ν is the highest power of 2 which is a factor of n.

Otherwise, and preferably, we may define n_2 to be

"The number of representations of n in the form—a power of 2 multiplied by *any number*."

Observe that the phrase "*any number*" occurs in the definition because the exponents of q in the denominator of the function, which we are transforming, involve the whole of the integers $1, 2, 3, \dots.$

If we had been transforming the function

$$\prod\limits_{0}^{\infty}(1 - q^{2s+1})^{-1}$$

the exponent of $(1 + q^n)$ would have been unity because any number has only *one* representation in the form

"a power of 2 multiplied by an *uneven* number."

We now consider the factor

$$(1 + q^s)^{s_2}, \text{ of the transformed function,}$$

and find $\quad q\dfrac{d}{dq}\log(1 + q^s)^{s_2} = s_2 \left\{ \dfrac{sq^s}{1 - q^{2s}} - \dfrac{sq^{2s}}{1 - q^{2s}} \right\}.$

The general term in the expansion of $s_2 \dfrac{sq^s}{1-q^{2s}}$ is

$$s_2 s q^{s(2k+1)},$$

indicating that the coefficient of q^n in

$$\sum_1^\infty s_2 \frac{sq^s}{1-q^{2s}}$$

is

$$\Sigma d_2 d,$$

where d is a divisor of n whose conjugate is uneven.

I write $\qquad \Sigma d_2 d = \Delta^{(1)}(n_2, n)\,{}^*.$

Similarly the coefficient of q^n in

$$\sum_1^\infty s_2 \frac{sq^{2s}}{1-q^{2s}}$$

is

$$\Sigma d_2 d,$$

where d ranges over the divisors of n which have even conjugates.

I write $\qquad \Sigma d_2 d = D^{(1)}(n_2, n).$

Hence

$$q \frac{d}{dq} \log \prod_1^\infty (1+q^s)^{s_2} = \sum_1^\infty \left\{ \Delta^{(1)}(n_2, n) - D^{(1)}(n_2, n) \right\} q^n$$

$$= \sum_1^\infty \zeta^{(1)}(n_2, n)\, q^n\,{}^\dagger.$$

Moreover, after Euler,

$$q \frac{d}{dq} \log \prod_1^\infty (1-q^s)^{-1} = \Sigma \sigma(n)\, q^n,$$

where $\sigma(n)$ is the sum of the divisors of n.

Thus we have the arithmetical identity

$$\zeta^{(1)}(n_2, n) = \sigma(n).$$

As a verification for $n = 20$:

Divisors	1,	2,	4,	5,	10,	20,
d_2	1,	2,	3,	1,	2,	3,
Conjugates	20,	10,	5,	4,	2,	1;

$$\Delta^{(1)}(n_2, n) = 3\,.\,4 + 3\,.\,20 = 72,$$

$$D^{(1)}(n_2, n) = 1\,.\,1 + 2\,.\,2 + 1\,.\,5 + 1\,.\,20 = 30,$$

$$\zeta^{(1)}(n_2, n) = 72 - 30 = 42 = 1 + 2 + 4 + 5 + 10 + 20 = \sigma(20).$$

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* Glaisher denotes by $\Delta'(n)$ the sum of the divisors whose conjugates are uneven.
,, $D'(n)$,,   ,,   ,,   even.
† Glaisher denotes $\Delta'(n) - D'(n)$ by $\zeta'(n)$.

5. I next apply the ternary scale formula

$$\frac{1}{1-q} = (1+q+q^2)(1+q^3+q^6)(1+q^9+q^{18})\ldots(1+q^{3^k}+q^{2.3^k})\ldots$$

and find that $\overset{\infty}{\underset{1}{\Pi}}(1-q^n)^{-1} = \overset{\infty}{\underset{1}{\Pi}}(1+q^s+q^{2s})^{s_3}$,

where s_3 is the number of representations of s in the form

"a power of 3 multiplied by *any number*."

We have

$$q\frac{d}{dq}\log(1+q^s+q^{2s})^{s_3} = \frac{s_3}{1-q^{3s}}(sq^s+sq^{2s}-2sq^{3s}).$$

The coefficient of q^n in

$$\overset{\infty}{\underset{1}{\Sigma_s}}\frac{s_3}{1-q^{3s}}(sq^s+sq^{2s})$$

is $\Sigma d_3 d$,

where d ranges over those divisors of n whose conjugates are *not* of the form 0 mod 3.

I write $\Sigma d_3 d = \Delta^{(2)}(n_3, n)$.

And the coefficient of q^n in

$$\overset{\infty}{\underset{s}{\Sigma_s}}\frac{s_3}{1-q^{3s}}\cdot sq^{3s}$$

is $\Sigma d_3 d$,

where d ranges over those divisors of n whose conjugates *are* of the form 0 mod 3.

I write $\Sigma d_3 d = D^{(2)}(n_3, n)$,

and thence $\zeta^{(2)}(n_3, n) = \Delta^{(2)}(n_3, n) - 2D^{(2)}(n_3, n) = \sigma(n)$.

6. In the general case of order r

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\ldots} = \overset{\infty}{\underset{1}{\Pi}}(1+q^s+q^{2s}+\ldots+q^{(r-1)s})^{s_r},$$

where s_r is the number of representations of s in the form

"a power of r multiplied by any number."

We have

$$q\frac{d}{dq}\log(1+q^s+q^{2s}+\ldots+q^{(r-1)s})$$

$$= \frac{s_r}{1-q^{rs}}(sq^s+sq^{2s}+\ldots+s^{(r-1)s}-(r-1)sq^{rs}).$$

The coefficient of q^n in

$$\sum_{1}^{\infty} \frac{s_r}{1-q^{rs}} (sq^s + sq^{2s} + \ldots + s^{(r-1)s})$$

is

$$\Sigma d_r d,$$

where d ranges over the divisors of n whose conjugates are *not* of the form $0 \bmod r$.

I write $\qquad \Sigma d_r d = \Delta^{(r-1)}(n_r,\, n),$

and the coefficient of q^n in

$$\sum_{1}^{\infty} \frac{s_r s q^{rs}}{1-q^{rs}}$$

is

$$\Sigma d_r d,$$

where d ranges over the divisors whose conjugates *are* of the form $0 \bmod r$.

I write $\qquad \Sigma d_r d = D^{(r-1)}(n_r,\, n),$

and thence

$$\zeta^{(r-1)}(n_r,\, n) = \Delta^{(r-1)}(n_r,\, n) - (r-1)D^{(r-1)}(n_r,\, n) = \sigma(n).$$

The verification for $r = 5$, $n = 20$ is

Divisors	1,	2,	4,	5,	10,	20,
d_5	1,	1,	1,	2,	2,	2,
Conjugates	20,	10,	5,	4,	2,	1;

$$\Delta^{(4)}(n_5,\, n) = 2.5 + 2.10 + 2.20 = 70,$$
$$D^{(4)}(n_5,\, n) = 1.1 + 1.2 + 1.4 = 7,$$
$$\zeta^{(4)}(n_5,\, n) = 70 - 4.7 = 42 = \sigma(20).$$

The arithmetical functions

$$\Delta^{(r-1)}(n_r,\, n), \;\; D^{(r-1)}(n_r,\, n), \;\; \zeta'(n_r,\, n) = \Delta^{(r-1)}(n_r,\, n) - (r-1)D^{(r-1)}(n_r,\, n),$$

which present themselves, appear to be new to the subject.

7. Other forms of arithmetical functions appear when other generating functions connected with the enumeration of partitions are considered. To give a general idea of their nature I will consider a generating function which, in consequence of the researches of Rogers and Ramanujan, has been much in evidence within the last few years. It is

$$\prod_{0}^{\infty} (1 - q^{5m+1}_{+4})^{-1} = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\ldots}.$$

Applying the formula for the r^{ary} scale of enumeration, we find

$$\overset{\infty}{\underset{1}{\Pi}} (1 + q^s + q^{2s} + \ldots + q^{(r-1)s})^{f(s)},$$

where $f(s)$ denotes the number of representations of s in the form

" a power of r multiplied by a number of the form $\pm 1 \bmod 5$."

When we operate with $q\dfrac{d}{dq}\log$ we meet with two numerical functions

$$\text{(i)} \quad \Sigma f(d) . d = f_1(n),$$

where d ranges over the divisors of n whose conjugates are *not* of the form $0 \bmod r$, and

$$\text{(ii)} \quad \Sigma f(d) . d = f_2(n),$$

where the range is over those divisors whose conjugates *are* of the form $0 \bmod r$;

$$f_1(n) - (r-1) f_2(n) = g(n),$$

where $g(n)$ is the number of partitions of n into parts of the form $5m \pm 1$.

8. The numerical function n_r is such that

$$\Sigma n_r x^n$$

$$= \frac{x + x^2 + \ldots + x^{r-1}}{1 - x^r} + 2\frac{x^r + x^{2r} + \ldots + x^{(r-1)r}}{1 - x^{r^2}} + 3\frac{x^{r^2} + x^{2r^2} + \ldots + x^{(r-1)r^2}}{1 - x^{r^3}} + \ldots,$$

the general term being

$$s\,\frac{x^{r^{s-1}} + x^{2.r^{s-1}} + \ldots + x^{(r-1)r^{s-1}}}{1 - x^{r^s}};$$

and it can be seen at once that

$$\Sigma n_r x^n - 2\Sigma n_r x^{rn} + \Sigma n_r x^{r^2 n} = \frac{x + x^2 + \ldots + x^{r-1}}{1 - x^r}.$$

9. I pass on to the factorial scale of notation for which

$$\frac{1}{1-q} = (1 + q^{1!})(1 + q^{2!} + q^{2.2!})(1 + q^{3!} + q^{2.3!} + q^{3.3!})\ldots,$$

and remark that, when application is made to

$$\overset{\infty}{\underset{1}{\Pi}} (1 - q^n)^{-1},$$

it becomes $\qquad \overset{\infty}{\underset{1}{\Pi}}\,\overset{\infty}{\underset{1}{\Pi}}\,\dfrac{1 - q^{t(s+1)!}}{1 - q^{t.s!}}.$

If we denote by f_n the number of representations of n in the form

"a factorial multiplied by *any number*,"

this may be written $\qquad \prod_1^\infty \dfrac{(1-q^n)^{f_n-1}}{(1-q^n)^{f_n}}$.

Operating as usual with $q\dfrac{d}{dq}\log$ we merely obtain the trivial result

$$\Sigma df_d - \Sigma d\,(f_d - 1) = \Sigma d = \sigma\,(n).$$

The numerical function Σdf_d ranges over all of the divisors of n, and I write

$$\Sigma df_d = \epsilon\,(n).$$

We have $\qquad \Sigma f_m q^n = \dfrac{q^{1!}}{1-q^{1!}} + \dfrac{q^{2!}}{1-q^{2!}} + \dfrac{q^{3!}}{1-q^{3!}} + \dots;$

$$q\frac{d}{dq}\log\prod_1^\infty(1-q^n)^{-f_n} = \frac{f_1 q}{1-q} + \frac{2f_2 q^2}{1-q^2} + \frac{3f_3 q^3}{1-q^3} + \dots + \frac{sf_s q^s}{1-q^s} + \dots$$

$$= \sum_1^\infty \epsilon\,(n)\,q^n,$$

and by integration

$$\prod_1^\infty (1-q^n)^{-f_n} = \exp.\sum_1^\infty \frac{1}{n}\,\epsilon\,(n)\,q^n.$$

Application to the Reciprocal of a Polynomial.

10. Consider

$$\frac{1}{(1-\alpha_1 q)(1-\alpha_2 q^2)(1-\alpha_3 q^3)\dots(1-\alpha_m q^m)} = \frac{1}{1-a_1 q+a_2 q^2+\dots+(-)^m a_m q^m},$$

and apply the formula

$$\frac{1}{1-q} = (1+q)\,(1+q^2)\,(1+q^4)\,(1+q^8)\dots$$

to each factor. We obtain

$$\prod_0^\infty (1+\alpha_1^{2^s} q^{2^s})\,(1+\alpha_2^{2^s} q^{2^s})\dots(1+\alpha_m^{2^s} q^{2^s}).$$

We express the symmetric functions of $\alpha_1,\,\alpha_2,\,\alpha_3,\,\dots,\,\alpha_m$ in the partition notation and find

$$\{1 + (1)\,q\ + (1^2)\,q^2 + (1^3)\,q^3\ + \dots + (1^m)\,q^m\}$$
$$\times\ \{1 + (2)\,q^2 + (2^2)\,q^4 + (2^3)\,q^6\ + \dots + (2^m)\,q^{2m}\}$$
$$\times\ \{1 + (4)\,q^4 + (4^2)\,q^8 + (4^3)\,q^{12}\ + \dots + (4^m)\,q^{4m}\}$$
$$\times\ \{1 + (8)\,q^8 + (8^2)\,q^{16} + (8^3)\,q^{24}\ + \dots + (8^m)\,q^{8m}\}$$
$$\times\ \dots\ ad\ inf.$$

This infinite product is equal to

$$1 + h_1 q + h_2 q^2 + h_3 q^3 + \ldots \ ad \ inf.,$$

where h_s is the homogeneous product sum of degree s of the quantities

$$\alpha_1, \ \alpha_2, \ \alpha_3, \ \ldots, \ \alpha_m.$$

Equating coefficients,

$h_1 = (1),$

$h_2 = (1^2) + (2),$

$h_3 = (1^3) + (1)(2),$

$h_4 = (1^4) + (1^2)(2) + (2^2) + (4),$

$h_5 = (1^5) + (1^3)(2) + (1)(2^2) + (1)(4),$

$h_6 = (1^6) + (1^4)(2) + (1^2)(2^2) + (1^2)(4) + (2^3) + (2)(4),$

$h_7 = (1^7) + (1^5)(2) + (1^3)(2^2) + (1^3)(4) + (1)(2^3) + (1)(2)(4),$

$h_8 = (1^8) + (1^6)(2) + (1^4)(2^2) + (1^4)(4) + (1^2)(2^3) + (1^2)(2)(4)$
$$\qquad\qquad + (2^4) + (2^2)(4) + (4^2) + (8),$$

etc.

h_n is expressible as a sum of products of symmetric functions, where each symmetric function involves *one* of the numbers $1, 2, 2^2,$ $2^3, \ldots$ and no other number. This is an algebraic analogue of the unique expression of all integers by means of the addition of powers of 2.

The number of terms in the expression of h_n is equal to the number of solutions in integers of the equation

$$x_0 + 2x_1 + 2^2 x_2 + \ldots + 2^s x_s + \ldots = n.$$

The fact is that every perfect partition has an algebraic analogue of this character.

If we take the perfect partition corresponding to

$$(1 + q + q^2 + \ldots + q^{k-1})(1 + q^k + q^{2k} + q^{3k} + \ldots),$$

we find that we reach a result

$$h_n = \Sigma P \cdot Q,$$

where P is a symmetric function whose partition expression contains no number greater than $k - 1$; and Q is one whose partition expression involves only numbers which are multiples of k. The expression of h_n in this form is unique.

The Prime Numbers of measurement on a scale. By Major P. A. MacMahon.

If we take a straight line of finite length l and divide it into n equal segments we obtain a scale or measuring rod which enables the measurement of any number of segments s where $0 \leqslant s \leqslant l$. If such measurements be the object in view it is clear that the rod is redundantly divided. There are more scale divisions than are necessary. Certain of the scale divisions may be obliterated. Thus a yard rod divided into three feet may have one scale division wiped out without interference with the measurements of one, two and three feet.

The problem which is here presented, in the case of a scale of finite length, has been discussed chiefly in connection with mathematical puzzles without leading to much of mathematical interest connected with the classical theory of numbers. The questions which arise are difficult but can be to some extent elucidated by tentative processes.

The maximum number of scale divisions that may be erased and the specification of the resulting scale of segments have not been determined as yet and it is clear that other questions quickly present themselves for solution.

The recent *History of the Theory of Numbers** in two volumes does not supply any references to scientific papers upon the subject.

The present communication deals with a scale of infinite length. This is, from one point of view, a simplification of the problem because we are practically freed from a boundary condition. In the scale of finite length if we take one end as origin, the other end presents a boundary condition which leads to difficulties immediately. In the case of the infinite rod these initial troubles are absent so that progress can be made up to a certain point, and it will be found, by a perusal of what follows, that an interesting system of numbers, of infinite extent, presents itself which appears to exhibit a certain analogy with the infinite series of prime numbers.

It is essential to specify precisely the problem of the infinite measuring rod. Beginning from the finite end, our zero point, we

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* L. E. Dickson of Chicago, published by the Carnegie Institute of Washington, D.C.

are going to insert certain dividing lines, in correspondence with a segment of unit length, so as to enable the measurement on the rod of a length of any integral number of segments where such number may be any integer 1, 2, 3, ... ∞ .

This measurement is to be made by *one* operation with two dividing lines of the rod. If we do not specify this condition, but allow a number of different measurements to be made and added together we are in face of a question which has already been completely solved. We have, for instance, only to take dividing lines which exhibit successive segments of lengths:

$$1, 2, 2^2, 2^3, \ldots \text{ad inf.}$$

to be enabled, by *one or more* measurements, to measure any length which is equal to any integral number of segments.

In fact the above written succession of numbers constitutes a "perfect" partition of the number infinity in that every number can be composed by selecting members of the series, once only, in only one way.

This is the simplest solution of a particular case of the general theorem which states that if

$$\alpha_1, \alpha_2, \alpha_3, \ldots$$

be *any integers at pleasure*, each $\geqslant 1$, a "perfect" partition of infinity is

α_1 numbers each equal to unity
α_2 ,,   ,, $1 + \alpha_1$
α_3 ,,   ,, $(1 + \alpha_1)(1 + \alpha_2)$
α_4 ,,   ,, $(1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3)$
                        etc.*

The simplest solution is obtained by putting

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = 1.$$

In regard to the problem now before us we are to measure any integral number of units of length by a *single* operation:

In order to arrive at a minimum number of dividing lines we further specify that, starting from the origin O, the lengths of the successive segments are to *increase* in length. The lengths are to be in ascending order of magnitude, but no two such lengths are to be equal.

Under these conditions it is clear that we require scale divisions

* *Combinatory Analysis*, Camb. Univ. Press, 1915, vol. I, p. 220.

at the points 1 and 3 in order to obtain segments of lengths 1, 2.
We can now measure three units of length between the origin and
the division at the point 3. We next require a division at the point
7 in order to measure a segment of length 4.

Proceeding in this way we obtain segments of successive lengths

$$1, 2, 4, 5, 8, 10, 14, \ldots$$

with divisions at the points

$$0, 1, 3, 7, 12, 20, 30, 44, \ldots$$

the segmental and divisional series of numbers respectively.

Observe that *every* number can be obtained

  (i) by adding successive numbers of the segmental series;

  (ii) by taking the difference of *two* properly selected numbers
of the divisional series.

The first 347 numbers of the segmental series are now given*.
This is the complete number $\leqslant 1000$. The corresponding divisional
numbers are also given.

In the segmental series the longest sequence that appears is
one of five from 629 to 633; the longest sequence that is absent
is one of ten from 448 to 457. In the first hundred 42 numbers
appear; in the tenth hundred there are 31. The density of the num-
bers appears to diminish very slowly from the results for the ten
hundreds:

1	2	3	4	5	6	7	8	9	10
42	33	32	39	37	33	35	32	33	31

Every integer is the result of the addition of a consecutive set
of numbers of the series, but the formation is not in every case
unique. The earliest example occurs in regard to the number 29
which appears as

$$2 + 4 + 5 + 8 + 10 \quad \text{and as} \quad 14 + 15.$$

In the classical theory of numbers we have on the one hand the
set of primes which serve to express every integer uniquely, and on
the other the set of square numbers with the theorem that every
integer is expressible, but not uniquely, as the sum of four or
fewer numbers of the set. The system of numbers before us seems
to present analogies with both of these theories. Since we have
already in the system of numbers $1, 2, 2^2, 2^3, \ldots$ what may be termed
the prime numbers of addition which involve the unique con-
struction by addition of every integer, it is not to be expected that
in the present question there will be the same unique character.
What is required is some method of dealing with the series of
numbers analytically. The divisional numbers are also available.

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* This number has been reduced to 42 on account of the cost of printing.

It would be interesting to determine how either set of numbers behaves at great distances from the origin of measurement.

Prime Numbers of Measurement.

Scale numbers.					
Ordinal number	Segmental number	Divisional number	Ordinal number	Segmental number	Divisional number
1	1	1	22	46	501
2	2	3	23	48	549
3	4	7	24	49	598
4	5	12	25	50	648
5	8	20	26	53	701
6	10	30	27	57	758
7	14	44	28	60	818
8	15	59	29	62	880
9	16	75	30	64	944
10	21	96	31	65	1009
11	22	118	32	70	1079
12	25	143	33	77	1156
13	26	169	34	80	1236
14	28	197	35	81	1317
15	33	230	36	83	1400
16	34	264	37	85	1485
17	35	299	38	86	1571
18	36	335	39	90	1661
19	38	373	40	91	1752
20	40	413	41	92	1844
21	42	455	42	100	1944

Note on Dr Burnside's recent paper on errors of observation. By Mr R. A. FISHER, Fellow of Gonville and Caius College.

[*Received* 16 July 1923.]

That branch of applied mathematics which is now known as Statistics has been gradually built up to meet very different needs among different classes of workers. Widely different notations have been employed to represent the same relations, and still more widely different methods of treatment have been designed for essentially the same statistical problem. It is therefore not surprising that Dr Burnside* writing on errors of observation in 1923 should have overlooked the brilliant work of "Student" in 1908†, which largely anticipates his conclusion.

Student's work is so fundamental from the theoretical standpoint, and has so direct a bearing on the practical conclusions to be drawn from small samples, that it deserves to be far more widely known than it is at present.

A set of n observations is regarded as a random sample from an indefinitely large population of possible observations, which population obeys the normal, or Gaussian, law of error, and is therefore characterised by two parameters, m, the mean, and σ, the standard deviation. The latter is related to the "precision constant," h, by the equation

$$h = \frac{1}{2\sigma^2},$$

and it is a matter of indifference, provided we steer clear of all assumptions as to *a priori* probability, which parameter is used. The frequency of observations in the range dx is given by

$$df = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx.$$

It is essential to remember that both m and σ are necessarily unknown; all that is known is the set of observations $x_1, x_2, \ldots x_n$. From these certain statistics may be calculated, which may be regarded as estimates of the unknowns, but are not to be confused

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* W. Burnside (1923), "On errors of observation," *Proceedings of the Cambridge Philosophical Society*, 21, pp. 482–7.

† Student (1908), "The probable error of a mean," *Biometrika*, 6, pp. 1–25.

with, or substituted for, them. For the normal distribution we have the two familiar statistics

$$\bar{x} = \frac{1}{n} S(x),$$

$$s^2 = \frac{1}{n} S(x - \bar{x})^2.$$

For each sample of n observations we shall obtain generally a different pair of values of $\bar{x}$ and s. In order to draw correct conclusions from any observed pair of values, it is necessary to know how these values are distributed in different samples from a single population.

If we regard the observations $x_1, x_2, \ldots x_n$ as coordinates in n-dimensional space, any set of observations will be represented by a single point, and the frequency element, in any volume element $dx_1 dx_2 \ldots dx_n$, will be

$$\frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2} S(x-m)^2} dx_1 dx_2 \ldots dx_n.$$

This may be expressed in terms of the statistics $\bar{x}$ and s by recognising the geometrical meaning of these two quantities, for if P be the point $(x_1, x_2, \ldots x_n)$, and PM be drawn perpendicular to the line

$$x_1 = x_2 = \ldots = x_n,$$

then PM will lie in the "plane" space, determined by $\bar{x}$,

$$S(x) = n\bar{x},$$

and M will be the point $(\bar{x}, \bar{x}, \ldots \bar{x})$.

Hence we see that $\bar{x}$ is constant in plane regions perpendicular to a fixed straight line, and the distance of M from the origin is $\bar{x}\sqrt{n}$; also that the distance PM is $s\sqrt{n}$, so that, for given values of $\bar{x}$ and s, P lies on a sphere in $n-1$ dimensions, of radius proportional to s; therefore the volume corresponding to $d\bar{x}\,ds$ will be proportional to

$$s^{n-2} ds\, d\bar{x},$$

and will be a region of constant density, proportional to

$$e^{-\frac{1}{2\sigma^2} S(x-m)^2}$$

$$= e^{-\frac{n}{2\sigma^2}(\bar{x}-m)^2} . e^{-\frac{ns^2}{2\sigma^2}}.$$

The frequency with which $\bar{x}$ and s fall into assigned elementary ranges $d\bar{x}, ds$ is therefore proportional to

$$e^{-\frac{n}{2\sigma^2}(\bar{x}-m)^2} d\bar{x} . s^{n-2} e^{-\frac{ns^2}{2\sigma^2}} ds,$$

from which it appears that the distribution of the two quantities is wholly independent, that of $\bar{x}$ being

$$df = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}}\, e^{-\frac{n}{2\sigma^2}(\bar{x}-m)^2}\, d\bar{x} \qquad \dots\dots(\text{I}),$$

and that of s

$$df = \frac{n^{\frac{1}{2}(\mathbf{n}-1)}}{2^{\frac{1}{2}(n-3)} \cdot \dfrac{n-3}{2}!} \cdot \frac{s^{n-2}}{\sigma^{n-1}}\, e^{-\frac{ns^2}{2\sigma^2}}\, ds \qquad \dots\dots(\text{II}).$$

It will be observed that the distributions both of $\bar{x} - m$ and of s depend upon σ, and, if σ is unknown, are not of direct service; but in statistical practice, including the practices ordinarily applied to errors of observation, it is the ratio of these two quantities which is of importance. If now

$$z = \frac{\bar{x}-m}{s},$$

we may substitute sz for $\bar{x} - m$, and $s\,dz$ for $d\bar{x}$, so that the simultaneous distribution of s and z is

$$df = \frac{n^{\frac{1}{2}n}}{2^{\frac{1}{2}(n-2)} \cdot \dfrac{n-3}{2}!\,\sqrt{\pi}} \cdot \frac{s^{n-1}}{\sigma^n}\, e^{-\frac{ns^2}{2\sigma^2}(1+z^2)}\, ds\,dz,$$

and integrating with respect to s from 0 to ∞, we have for the distribution of z

$$df = \frac{\dfrac{n-2}{2}!}{\dfrac{n-3}{2}!\,\sqrt{\pi}} \cdot \frac{dz}{(1+z^2)^{\frac{1}{2}n}} \qquad \dots\dots(\text{III}).$$

The distributions of s, (II), and of z, (III), were given by Student in 1908.

The traditional treatment of the probable error of the mean depends upon the distribution of $\bar{x}$, (I). The mean varies about its population value, m, in a normal distribution, with standard deviation $\sigma/\sqrt{n}$. If, therefore, σ were known, we could accurately assign to $\bar{x}$ the probable error, $\cdot6745\sigma/\sqrt{n}$, and test whether the observed value, $\bar{x}$, were in accord with any hypothetical value, m, by means of the probability integral of the normal curve

$$P = \int_x^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}t^2}\, dt, \quad x = \frac{(\bar{x}-m)\sqrt{n}}{\sigma}.$$

But if, in fact, σ is not known, and we only have an estimate of σ, such as s, then the above reasoning collapses, for the distribution of

$$\frac{\bar{x}-m}{s} = z$$

is not a normal distribution; the "probable error," whether calculated as the quartile distance, or as a conventional multiple of the standard deviation, ceases to supply a test of the significance of the departure of $\bar{v}$ from its hypothetical value, m. Such a test is supplied by the probability integral of the Type VII curve, which gives the actual distribution of z, that is by

$$P = \int_z^\infty \frac{\dfrac{n-2}{2}!}{\dfrac{n-3}{2}! \sqrt{\pi}} \cdot \frac{dt}{(1+t^2)^{\frac{1}{2}n}}.$$

Tables of this integral, for different values of z and n, have been given by Student * in 1917. Fuller tables are now in course of preparation. The slight difference between the above formula and that given by Dr Burnside is traceable to Dr Burnside's assumption of an *a priori* probability for the precision constant, whereas Student's formula gives the actual distribution of z in random samples.

* Student (1917), "Tables for estimating the probability that the mean of a unique sample of observations lies between $-\infty$ and any given distance of the mean of the population from which the sample is drawn," *Biometrika*, 11, pp. 414–17.

The fundamental theorem of Denjoy integration. By J. C. BURKILL, Trinity College.

The object of this note is to give a simple proof that the Denjoy integral has almost everywhere the integrand as derivative. The argument depends on a lemma first stated—in a slightly different form—by Vitali†.

VITALI'S LEMMA. *Let E be a linear set of finite outer measure.*

Suppose that with each point p of E is associated one or more closed intervals containing p as an interior or end point.

Then, given ϵ, we can choose a set $\mathcal{E}$ of a finite number of the associated intervals, non-overlapping, such that

$$m\mathcal{E} > \tfrac{1}{3}m^*E - \epsilon.$$

Let l_1 be the upper bound of the lengths of intervals associated with points of E. If $l_1 \geqslant \tfrac{1}{3}m^*E$, we can choose a single interval to satisfy the required inequality. If $l_1 < \tfrac{1}{3}m^*E$, choose an associated interval I_1 of length greater than $l_1 - \tfrac{1}{4}\epsilon$.

Then the set G_1 of points of E with which are associated intervals having points common with I_1 is included in an interval of length at most $3l_1$.

Let l_2 be the upper bound of the lengths of intervals associated with points of $E - G_1$. Choose an interval I_2 of length greater than $l_2 - \tfrac{1}{8}\epsilon$.

Then the set G_2 of points of $E - G_1$ with which are associated intervals having points common with I_2 is included in an interval of length at most $3l_2$.

Repeat this process. If, for some value of n, the set G_{n+1} contains no points, then

$$E \subset \sum_1^n G_i,$$

and so
$$m^*E \leqslant 3\sum_1^n l_i.$$

Therefore
$$m\sum_1^n I_i > \sum_1^n l_i - \epsilon \geqslant \tfrac{1}{3}m^*E - \epsilon.$$

If no G_n is null, we have two cases:

(i) Σl_i diverges,

(ii) Σl_i converges.

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† Vitali, "Sui gruppi di punti...," *Atti di Torino*, 43, p. 229 (1908). Also Carathéodory, *Vorlesungen über reelle Funktionen*, p. 299.

In case (i), choose n to make $\overset{n}{\underset{1}{\Sigma}} l_i > \frac{1}{3} m^* E$ and we have the result.

In case (ii),
$$\lim_{i \to \infty} l_i = 0.$$

Therefore
$$E \subset \overset{\infty}{\underset{1}{\Sigma}} G_i.$$

We have then
$$m^* E \leqslant 3 \overset{\infty}{\underset{1}{\Sigma}} l_i.$$

Hence
$$\overset{\infty}{\underset{1}{\Sigma}} m I_i > \overset{\infty}{\underset{1}{\Sigma}} l_i - \tfrac{1}{2} \epsilon \geqslant \tfrac{1}{3} m^* E - \tfrac{1}{2} \epsilon.$$

Choose a finite set $\mathscr{E}$ of the I_n such that
$$m\mathscr{E} > \tfrac{1}{3} m^* E - \epsilon,$$
and the lemma is proved.

Corollary 1. Suppose that with each point of E is associated a *sequence* of closed intervals whose lengths tend to zero.

If O is any open set containing E, then we can choose $\mathscr{E}$ such that
$$\mathscr{E} \subset O.$$

If, further, O is chosen so that
$$m O < m^* E + \epsilon,$$

then
$$m_* (\mathscr{E} - \mathscr{E} E) < \epsilon.$$

Corollary 2. The lemma is true in q dimensions if we work with q-dimensional cubes or spheres instead of intervals and replace the fraction $\frac{1}{3}$ by $1/3^q$.

We shall use the following notation:

P is a perfect set contained in (a, b).

(a_n, b_n) or u_n are the black intervals of P.

In each u_n is defined a function $y_n(x)$ such that
$$y_n(a_n) = 0,$$
$$v_n = y_n(b_n),$$
$$w_n = \text{the oscillation of } y_n(x) \text{ in } u_n.$$

We define $F(x)$ in (a, b):

If x is a point of P, $F(x) = \Sigma y_n(b_n)$ summed for values of n for which $b_n \leqslant x$.

If x is a point of (a_m, b_m), $F(x) = F(a_m) + y_m(x)$.

THEOREM 1. *If Σw_n converges, then $F'(x) = 0$ at almost every point of P.*

Let E be the set of points of P at which it is not true that $F'(x) = 0$.

For each positive integer k, let E_k be the set of points of P at which any derivate of $F(x)$ is numerically greater than $1/k$.

Then $E_k \subset E_{k+1}$ and $E = \lim_{k \to \infty} E_k$.

To prove that $mE = 0$, we have to prove that for each k

$$mE_k = 0.$$

Suppose that there is a value of k for which $m^*E_k > 0$. Given ϵ, choose N such that $\sum_{N+1}^{\infty} w_n < \epsilon$.

Remove from E_k any end-points of the intervals $u_1, \ldots u_N$; the outer measure is unchanged.

Then with each remaining point x_0 of E_k is associated a sequence of intervals (x_0, x), arbitrarily small, for which

$$(1) \qquad |F(x) - F(x_0)| > \frac{1}{k}|x - x_0|,$$

(2) (x_0, x) has no point common with any u_n for $n \leqslant N$.

Choose according to Vitali's lemma a finite set (c_i, d_i) $(i = 1, \ldots r)$ of the associated intervals, non-overlapping, lying within (a, b) and of total measure greater than

$$\tfrac{1}{3}m^*E_k - \epsilon.$$

From (1),

$$\Sigma |F(d_i) - F(c_i)| > \frac{1}{k}\Sigma |d_i - c_i| > \frac{1}{k}(\tfrac{1}{3}m^*E_k - \epsilon).$$

But from (2), $\quad \Sigma |F(d_i) - F(c_i)| \leqslant \sum_{N+1}^{\infty} w_n < \epsilon.$

Therefore $\qquad m^*E_k < 3(k+1)\epsilon.$

Since ϵ is arbitrary we must have $m^*E_k = 0$.

This is the main theorem in the proof that if $F(x)$ is the integral $\int_a^x f(x)\,dx$ according to the original definition of Denjoy[†], then $F'(x) = f(x)$ almost everywhere.

Denjoy's proof of Theorem 1 is based on the properties of a system of intervals of which no three intervals have common points[‡].

Khintchine[§] observed that the definition of $F(x)$ has a meaning under the wider condition of the convergence of $\Sigma|v_n|$ and stated a necessary and sufficient condition that the conclusion of Theorem 1 should be true, namely

† *Comptes Rendus*, 154, pp. 859, 1075 (1912).
‡ *Jour. de math.*, ser. 7, tome 1, p. 138 (1915). The argument is reproduced by Hobson, *Theory of Functions*, 2nd edition, 1, p. 636.
§ *Comptes Rendus*, 162, p. 287 (1916).

THEOREM 2. *If* (1) $\Sigma \mid v_n \mid$ *converges, then the necessary and sufficient condition that* $F'(x) = 0$ *at almost every point of* P *is that at almost every point* p *of* P, (2) $\lim\limits_{n \to \infty} \dfrac{w_n}{d_n} = 0$, *where* d_n *is the distance of* p *from* u_n.

The proof that the condition is necessary is immediate.

In order to prove it sufficient, as in Theorem 1, we have only to shew that if E_k is the set of points of P at which any derivate of $F(x)$ is numerically greater than $1/k$, then $mE_k = 0$.

Suppose that there is a value of k for which $m^*E_k > 0$. Given ϵ, choose N such that $\sum\limits_{N+1}^{\infty} \mid v_n \mid < \epsilon$.

Remove from E_k any end-points of the intervals $u_1, \dots u_N$ and any points at which the hypothesis (2) is untrue; the outer measure is unchanged.

Then with each remaining point x_0 of E_k is associated a sequence of intervals (x_0, x), arbitrarily small, for which

(1) $\mid F(x) - F(x_0) \mid > \dfrac{1}{k} \mid x - x_0 \mid$,

(2) (x_0, x) has no point common with any u_n for $n \leqslant N$,

(3) $\mid w_n \mid < \epsilon \mid x - x_0 \mid$ if (x_0, x) has a point common with u_n.

Choose according to Vitali's lemma a finite set (c_i, d_i) $(i = 1, \dots r)$ of the associated intervals, non-overlapping, lying within (a, b) and of total measure greater than

$$\tfrac{1}{3} m^*E_k - \epsilon.$$

From (1),

$$\Sigma \mid F(d_i) - F(c_i) \mid > \frac{1}{k} \Sigma \mid d_i - c_i \mid > \frac{1}{k} (\tfrac{1}{3} m^*E_k - \epsilon).$$

But from (2),

$$\Sigma \mid F(d_i) - F(c_i) \mid \leqslant \sum_{N+1}^{\infty} \mid v_n \mid + \epsilon \Sigma \mid d_i - c_i \mid < \epsilon (1 + b - a).$$

Therefore $\qquad m^*E_k < 3\epsilon \{k + 1 + k(b - a)\}$.

Since ϵ is arbitrary we must have $m^*E_k = 0$.

Denjoy's original definition of the integral was extended independently by Khintchine and by Denjoy himself. It is not true that an integral according to the extended definition has at almost every point a derivative equal to the integrand. Khintchine however discovered that the statement is true if we replace the ordinary derivative by an *approximate (or asymptotic)* derivative.

Definition. $F(x)$ has at x_0 an approximate derivative $\phi(x_0)$ if there exists a measurable set E whose density at x_0 is 1 such that

$$\frac{F(x) - F(x_0)}{x - x_0} \to \phi(x_0)$$

as $x \to x_0$ by values in E.

We then have the following analogue of Theorem 1.

THEOREM 3. *If* $\Sigma |v_n|$ *converges, then at almost every point of* P, $F(x)$ *has approximate derivative zero.*

Let E_k be the set of points x_0 of P at which

$$|F(x) - F(x_0)| > \frac{1}{k}|x - x_0|$$

for a set of points x whose density at x_0 is greater than $1/k$.

Then we have to prove that $mE_k = 0$.

Suppose that there is a value of k for which $m^*E_k > 0$. Given ϵ, choose N such that $\sum\limits_{N+1}^{\infty} |v_n| < \epsilon$.

Remove from E_k any end-points of the intervals $u_1, \ldots u_N$, and any points at which P has not density 1; the outer measure is unchanged.

Then with each remaining point x_0 of E_k is associated a sequence of intervals (x_0, x), arbitrarily small, for which

(1) $|F(x) - F(x_0)| > \frac{1}{k}|x - x_0|$,

(2) (x_0, x) has no point common with any u_n for $n \leqslant N$,

(3) x is a point of P.

Choose according to the lemma a finite set (c_i, d_i) $(i = 1, \ldots r)$ of the associated intervals, non-overlapping, and of total measure greater than

$$\tfrac{1}{3}m^*E_k - \epsilon.$$

From (1),

$$\Sigma |F(d_i) - F(c_i)| > \frac{1}{k}\Sigma |d_i - c_i| > \frac{1}{k}(\tfrac{1}{3}m^*E_k - \epsilon).$$

But from (2) and (3),

$$\Sigma |F(d_i) - F(c_i)| \leqslant \sum\limits_{N+1}^{\infty} |v_n| < \epsilon.$$

Hence $m^*E_k = 0$, and the theorem is proved.

Extensions of a theorem of Segre's, and their natural position in space of seven dimensions. By Mr C. G. F. JAMES, Trinity College, Cambridge.

It is a well-known theorem of four-dimensional geometry that *the planes which meet four lines also meet a fifth line, determined by the first four*[*]. In this paper we prove a similar theorem, namely that *those trisecant planes of a rational normal quartic in* [4][†], *which meet a line, all meet a second such curve*; and reciprocally all trisecant planes of this second curve which meet the same line, also meet the first. We shall see that this is one of a set of associated theorems, one in each space from [4] to [7] (inclusive), any one of which is deducible from any one of the others, but that, for reasons which will appear, the discussion in [7] is much the simplest, and as all the others can be deduced by taking suitable sections, it must be regarded as the fundamental one of the set. We then see how it is possible to deduce from these theorems the above-mentioned theorem of Segre's, and in fact it is possible to derive in a few lines the dual figure to that of the configuration of lines on the 10-nodal cubic form[‡] in [4]. We denote by a form a construct in [n] of $n-1$ dimensions. It is also possible to obtain a number of intermediate cases.

§ 1. The equation

$$\left\| \begin{array}{cccc} a_{1x} & a_{2x} & a_{3x} & a_{4x} \\ b_{1x} & b_{2x} & b_{3x} & b_{4x} \end{array} \right\| = 0, \qquad \ldots\ldots(1)$$

(the elements being linear functions of $x_1, \ldots x_{r+1}$), represents in [r] a quartic construct of dimension $(r-3)$, say[§] V^4_{r-3}; for its section by a [3] consists of four points. V^4_{r-3} is an [$r-4$]-scroll, since it contains the [$r-4$] given by $a_{ix} + \lambda b_{ix} = 0$ $(i = 1, \ldots 4)$, as λ varies. It further contains, apart from the lines in these [$r-4$], ∞^3 directrix lines for $r = 7$, and ∞^1 for $r = 6$. Thus in the first case the equation may be taken to be

$$\left\| \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{array} \right\| = 0,$$

* Segre, "Sull' Incidenza di rette e piani...," *Rend. Circ. Mat. di Palermo*, t. II, 1888, p. 45, n. 4.

† We shall denote by [a] a linear space of dimension a.

‡ Segre, "Sulle varietà cubiche...," *Mem. Torino*, ser. (2), XXXIX, 1889, p. 3, n. 24.

§ V^β_α denotes a construct of dimension α and order β.

and contains the lines joining the points $(\alpha, \beta, \gamma, \delta, 0, 0, 0, 0)$ and $(0, 0, 0, 0, \alpha, \beta, \gamma, \delta)$, for any values of $\alpha \ldots \delta$. So also in the second case it may be taken as

$$\left\| \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_5 & x_6 & x_7 \end{array} \right\| = 0,$$

and contains the lines joining $(0, \beta, \gamma, 0, 0, 0, 0)$ and $(0, 0, 0, 0, \beta, \gamma, 0)$. In each case these lines meet every generating $[r-4]$, and no two of them can meet each other.

There is no need to consider spaces of dimension greater than seven, as we have only eight linear forms at our disposal. In such spaces the matrix (1) will represent a conical construct, and the theorems obtained will be, in the abstract sense, the same as for [7].

Now the equations

$$\overset{4}{\underset{1}{\Sigma}} \alpha_i a_{ix} = \overset{4}{\underset{1}{\Sigma}} \alpha_i b_{ix} = 0 \qquad \ldots\ldots(2)$$

represent a $[r-2]$, which cuts V^4_{r-3} in a V_{r-4} of the third order, for it cuts V^4_{r-3} where it cuts the V^3_{r-2} given by

$$\left\| \begin{array}{ccc} a_{2x} & a_{3x} & a_{4x} \\ b_{2x} & b_{3x} & b_{4x} \end{array} \right\| = 0.$$

We shall call these $[r-2]$ the cubic-$[r-2]$ of the V^4_{r-3}. Further (2) represents the most general such $[r-2]$, and their system ∞^3 is seen to be birationally representable on a [3], which we shall call $S_{(a)}$. Now the cubic-$[r-2]$ which pass through a fixed point (x) are represented by lines in $S_{(a)}$, as is immediately seen, but as the lines of $S_{(a)}$ are ∞^4 only, *it follows that these $[r-2]$ must all pass through the same set of ∞^{r-4} points*, which must be linear. It can in fact be immediately seen to be that $[r-4]$ through (x) which cuts V^4_{r-3} in a V^2_{r-5}. Thus in [5] the section of the V^4_2 by a cubic space through (x) being a cubic curve, one of its chords passes through (x). Now this chord is unique*, and must therefore be common to all the spaces. From this the general case follows at once. In the case of the figure in [4], there is no corresponding result; but if (x) should lie on the cubic form of chords of the quartic curve† concerned, then the corresponding planes will pass through the chord on which (x) lies.

We pass on at once to consider *those cubic-$[r-2]$ which meet a line l*, which will be taken to join the points $(\xi)(\eta)$. The $\infty^2[r-2]$ concerned will pass by ∞^1's through the $[r-4]$, which, passing through the single points of l, meet V^4_{r-3} in V^2_{r-5}, and

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* Or at most one of a finite number, but the very argument shews that it is necessarily unique.

† Segre, "Sulle varietà...," n. 43.

which will be seen to generate a second construct $^{(1)}V_{r-3}^4$ (which we shall call *the projecting construct*). In [7] *this V_4^4 contains ∞^3 lines, as we have seen, any one of which can replace l, and is met by all the ∞^2* [5], *and is of general character.* Similarly in [6] we have ∞^1 such lines, and the V_3^4 is of general type. In [5] we have V_2^4, but which is no longer of general character, in that it possesses a rectilinear directrix. *Finally in* [4] *the projecting construct is replaced by the degenerate quartic composed of l and the three chords of C_4 meeting l*.* Hence generally the V_{r-4} common to V_{r-3}^4 and its projecting construct is of the sixth order, for that for the figure in [4] arises by a suitable section.

Now the ∞^2 cubic-spaces meeting a line will be represented in $S_{(4)}$ by points on the quadric†

$$\begin{vmatrix} \Sigma\alpha_i a_{i\xi} & \Sigma\alpha_i b_{i\xi} \\ \Sigma\alpha_i a_{i\eta} & \Sigma\alpha_i b_{i\eta} \end{vmatrix} = 0, \qquad i = 1, \dots 4,$$

which is not in general a cone, and hence contains two systems of generators. To the single lines of one correspond $[r-2]$ through the $[r-4]$ of the projecting construct, while *to the lines of the second correspond sets of* $\infty^1 [r-2]$ *passing through the $[r-4]$ of a second series generating a second V_{r-3}*. This second construct is also of the fourth order, as we shall shew, and hence we may finally state as the results to be proved:

I. *Those cubic-$[r-2]$ of a quartic $[r-4]$-scroll in $[r]$ which meet a line*, and therefore ∞^i lines, all skew ($i = 3$ for $r = 7$; $i = 1$ for $r = 6$, and zero for $r < 6$), *also pass through the $[r-4]$ of a second such scroll* $^{(2)}V_{r-3}^4$.

II. *The relation of the two scrolls is symmetrical* in that the cubic-$[r-2]$ of the second scroll meeting any one of the ∞^1 lines, meet them all, and meet the first scroll in $[r-4]$‡.

III. *Further in* [7] *and* [6] *the relations of the three scrolls, V, $^{(1)}V$, $^{(2)}V$, are mutually symmetrical*, in that deriving from any one of them, the associated construct with respect to a directrix line of a second, we get the third. This will not hold in the lower spaces, unless the original V is of special character§.

IV. *The three constructs meet in a V_{r-4}^6.* Two of them may

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* The normal quartic is of rank 6.
† We now see a direct geometrical verification of the multiplicity of directrix lines on a quartic $[r-4]$-scroll in [6] or [7], since the number of lines in [6], [7] is ∞^{10}, ∞^{12} respectively, while the quadrics of [3] are ∞^9 only. Hence each such must arise for (at least) ∞^1, ∞^3 lines respectively.
‡ This second part follows generally from any one case, when we have proved the first, but the first part can only be inferred directly or from the theorem for a *higher* space.
§ It holds in a partial sense. Thus in [4] we have our two curves C_4, C_4' and the third curve consists of the line l and the three common chords meeting it. All planes meeting these chords and C_4 (or C_4') meet C_4 (or C_4).

coincide giving rise to some limiting cases, but it is not possible for all three to do so.

§ 2. It is clear that we may deduce all these theorems by taking a section of the figure in [7]. However the case of [4] presents certain specialities, and on account of its special interest, we propose to prove some at least of these theorems directly. We shall then omit the intermediate cases, the direct methods being seen immediately, and discuss the case of [7]. We may take as the equation of our V_{r-3}^4

$$\left\| \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ x_{r-2} & x_{r-1} & x_r & x_{r+1} \end{array} \right\| = 0,$$

and *our quartic curve C_4 in* [4] is accordingly

$$\left\| \begin{array}{cccc} x_1 & x_3 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{array} \right\| = 0. \qquad \ldots\ldots(3)$$

The trisecant planes are given by

$$\sum_1^4 \alpha_i x_i = \sum_1^4 \alpha_i x_{i+1} = 0, \qquad \ldots\ldots(4)$$

and those meeting a line $(\xi)\,(\eta)$ are represented by

$$\left| \begin{array}{cc} \sum\limits_1^4 \alpha_i \xi_i & \sum\limits_1^4 \alpha_i \xi_{i+1} \\ \sum\limits_1^4 \alpha_i \eta_i & \sum\limits_1^4 \alpha_i \eta_{i+1} \end{array} \right| = 0. \qquad \ldots\ldots(5)$$

These quadrics form an ∞^6 contained in a linear ∞^8, since the equation is linear in the coordinates p_{ij} of l. This need not be written down. The second regulus of lines is given by

$$\sum_1^4 \alpha_i (\xi_i + \mu \xi_{i+1}) = \sum_1^4 \alpha_i (\eta_i + \mu \eta_{i+1}) = 0. \qquad \ldots\ldots(6)$$

It will be possible to identify this line with (4), giving a point through which pass ∞^1 of the trisecant planes, one for each point of l. This gives (for $i = 2, 3, 4$):

$$\left. \begin{array}{l} x_1 = \xi_1 + \mu \xi_2 + \tau (\eta_1 + \mu \eta_2), \\ x_i = \xi_i + \mu \xi_{i+1} + \tau (\eta_i + \mu \eta_{i+1}) = \kappa (\xi_{i-1} + \mu \xi_i) + \lambda (\eta_{i-1} + \mu \eta_i), \\ x_5 \qquad\qquad\qquad\qquad\quad = \kappa (\xi_4 + \mu \xi_5) + \lambda (\eta_4 + \mu \eta_5). \end{array} \right\} \\ \ldots\ldots(7)$$

Solving the two sets of four equations:

$$\left. \begin{array}{l} 1,\ \mu,\ \tau,\ \tau\mu = \| x_1;\ \xi_1\ \xi_2\ \eta_1\ \eta_2 \|/\Delta \equiv X_1, \ldots X_4 \\ \kappa,\ \kappa\mu,\ \lambda,\ \lambda\mu = \| x_2;\ \xi_1\ \xi_2\ \eta_1\ \eta_2 \|/\Delta \equiv Y_1, \ldots Y_4 \end{array} \right\} \ \ldots\ldots(8)$$

where *two conventions* are employed, which we use hereafter without explanation.

(1) The i-th term on the left of each equation is equal to the result of substituting the first column in the $(i+1)$-th. This is indicated by the semi-colon.

(2) A term μ_i stands for a column μ_i, μ_{i+1}, ... μ_{i+3} in this case; in general, terminating with such an element as to give the correct number of rows, and Δ denotes $|\xi_1\ \xi_2\ \eta_1\ \eta_2|$. Eliminating between the (7) we get as locus of (x)

$$\left\| \begin{array}{cccc} X_1 & X_3 & Y_1 & Y_3 \\ X_2 & X_4 & Y_2 & Y_4 \end{array} \right\| = 0.$$

Thus all trisecant planes of a normal quartic C_4 meeting a line meet a second such quartic C_4' once. C_4' cannot meet l, for this would imply a common generator of the two reguli. The number of points in which it meets C_4 is given by the number of planes[*] $\sum_1^4 \alpha_i \theta^{i-1} = 0$, corresponding to trisecant planes through the single points of C_4, which touch the quadric (so as to contain a generator of the second regulus), and this number is six. We shall see that these are the intersections with C_4 of its chords meeting l[†].

The planes which pass through a chord of C_4 are represented by points of a line of the congruence $(3, 1)$ of lines which lies in two osculating planes $\Sigma a_i \theta^{i-1} = 0$, $\Sigma a_i \phi^{i-1} = 0$ of a certain cubic curve. This result we shall use later.

§ 3. *For the lines of a quartic system ∞^5 of lines given by expressing that (5) reduces to a cone, the second regulus coincides with the first*, and the second quartic disappears. In its analytical expression it will degenerate into the line l, and the three chords of C_4 meeting l. The system consists, in fact, of all lines lying in a trisecant plane.

§ 4. *In order to prove our reciprocal theorem, namely that the trisecant planes of C_4' which meet l also meet C_4*, we require a new determination of C_4'. If we solve the three middle equations in (7) for τ, κ, λ and substitute, we obtain a parametric representation of the curve C_4', namely

$$\left. \begin{array}{l} x_i = B\,(\xi_i + \mu\xi_{i+1}) + D\,(\eta_i + \mu\eta_{i+1}), \quad i = 1, \ldots 4 \\ x_5 = -\,A\,(\xi_4 + \mu\xi_5) - C\,(\eta_4 + \mu\eta_5) \end{array} \right\}, \quad \ldots\ldots(9)$$

where $A : B : C : D = \| \alpha_1 + \mu\alpha_2 \quad \alpha_2 + \mu\alpha_3 \quad \beta_1 + \mu\beta_2 \quad \beta_2 + \mu\beta_3 \|$.

---

[*] Since for a point on C_4, $x_i = \theta^{i-1}$.

[†] We can also get this number as follows. The number of trisecant planes through a general line is zero, the number through a point P meeting a line is two. Suppose C_4' is a quartic meeting C_4 x times, and suppose it breaks into four lines with three intersections. Then the number of trisecant planes through P meeting C_4' is $8 - 3a - xb$, where a equals the number of planes through a line and b equals the number through a unisecant of C_4 elsewhere bisecant, or $8 - x\,(h-2) = 8 - x$ (h being the number of apparent nodes), and this number must equal two.

It is now practically essential to choose a special system of reference. The systems of reference giving the canonical parametric representation of C_4 are ∞^2 in number, namely, we select two points of C_4 to correspond to zero and infinity. If these points be taken as the points of contact of two osculating planes meeting l, we may take as coordinates of (ξ) and (η)

$$(a,\ a',\ 1,\ 0,\ 0)\ ;\ (0,\ 0,\ 1,\ b',\ b).\qquad \ldots\ldots(10)$$

We shall merely sketch the subsequent algebra. We find

$$x_1 = -a^2 - (a^2b' + aa')\,\mu + (a^2\beta - aa'b')\,\mu^2 + (2aa'\beta + ab')\,\mu^3 + a'\,(a'\beta + b')\,\mu^4,$$
$$x_2 = -aa' - (aa'b' + a'^2)\,\mu + (aa'\beta - a'^2b')\,\mu^2 + (a + a'^2)\,\beta\mu^3 + a'\beta\mu^4,$$
$$x_3 = -a'^2 + b'\,(a - a'^2)\,\mu + (ab - a'b' - a'^2b'^2)\,\mu^2 + (2a'\beta - a'b)\,\mu^3 - b'^2\mu^4,$$
$$x_4 = b'a + (b + b'^2)\,a\mu + (bb'a - b'^2a')\,\mu^2 - (bb'a' + b'^2)\,\mu^3 - bb'\,\mu^4,$$
$$x_5 = b'\,(b'a + a') + (2bb'a + a'b)\,\mu + (b^2a - bb'a')\,\mu^2 - (b^2a' + bb')\,\mu^3 - b^2\mu^4.\ \ldots\ldots(11)$$

x_1, x_2, x_3 pass into x_5, x_4, x_3 by interchange of a and b, a' and b', α and β, and $1/\mu$ and μ. In them we have used $\alpha = a - a'^2$, $\beta = b - b'^2$, and we shall later use [*]

$$\Delta \equiv a'^2b'^2 - ab'^2 - ba'^2 + ab - a'b'.$$

The determinant of these expressions reduces to Δ^5, so that $\Delta = 0$ would appear to be the condition that C_4' shall be a space quartic, but actually the condition that (5) shall be a cone (see § 4) is $\Delta^2 = 0$; so that C_4', if non-degenerate, always belongs to [4].

We have to solve these equations for $1 : \mu : \ldots : \mu^4$. We first define the spaces X_1, X_3, X_5 by

$$\Delta X_1 = ax_2 - a'x_1, \quad \Delta X_5 = bx_4 - b'x_5,$$
$$\Delta X_3 = a'b'x_3 - b'x_2 - a'x_4,$$

so that three of our equations can be replaced by

$$X_1 = a\mu^3 + a'\mu^4, \quad X_5 = b' + b\mu, \quad X_3 = a'\mu + a'b'\mu^2 + b'\mu^3.$$

The solution is now immediate. We find

$$a'\,\Delta\mu = (ab - a'b' - ba'^2)\,X_3 - a'x_4 - bb'X_1 + a'\alpha X_5,$$
$$b'\,\Delta\mu^3 = (ab - a'b' - ab'^2)\,X_3 - b'x_2 - aa'X_5 - b'\beta X_1,$$

and the rest are easily written down. We shall write $\mu^i = \xi_{i+1}$, and we wish to shew that if we derive a system of reference $\eta_1 \ldots \eta_5$ from $\xi_1 \ldots \xi_5$, as the latter was derived from $x_1 \ldots x_5$, using the same line l, it will coincide with the first. Now the determining points (10) for l become in the new system

$$(0,\ 0,\ 1,\ -a',\ a),\ (b,\ -b',\ 1,\ 0,\ 0)$$

and the proof is completed by observing that the relations between the (x) and the (ξ) are unaltered if we interchange a and b, a' and b', and change the signs of the latter pair of letters[†].

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[*] The same Δ as previously defined.

[†] *I.e.* the expressions of the ξ in terms of the x are exactly the same as those of the x in terms of the ξ, with these changes.

§ 5. If we carry through the details of § 4 for the line joining the vertices A_2, A_4 of the pentahedroid of reference we shall find that C_4' coincides with C_4. *We now propose to shew that this is the unique case in which this is so.* It is necessary to replace the points (10) determining l by

$$(a,\ a',\ a'',\ 0,\ 0),\quad (0,\ 0,\ b'',\ b',\ b).$$

This will be allowed for by making our equations everywhere homogeneous in the a, and in the b, by insertion of a'', b''. It must, by hypothesis, be possible to find a linear transformation of μ transforming (11) into

$$x_i = \mu^{i-1}, \qquad\qquad i = 1, \dots 5.$$

In other words the equation $x_i = 0$, as written, must have $i - 1$ equal roots α (say) and $5 - i$ roots β (say).

If we then write

$$\mu' = \rho\,(\mu - \alpha)/(\mu - \beta), \qquad\qquad \dots\dots(\text{A})$$

we shall have

$$x_i = \kappa_i\,(\mu')^{i-1}\,(\mu - \beta)^4/\rho^{i-1}, \qquad i = 1, \dots 5.$$

Hence a second and last set of conditions must be

$$\kappa_i = -\,\rho^{i-1}\tau, \qquad\qquad i = 1, \dots 5.$$

If then we identify the x_i of (11) with $\kappa_i\,(x - \alpha)^{i-1}\,(x - \beta)^{5-i}$ we shall obtain the required conditions. These are **21** in number, but they are not independent, as we know at least one solution exists. We will suppress the details of the identification, but we find that we must have $\alpha = 0$, $\beta = \infty$ or inversely, and that either leads to

$$a = a'' = b = b'' = 0,$$

these conditions being sufficient. We could on geometrical grounds have foreseen that the transformation (A) was necessarily equivalent to the identical transformation of C_4.

The system of ∞^2 trisecant planes meeting l now consists of a series of sets of ∞^1 planes passing through the single lines of a scroll, whose generators meet C_4 once. These lines must be directrices of the chord system of the C_4*.

We have one such line l for each canonical representation of C_4. *The system ∞^2 of these lines consists of those lines which meet two tangents of C_4 such that the point of intersection with either lies in the osculating space* [3] *at the point of contact of the other.* The line lies in the plane common to the osculating spaces. We must also observe that the six osculating planes meeting such a line coincide by threes in the two osculating planes at these points.

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* Cf. Segre, "Sulle varietà...."

If we take θ, ϕ as the parameters of the points of contact of these tangents etc., we find for the points of intersection of l with them respectively,

$$(1, \quad \tfrac{1}{4}(3\theta+\phi), \quad \tfrac{1}{2}\theta(\theta+\phi), \quad \tfrac{1}{4}\theta^2(\theta+3\phi), \quad \phi\theta^3),$$

and

$$(1, \quad \tfrac{1}{4}(\theta+3\phi), \quad \tfrac{1}{2}\phi(\theta+\phi), \quad \tfrac{1}{4}\phi^2(3\theta+\phi), \quad \phi^3\theta).$$

The number of such lines in a space [3] is equal to one-half the number of points* common to the curves represented by the same linear combinations of these two rows of quantities, inserting a homogeneity factor ψ, and taking (ϕ, θ, ψ) as coordinates in a plane, and is therefore three. *The lines themselves describe a form* F, which is given parametrically (a point of F being represented by two points of the space), by

$$x_1=\psi^4, \quad x_2=\psi^3\{\tfrac{1}{4}(3\theta+\phi)+\tfrac{1}{2}\tau\}, \quad x_3=\psi^2\{\tfrac{1}{2}\theta(\theta+\phi)+\tfrac{1}{2}\tau(\theta+\phi)\},$$
$$x_4=\psi\{\tfrac{1}{4}\theta^2(\theta+3\phi)+\tfrac{1}{4}\tau(\theta^2+4\theta\phi+\phi^2)\}, \quad x_5=\phi\theta^3+\tau\phi\theta(\phi+\theta).$$

For a fixed value of θ the locus of the point of intersection of l with the second tangent is a cubic in the osculating space, touching the quartic at (θ), and its points are in $(1, 1)$ correspondence with the tangent, since through each point of the latter passes one osculating space, excluding that of θ. Hence the section contains the quartic scroll generated by the joining lines. All tangents to C_4 belong to F, and the sextic scroll they generate is its only multiple surface S^6, as is seen by considering the section by a plane such as $x_4=x_5=0$. Since one of our lines is represented by a pair of lines for θ, ϕ constant, it is seen that through an arbitrary point of F only one line passes, but through a point of S^6 two pass, namely the tangent and one other. The singularity of S^6 appears to be of some complicated nature, which we have not determined. Finally the order of F is given by considering the intersections of three members of the linear family defined by the parametric representation. These are quartic surfaces passing through $g\,(\psi=\theta=0)$, $g'\,(\psi=\phi=0)$ and having at

$$P\,(\psi=\theta=\phi=0), \quad Q\,(\tau=\psi=\theta=0) \text{ and } R\,(\theta-\phi=\tau+\theta=\psi=0)$$

multiple points of order 3, 2, 1 respectively. Three such surfaces intersect in 26 points, as may be seen by breaking one into four planes, passing respectively through g, g', P, and Q. *Hence* F *is of order* 13.

§ 6. We pass on then to consider the V_4^4 of [7] which we may take as

$$(V_4^4) \qquad\qquad \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{vmatrix} = 0, \qquad\qquad \ldots\ldots(12)$$

the equations $\overset{4}{\underset{1}{\Sigma}}\alpha_i x_i = 0$, $\overset{4}{\underset{1}{\Sigma}}\alpha_i x_{i+4} = 0$ representing [5] which cut V_4^4 in V_3^3. These [5] we shall denote by ϖ.

It is clear that the ϖ through a point (ξ) all pass through a [3] given by

$$x_i = \kappa\xi_i + \mu\xi_{i+4}; \quad x_{i+4} = \kappa'\xi_i + \mu'\xi_{i+4}, \quad (i=1, \ldots 4), \quad \ldots\ldots(13\,a)$$

which we have seen is the [3] through (ξ) cutting V_4^4 in a V_2^2. Generally for a linear relation

$$\Sigma\alpha_i a_i = \Sigma\alpha_i b_i = 0, \qquad\qquad i=1, \ldots 4,$$

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* Since the points are paired so that $\theta=\alpha$, $\theta=\beta$ gives the same line as $\theta=\beta$, $\phi=\alpha$.

we have the ϖ through a [3] given by

$$x_i = \kappa a_i + \mu b_i, \quad x_{i+4} = \kappa' a_i + \mu' b_i.$$

Consider then the ϖ which meet a line l joining (ξ) (η). They are represented by a quadric whose second regulus is

$$\Sigma \alpha_i (\xi_i + \tau \xi_{i+4}) = \Sigma \alpha_i (\eta_i + \tau \eta_{i+4}) = 0,$$

so that *the ∞^2 ϖ in question pass by ∞^1's through the* [3] *of a scroll of* ∞^1 [3]'s *given by*

$$\begin{aligned} x_i &= \kappa (\xi_i + \tau \xi_{i+4}) + \mu (\eta_i + \tau \eta_{i+4}) \\ x_{i+4} &= \kappa' (\xi_i + \tau \xi_{i+4}) + \mu' (\eta_i + \tau \eta_{i+4}) \end{aligned} \Bigg\} . \quad \dots\dots(13)$$

Solving (13)

$$\kappa, \ \kappa\tau, \ \mu, \ \mu\tau = \| \ x_1; \ \xi_1 \ \xi_5 \ \eta_1 \ \eta_5 \ \| \ /\Delta \equiv X_1, \ X_5, \ X_2, \ X_6,$$

$$\kappa', \ \kappa'\tau, \ \mu, \ \mu'\tau = \| \ x_5; \ \xi_1 \ \xi_5 \ \eta_1 \ \eta_5 \ \| \ /\Delta \equiv X_3, \ X_7, \ X_4, \ X_8,$$

where $\Delta = | \ \xi_1 \ \xi_5 \ \eta_1 \ \eta_5 \ |.$

Eliminating we obtain as the scroll in question:

$$^{(2)}V_4^4 \qquad \left\| \begin{matrix} X_1 & X_2 & X_3 & X_4 \\ X_5 & X_6 & X_7 & X_8 \end{matrix} \right\| = 0. \qquad \dots\dots(14)$$

This proves Theorem I of § 1.

We may take the $X_1 \dots X_8$ as defining a new coordinate system, in which (ξ) (η) are given by

$$X_1 = X_7 = 1; \quad X_2 = X_8 = 1,$$

the remainder being zero. *The great simplification which arises in* [7] *is due to the fact that the elements of* (12) *and* (14) *are in each case linearly independent.*

In the same way we find from (13a) that the scroll generated by the [3] through which pass all ϖ through the single points of l is

$$^{(1)}V_4^4 \qquad \left\| \begin{matrix} X_1 & X_5 & X_3 & X_7 \\ X_2 & X_6 & X_4 & X_8 \end{matrix} \right\| = 0, \qquad \dots\dots(15)$$

$^{(1)}V_4^4$ meets V_4^4 in a sextic construct V_3^6, locus of the V_2^2 (as is seen by considering a [4] section).

Repeating the same analysis for $^{(2)}V_4^4$ in association with l we find that its cubic-[5], meeting l, pass through the [3] of the scroll

$$\left\| \begin{matrix} X_1 & X_2 & X_5 & X_6 \\ X_3 & X_4 & X_7 & X_8 \end{matrix} \right\| = 0, \qquad \dots\dots(16)$$

but this is V_4^4 since the elements of its rows are the same linear functions of $x_1 \dots x_4$; $x_5 \dots x_8$ respectively. *Hence the reciprocal Theorem II of* § 1 *is proved.*

It is clear that we shall get the same $^{(2)}V_4^4$ and $^{(1)}V_4^4$ by choosing any one of the ∞^3 lines of the latter. Hence also $^{(1)}V_4^4$ must bear

the same relation to $^{(2)}V_4^4$ as to V_4^4, and meets it in a V_3^6, which is, in fact, the same construct, being the intersection of (14) and the [3]-cone:

$$X_1 X_4 - X_2 X_3 = 0, \qquad \dots\dots(17)$$

since both $^{(1)}V_4^4$ and $^{(2)}V_4^4$ can be defined by (17) and two determinants from (14). *Hence* V_4^4, $^{(1)}V_4^4$ *and* $^{(2)}V_4^4$ *meet in a* V_3^6, generated by ∞^1 V_2^2 of a linear system. To see this consider the case in [5]. We have ∞^1 pairs of points on a curve C_6, such that the joining lines meet l. If this series were not linear, then l would lie in a trisecant plane, and this is not the case, since the trisecant planes consist of the planes in the cubic-[3] only.

This series has four united points, since such is the number of tangents to V_2^4 meeting l*. Hence C_6 is of genus unity†. The same is true of an arbitrary section of V_3^6 of [7]. (See § 9.) This gives Theorem IV of § 1, and includes the theorem stated without proof that C_4 and C_4' cut in the six intersections of their common chords meeting l. V_3^6 cuts the generators of each scroll in V_2^2.

§ 7. Finally we must shew that *if we select a line* l' *of* (*say*) V_4^4, *then* V_4^4 *is the scroll projecting either* $^{(1)}V_4^4$, $^{(2)}V_4^4$, *and that these latter constructs are associated with respect to* l', *precisely as were* V_4^4 *and* $^{(2)}V_4^4$ *with respect to* l. This is Theorem III of § 1. This is a matter of a moment: we need only observe that the line of V_4^4 which corresponds to l on $^{(1)}V_4^4$ [in the sense that they correspond in the collineation:

$$(X_1\,X_2\,X_3\,X_4\,X_5\,X_6\,X_7\,X_8) = (X'_1\,X'_3\,X'_5\,X'_7\,X'_2\,X'_4\,X'_6\,X'_8)]$$

joins the points $X'_1 = X'_6 = 1$ and $X'_3 = X'_3 = 1$, the remainder in each case being zero; and modify the details accordingly.

§ 8. As already remarked we get the corresponding theorems in lower spaces by taking a section through l. Other results are given by taking an arbitrary section. We thus foresee *the existence* (*say*) *in* [4] *of sets of three normal quartics through the same six points, and such that the trisecant planes of any one meeting one of the others meets the third as well.*

Taking the first curve as our initial C_4, the ∞^2 planes in question are represented by points of a quadric, which is now in no way prescribed, so that *we may select two of the curves arbitrarily, the third being thereby determined.* Any two of the curves may coincide (corresponding to a representative cone), but it can be seen on geometrical grounds that all three cannot.

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* Severi, "Sulle intersezioni delle varietà algebriche...," *Mem. Torino*, ser. (2), t. LII, p. 61, 1903, § 10, n. 24, or directly since the projection of V_2^4 from l is a quartic surface with a nodal cubic, and the tangents in question give the cuspidal points of this curve.

† Severi-Löffler, *Vorlesungen über Algebraische Geometrie*, 1921, p. 168.

In this form we can extend the problem to curves of type

$$\| \ a_{ix}^r \quad b_{ix}^s \ \| = 0, \qquad\qquad i = 1, \dots 4,$$

a_x^r denoting an homogeneous polynomial of order r, etc. We have ∞^3 surfaces of order rs, which are $rs(r^2 + rs + s^2)$ secant to the curve, which is of order $(r+s)(r^2 + s^2)$. We shall have ∞^9 sets of pairs of associated curves, such that the surfaces meeting the one also meet the other, but there is no reason to suppose that there are any cases of particular interest.

§ 9. It is suggested by the theory of the Segre configuration from which we started that interesting results might be expected from a consideration of *the tangential system defined by our cubic-[5]*, that is to say the* Ω_2 of [6] each of which passes through one of these [5]. Now such a space will be given by

$$(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4) + \lambda(\alpha_1 X_3 + \alpha_2 X_4 + \alpha_3 X_7 + \alpha_4 X_8) = 0$$

(using the new system of reference), subject to

$$\alpha_1 \alpha_4 - \alpha_2 \alpha_3 = 0.$$

We shall denote the matrix

$$\left\| \begin{array}{cccc} U_i & U_j & U_k & U_l \\ U_\alpha & U_\beta & U_\gamma & U_\delta \end{array} \right\| \text{ by } \left\| \begin{array}{cccc} i & j & k & l \\ \alpha & \beta & \gamma & \delta \end{array} \right\|$$

the U_i being [6]-coordinates corresponding to the (X). We find that our Ω_3 is given by

$$\left\| \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array} \right\| = \left| \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right| = 0,$$

and is therefore the Ω_3^6, dual to the V_3^6 previously met with, and common to the three Ω_4^1,

$$\left\| \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array} \right\| = 0; \quad \left\| \begin{array}{cccc} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{array} \right\| = 0; \quad \left\| \begin{array}{cccc} 1 & 5 & 3 & 7 \\ 2 & 6 & 4 & 8 \end{array} \right\| = 0. \ \dots\dots(18)$$

We find that *we get precisely the same Ω_3^6 starting with any one of V, $^{(1)}V$, $^{(2)}V$ and taking its cubic-[5] meeting a directrix line of either of the other two* (and therefore all such lines of both). It therefore contains three series of ∞^2 Ω_4^1 and by duality three series of ∞^1 Ω_3^2. The envelope of each system of ∞^3 [6] is on the corresponding V_4^1.

It is also worth while to notice *that the* [6] *through all the cubic-[5] of $^{(2)}V_4^1$, V_4^1, $^{(1)}V_4^1$ form the respective systems in* (18), which systems we shall therefore call $^{(2)}\Omega_4^1$, Ω_4^1, $^{(1)}\Omega_4^1$; the single pencils through the individual [5] correspond to the directrices on the V. Conversely *the lines of the V are such that through each of them pass the* [6] *of an Ω_3^2 in the corresponding Ω_4^1 and those of these lines on one of the V which meet a* [5] *base of one Ω_4^1 of a second Ω_4^1 meet them all, and meet all the bases* [5] *of the third. Thus we have (for*

* An Ω_i^j of [n] denotes a system of ∞^1 [n−1] of class j.

example) on [1] V_4^4 ∞^2 *lines, which meet every cubic-*[5] *of both* V_4^4 *and*
[2] V_4^4. These ∞^2 lines lie by ∞^1's in the base [3] of the ∞^1 Ω_3^1
generating Ω_4^4 or [2] V_4^4 which bases are in fact the generating
[3] of V_4^4 or [2] V_4^4, and in fact *they coincide with one of the three*
systems, which we know must exist on V_3^6 (by duality to Ω_3^6).
Inversely it can be verified that the lines of a V_4^4 meeting a [5]
form a V_3^6.

The figure is now completely self-dual, V_4^4 and Ω_4^4, etc. are
respectively the points and tangent [6] of the same construct*, its
cubic-[5] are dual to its lines, and the V_3^6, common to the three
V_4^4, to the common Ω_3^6. There are three systems of ∞^2 lines on
V_3^6, and three systems of ∞^2 Ω_1^1 in Ω_3^6, such that a line of one
system meets all the bases of the two non-corresponding systems
of Ω_1^1.

§ 10. *The study of* V_3^6 (*or* Ω_3^6) *may be completed by the use of*
its parametric representation.

If we write $X_1 = \lambda X_2, = \nu X_3 = \mu X_5$ and use the equations of
the various V_4^4, we obtain as the representation of V_3^6, introducing
a homogeneity factor

$$X_1 = \lambda\mu\nu$$
$$X_2 = \mu\nu\tau, \quad X_3 = \mu\lambda\tau, \quad X_5 = \nu\lambda\tau$$
$$X_4 = \mu\tau^2, \quad X_7 = \nu\tau^2, \quad X_6 = \lambda\tau^2$$
$$X_8 = \tau^3.$$

This representation is birational. To [6]-sections of V_3^6 correspond
cubic surfaces with three base lines $\tau = \lambda = 0$, $\tau = \mu = 0$, $\tau = \nu = 0$
and double base points at their intersections A_μ, A_ν, A_λ. To A_λ
corresponds the quadric $X_3 X_5 = X_1 X_7$, $X_2 = X_4 = X_6 = X_8 = 0$, etc.,
to a point $(0, 1, \alpha, 0)$ of $A_\mu A_\nu$ corresponds the line h_λ given by

$$X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0,$$

thus each one of its points corresponds to every point of this
line, etc.

Inversely, to the point A_1 corresponds every point of $\tau = 0$. *To*
[5]-*sections of* V_3^6 *correspond* the intersection of two such cubic
surfaces, namely sextic curves having A_λ, A_μ, A_ν as actual nodes,
and therefore of genus 1. *Thus the* [4]-*sections of* V_3^6 *are of genus*
unity. Let us now consider some applications of this method.

V_3^6 *contains three sets of* ∞^2 *lines*, represented by lines through
A_μ, A_ν, A_λ respectively, and ∞^6 space cubics given by such cubics
through A_μ, A_ν, and A_λ. There is also a set of ∞^4 such curves
represented by general lines of the [3]. *There are six sets of* ∞^4
conics represented by plane cubics having a node at A_μ (say) and
passing through one other base point, and three sets of ∞^3 repre-
sented by lines meeting $A_\mu A_\nu$, etc.

---

* In the sense that the [6] through the [3] generating V_4^4 form the system Ω_3^6.

There are three systems of quadrics represented by planes through lines such as $A_\mu A_\nu$. The reguli on each belong to distinct systems. Two quadrics of different systems share a line, but two of the same system have no points in common outside the common singular line h_λ. V_3^6 *contains no planes.* These results agree with what would be expected.

§ 11. This duality completely disappears in lower space. We may in each such case consider the system Ω_3 of $[r-1]$ through a cubic-$[r-2]$ by the same method, and in each case it will contain three systems of ∞^2 Ω_1^1, the class of which must be six in all cases (for the process is dual to that of projecting V_3^6 from a point, line, etc., by a V_{10-r}^6 of a conical nature). The case of a $[6]$-section is the most interesting, for the $[6]$ may be selected arbitrarily. In $[6]$ we may take as our V_3^4

$$\left\| \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_5 & x_6 & x_7 \end{array} \right\| = 0, \qquad \ldots\ldots(19)$$

and the tangential system of the cubic-$[4]$ is given by

$$u_i = \alpha_i + \lambda \alpha_{i-3}, \text{ if } \alpha_j \equiv 0, \text{ for } j \geqslant 5 \text{ and } j \leqslant 0.$$

Eliminating between these and the equation of the representative quadric for a line joining $(\xi)(\eta)$ we obtain

$$u_1 u_5^2 - u_2 u_5 u_7 + u_7 u_1^2 = u_5 u_3 - u_2 u_6 = 0,$$

$$\left| \begin{array}{cc} u_5(u_1\xi_1 + u_2\xi_2 + u_3\xi_3) + u_2 u_7\xi_4 & u_5(u_1\xi_2 + u_2\xi_3 + u_3\xi_4) + u_2 u_7\xi_5 \\ u_5(u_1\eta_1 + u_2\eta_2 + u_3\eta_3) + u_2 u_7\eta_4 & u_5(u_1\eta_2 + u_2\eta_3 + u_3\eta_4) + u_2 u_7\eta_5 \end{array} \right| = 0.$$

These share $u_2 = u_5 = 0$, $u_1 = u_2 = u_3 = 0$, $u_5 = u_6 = u_7 = 0$ (multiply), and represent a construct of the sixth class.

The same method may be applied to the Ω_3^6 in $[5]$ or in $[4]$. In this last case *the resulting Ω_3^6 corresponds (in the dual form) to Segre's well-known cubic form with* 10 *nodes. We have seen however that it contains only three systems of Ω_1^1 instead of six, namely those whose bases consist of trisecant planes of C_4 meeting l and therefore C_4', of trisecant planes of C_4' meeting l and C_4, and finally of planes meeting the three chords of C_4 (or C_4'), which meet l and C_4 (and therefore C_4').*

§ 12. To close the general problem we must consider the envelope of the $[r-1]$, which pass through the $[r-2]$ of our various systems ∞^2. This will be a form of developable character, being touched along $[r-4]$ by the various $[r-1]$.

Taking in the first place $r = 7$, and the construct V_3^4, we have to consider the envelope of

$$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_5 + \alpha_4 X_6 + \lambda(\alpha_1 X_3 + \alpha_2 X_4 + \alpha_3 X_7 + \alpha_4 X_8) = 0,$$

subject to

$$\alpha_1 \alpha_4 - \alpha_2 \alpha_3 = 0.$$

We find by the usual rules

$$4(X_1 X_6 - X_2 X_5)(X_3 X_8 - X_4 X_7) = (X_1 X_8 - X_2 X_7 - X_4 X_5 + X_3 X_6)^2, \qquad \ldots\ldots(20)$$

a quartic form F_6^4, having $^{(1)}V_4^4$ and $^{(2)}V_4^4$ as double constructs. It may be verified that we get the same form starting with $^{(1)}V_4^4$, and $^{(2)}V_4^4$. Hence *the envelope in question is a form F_6^4, having the three associated V_4^4 as nodal constructs.* Its equation is conserved by a substitution group of order $3(4!)$ on the letters (X), representing as many collineations. It contains three sets of ∞^3 [3], each common to a [6] of one system, and all its consecutive [6]. These are given by, for one system,

$$\left\| \begin{array}{cccc} X_1 + \lambda X_3 & X_2 + \lambda X_4 & X_5 + \lambda X_7 & X_6 + \lambda X_8 \\ \alpha_4 & -\alpha_3 & -\alpha_2 & \alpha_1 \end{array} \right\| = 0, \qquad \ldots\ldots(21)$$

with $\Sigma \alpha_1 X_1 = \Sigma \alpha_1 X_3 = 0$, and the above relation between the (α), its effect being to reduce to four the number of equations involved.

Omitting the intermediate cases we pass to the case of [4], where the envelope is now a form of general character.

We seek the envelope of

$$\overset{4}{\underset{1}{\Sigma}} \alpha_i x_i + \lambda \overset{4}{\underset{1}{\Sigma}} \alpha_i x_{i+1} = 0,$$

subject to a relation (5), which we shall write

$$\phi(\alpha_1\, \alpha_2\, \alpha_3\, \alpha_4) \equiv (A, B \ldots W\!\!\!\big)\!\big(\alpha_1\, \alpha_2\, \alpha_3\, \alpha_4)^2 = 0, \qquad \ldots\ldots(22)$$

in the usual notation. We shall denote its discriminant by Δ and the form constructed from the reciprocal coefficients by $\phi'(\alpha_1\, \alpha_2\, \alpha_3\, \alpha_4)$. Then to determine the envelope we have to eliminate (α) and (λ) between these and

$$x_i + \lambda x_{i+1} + \tfrac{1}{2}\mu \frac{\partial \phi}{\partial x_i} = 0, \qquad\qquad i = 1, \ldots 4,$$

$$\Sigma \alpha_i x_{i+1} = 0.$$

Solving the first set

$$-\Delta \mu \alpha_i = \tfrac{1}{2}\left\{ \frac{\partial}{\partial x_i} \phi'(x_1 \ldots x_4) + \lambda \frac{\partial}{\partial x_{i+1}} \phi'(x_2 \ldots x_5) \right\}, \qquad i = 1, \ldots 4.$$

Substituting and eliminating (α):

$$(x_1 \ldots x_4)\, \phi'(x_2 \ldots x_5) - \left\{ \overset{4}{\underset{1}{\Sigma}} x_i \frac{\partial}{\partial x_{i+1}} \phi'(x_2 \ldots x_5) \right\} \left\{ \overset{4}{\underset{1}{\Sigma}} x_{i+1} \frac{\partial}{\partial x_i} \phi'(x_1 \ldots x_4) \right\} = 0,$$

or

$$4\phi'(x_1 \ldots x_4)\, \phi'(x_2 \ldots x_5) - \left\{ \overset{4}{\underset{1}{\Sigma}} x_i \frac{\partial}{\partial x_{i+1}} \phi'(x_2 \ldots x_5) \right\}^2 = 0, \qquad \ldots\ldots(23)$$

the envelope F_3^4 in question. From the previous theory this form has as nodal curves C_1, C_1', l, and the three common chords of C_4,

C_4', meeting l, and these only. Its system of tangent spaces contains three systems ∞^2 of pencils. We proceed to draw some important corollaries from the existence of this form.

In the first place consider a [3]-section through l. It is a quartic surface with a double line, and 8 nodes, which, it is known[*], must be associated in pairs such that the joining lines meet l. *Hence C_4 and C_4' are in $(1, 1)$ correspondence such that the joining line meets l.* The system of all these joining lines belongs to F_3^4.

Further each trisecant plane of C_4 or C_4' meeting l, touches F_3^4 along a conic, since its section contains five nodes. The same is true of the planes meeting the three chords, C_4, and therefore C_4'. Thus F_3^4 contains three systems of ∞^2 conic tropes. The configuration of nodes and tropes on the Plücker Surface can be completed without difficulty. The surface in question is identical with the focal surface of the congruences, $(2, 2)$, sections of the systems of planes. In fact the section of the complete system of trisecant planes of C_4 or C_4', or that of planes meeting the three chords, is a tetrahedral complex[†].

The form F_3^4 contains further systems of lines, for a section by a space through l, and a pair of the chords contains three nodal lines, and is therefore ruled. There are three such systems of lines.

A general section of F_3^4 is a quartic surface with 12 nodes, which is the focal surface of three congruences of order two and therefore of class six[‡]. They must be of the second type[§], the four four-fold singular points being the section of that curve met three times by each plane of the system in question. The class may be verified from the fact that the trisecant planes of C_4 (or C_4') which meet a plane in lines, are represented by points on a cubic curve.

Similar results hold for the quartic form section of the form in [7] by an arbitrary space, a form with three nodal quartic curves; and for the other special cases of §§ 12, 13.

§ 13. It was suggested to me by Prof. Baker that the theory developed above might include as a special case the special theorem of Segre's mentioned in the introduction. We proceed to see how this arises, and how we can deduce at once *the figure dual to that of the lines on the* 10 *nodal cubic form*‖ of [4]. This is done by allowing C_4 to degenerate into four lines with three intersections. Two cases are possible.

(1) No line meets more than one other. It can be shewn, by a process analogous to that about to be developed, that this configuration gives a trivial result.

[*] Salmon-Rogers, *Geometry of Three Dimensions*, edn. 5, vol. II, p. 218.

[†] The fundamental cross ratio of this complex is that of the four points in which the space cuts C_4 (or C_4' etc.).

[‡] Jessop, *Treatise on the Line Complex*, § 249, p. 276.

[§] *Id.* p. 280.　　　　　　　　　　　　‖ Segre, *loc. cit.*

(2) *One line*, 0, *meets the other three* 1, 2, 3. For symmetry we denote l by 0″, and the three chords of C_4 meeting l by 1″, 2″, 3″. These are the lines meeting 0″, 2, 3, etc. Now C'_4 must contain a part in each of the spaces (0, 1, 2), etc., and also a part met by planes meeting 1, 2, 3. Each such part must therefore be a line. It is convenient to draw a symbolic figure, in which all the intersections represent actual intersections and we shall denote by (a, β) the intersection of these lines.
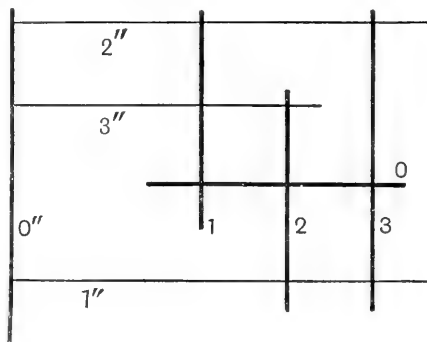
Fig. 1

Thus all the planes meeting 0″, 1, 2, 3 *meet a fifth line* which we shall afterwards call 0′. Also the complete C'_4 must consist of four lines passing through the six points (i, j''). Consideration of the symmetrical relations of the lines 1, 2, 3, etc. shews that the lines will be 1′ joining (2, 3″), (2″, 3); 2′, 3′ similarly defined and 0′ meeting 1′, 2′, 3′. We have now in the figure 12 lines falling into three sets of five

$$1, 2, 3, 0', 0''; \quad 1', 2', 3', 0, 0''; \quad 1'', 2'', 3'', 0, 0', \quad \ldots\ldots(A)$$

such that the planes meeting any four of the same set meet the fifth. Consider now the repartitioning of the original four lines into 1 ; 0″, 2, 3 (Fig. 2). This must give a configuration of the same type. We obtain as components of the C'_4 the lines κ_3 joining 3″0″, 03, κ_2 joining 2″0″, 02 and 1′, so that 0′ must meet κ_2, κ_3. We get a new quintuple of lines

$$1, 1', 1'', \kappa_2, \kappa_3. \quad\quad \ldots\ldots(B)$$
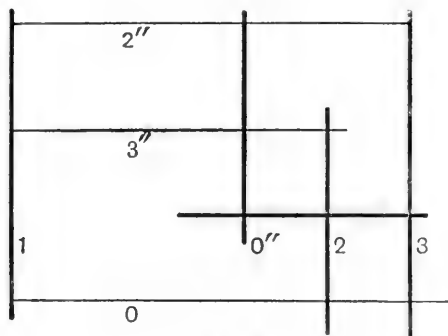
Fig. 2

Similarly, introducing κ_1, joining $1''0''$, 01, we get the remaining two quintuples

$$2,\ 2',\ 2'',\ \kappa_1,\ \kappa_3;\quad 3,\ 3',\ 3'',\ \kappa_1,\ \kappa_2 \qquad\qquad \ldots\ldots(C)$$

and κ_1 meets $0'$. By taking the partition $1''$; $0''$, $2'$, $3'$ we see that κ_i meets i' where it meets $0'$. The tableau A, B, C is to be compared with that given by Castelnuovo, "Ricerche nella Geometria della retta nello spazio S_4," *Atti Ist. Veneto*, Ser. 7, Vol. II, Part I, 1890, n. 11, or Segre, *loc. cit.* n. 24 (in each case in dual form). As a verification of the method we may consider partitions such as 2; $0'$, $0''$, 1 which supply no new conditions.

§ 14. There are *a number of other degenerate normal quartics which furnish special theorems of varying interest.*

We may have

(1) *A pair of conics* C_2, C_2' meeting in one point. It could be foreseen that C_4' breaks into two conics in the same two planes, so that the theorem is not significant. This may be verified by taking as the conics the curves

$$x_4 = x_5 = x_1 x_3 - x_2^2 = 0;\quad x_2 = x_3 = x_1 x_5 - x_4^2 = 0,$$

the vertex A_1 being the point common to the conics. Then the system of planes meeting C_2 twice and C_2' once is given by

$$a_1 x_2 + a_2 x_3 = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_5 = 0,$$

and the details proceed as before. The conic obtained is

$$x_2 = x_3 = 0,\quad x_4 = \eta_4\,(\xi_2 + \mu\xi_3),\quad x_5 = -\,\xi_5\,(\eta_2 + \mu\eta_3),$$
$$x_1 = (\xi_1 + \mu\xi_2)\,(\eta_2 + \mu\eta_3) - (\xi_2 + \mu\xi_3)\,(\eta_1 + \mu\eta_2),$$

where (ξ) is the point where l meets $x_4 = 0$, and (η) where it meets $x_5 = 0$.

This case includes that of the first system of four lines, and that of a conic C_2 and two lines g, g' where g meets C_2 and g'.

(2) *A plane cubic,* and a line which does not meet it. This case again is without significance.

(3) *A space cubic* C_3 *and one of its unisecants* g. We shall see that *the planes meeting C_3 twice and g once meet as well a second curve C_3'.* Let the chord of C_3 meeting l be called $d_{(0)}$, and the lines meeting g and C_3, $d_{(1)}$ and $d_{(2)}$ respectively. We shall denote their intersections by R_1, R_2; P_1, Q_1; P_2, Q_2 (the P lying on g). Then C_4' must contain a part in ϖ, the space of C_3, and is to pass through the six points $R_1 \ldots Q_2$. Consideration of the symmetry of the figure indicates that C_4' consists of the line $Q_1 Q_2$ or g', and a cubic curve through R_1, R_2, P_1, P_2, and therefore lying in the space of g and $d_{(0)}$, and meeting g', where g' meets this space.

All these results may be verified by taking C_3 as

$$x_5 = \begin{Vmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{Vmatrix} = 0,$$

and g as joining A_1 and A_5. The system of planes meeting C_3 twice and g once is

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_5 = a_1 x_2 + a_2 x_3 + a_3 x_4 = 0.$$

Taking l as joining (ξ) (η) with $\eta_5 = 0$ we obtain explicitly the representation of C_3. Also $d_{(0)}$ being the line

$$\begin{Vmatrix} x_1 & x_2 & \eta_1 & \eta_2 \end{Vmatrix} = x_5 = 0,$$

we find that C_3' lies in the space of $d_{(0)}$ and g, namely

$$\begin{vmatrix} x_2 & \eta_1 & \eta_2 \end{vmatrix} = 0.$$

The relation of C_3 with C_3' is again symmetrical.

(4) Finally we have the figure *composed of a conic C_2 and two unisecants*, g_1, g_2. C'_4 must contain parts in the spaces of C_2 and g_i, and a residual part met by all planes meeting C_2, g_1, g_2, and l, so that these parts must be respectively two lines and a conic C'_2. Let $d_{(0)}$ denote the line meeting l, g_1, g_2 (the latter lines in Q_1, Q_2) and let $d_{(1)}$, $d_{(2)}$ meet l, C_2 and g_1, g_2 respectively, meeting these lines in P_1, P_2 and the conic in R_1, R_2. Then as before C'_4 consists of $Q_1 R_2$, $Q_2 R_1$, and a conic through P_1 and P_2, meeting these lines where they meet the spaces defined by $P_1 P_2$ and $Q_2 R_1$, $Q_1 R_2$ respectively. As usual, *the relation of the two conics is symmetrical.*

Algebraically we may take the conic as

$$x_1 x_3 - x_2^2 = x_4 = x_5 = 0,$$

and the lines as $A_1 A_4$, $A_3 A_5$. The system of planes is

$$a_1 x_1 + a_2 x_2 + a_3 x_4 = a_1 x_2 + a_2 x_3 + a_4 x_5 = 0,$$

and we find a parametric representation for C'_2 in the usual way, taking $\xi_5 = 0$, $\eta_4 = 0$, namely

$$x_1 = (\xi_1)(\eta_1), \quad x_2 = (\xi_2)(\eta_1), \quad x_3 = (\xi_2)(\eta_2),$$
$$x_4 = \xi_4(\eta_1), \quad x_5 = \mu \eta_5(\xi_2),$$

where (a_i) denotes $(a_i + \mu a_{i+1})$. Its plane is

$$|\ x_1 \quad \xi_1 \quad \xi_2\ | = 0, \quad |\ x_2 \quad \eta_1 \quad \eta_2\ | = 0,$$

where a_i stands for a column $(a_i \ a_{i+1} \ a_{i+2})$.

§ 15. *We must now consider the possibility of certain extensions of our theorems.* Consider the construct represented by

$$\left\| \begin{array}{ccc} a_{1x} & \ldots & a_{mx} \\ b_{1x} & \ldots & b_{mx} \end{array} \right\| = 0. \qquad \ldots\ldots(24)$$

In $[n]$ this represents a V^m_{n-m+1}, generated by $\infty^1 [n-m]^*$. The equations

$$\sum_1^m \alpha_i a_{ix} = \sum_1^m \alpha_i b_{ix} = 0$$

represent the $\infty^{m-1} [n-2]$, which cut V^m_{n-m+1} in V^{m-1}_{n-m}, so that they can be represented rationally on a $[m-1]$, those through a point being represented by a $[m-3]$. Now the $[m-3]$ of $[m-1]$ are $\infty^{2(m-2)}$, so that an arbitrary $[m-3]$ will not represent the $[n-2]$ through a point of $[n]$ unless

$$2(m-2) \leqslant n,$$

in which case all the $[n-2]$ will pass through $\infty^{n-2(m-2)}$ points of $[n]$, which must form a $[n-2m+4]$. We have also $n \leqslant 2m-1$.

Assuming these inequalities to hold, consider the ∞^{m-2} of the

---

* Thus for $m=3$ we have a V^3_3 of $[5]$ or a V^3_2 of $[4]$. Taking the former as $\| x_1 . x_4 \| = 0$, it contains ∞^2 directrix lines joining

$$(\alpha, \beta, \gamma, 0, 0, 0) \text{ and } (0, 0, 0, \alpha, \beta, \gamma);$$

and in the second case $\| x_1 . x_3 \| = 0$ contains one directrix joining A_2 and A_4.

above system which meet the line l, joining (ξ) (η). They are represented by

$$
\begin{vmatrix}
\overset{m}{\underset{1}{\Sigma}} \alpha_i a_{i\xi} & \overset{m}{\underset{1}{\Sigma}} \alpha_i b_{i\xi} \\[2mm]
\overset{m}{\underset{1}{\Sigma}} \alpha_i a_{i\eta} & \overset{m}{\underset{1}{\Sigma}} \alpha_i b_{i\eta}
\end{vmatrix} = 0, \qquad \ldots\ldots(25)
$$

a $[m-5]$ quadric cone. This is known to contain two systems of $\infty^1 [m-3]$. The $[m-3]$ of the first represent the sets of ϖ through the single points of l, and therefore through as many $[n-2m+4]$. *The existence of the second system shews that these* $[n-2m+4]$ *may be re-grouped into a second system ∞^1 of sub-systems through the* $[n-2n+4]$ *of a second* V_{n-2m+5}. We proceed to shew that we do not hereby gain any essentially new theorems. It will be sufficient to illustrate this with the case $m = 5$. As indicated above we must take $n \geqslant 6$. Taking the lowest case we take the V_2^5 of $[6]$

$$
\left\| \begin{array}{ccc} x_1 & \ldots & x_5 \\ x_3 & \ldots & x_7 \end{array} \right\| = 0,
$$

and consider the $[4]$ which cut it in V_1^4. Those through a point are represented by a plane in the representative $[4]$, and conversely to points of any plane correspond $[4]$ through a point uniquely determined thereby. We seek the point corresponding to

$$
\overset{5}{\underset{1}{\Sigma}} \alpha_i (\xi_i + \mu \xi_{i+2}) = \overset{5}{\underset{1}{\Sigma}} \alpha_i (\eta_i + \mu \eta_{i+2}) = 0.
$$

Identifying this with

$$
\overset{5}{\underset{1}{\Sigma}} \alpha_i x_i = \overset{5}{\underset{1}{\Sigma}} \alpha_i x_{i+2} = 0
$$

we get

$$
\begin{aligned}
x_i &= (\xi_i + \mu \xi_{i+2}) + \tau (\eta_i + \mu \eta_{i+2}), & i &= 1, \ldots 5, \\
x_j &= \kappa (\xi_j + \mu \xi_{j+2}) + \tau' (\eta_j + \mu \eta_{j+2}), & j &= 3, \ldots 7.
\end{aligned}
$$

Thus the locus of the (x) is a normal quartic curve in the $[4]$ given by

$$
\mid x_1 \ \xi_1 \ \xi_3 \ \eta_1 \ \eta_3 \mid = \mid x_3 \ \xi_1 \ \xi_3 \ \eta_1 \ \eta_3 \mid = 0,
$$

which is, in fact, that $[4]$ of our system which passes through l. This special $[4]$ will cut all the other $[4]$ in planes which are trisecant to the section of V_2^5. Hence *the theorem obtained is for all purposes identical with that for* $m = n = 4$, and in precisely the same way all the other cases will reduce to one of the set for $m = 4$, with which we have dealt.

§ 16. *Appendix.* We propose now to return to the normal quartic curve of $[4]$, and consider for what lines l the representative

quadric breaks into a pair of planes. Clearly this will happen for, and only for, a line meeting C_4. The resulting quadric breaks into a plane

$$\alpha_1 + \alpha_2\phi + \alpha_3\phi^2 + \alpha_4\phi^3 = 0 \qquad \ldots\ldots(26)$$

(of a certain developable of the third class, ϕ being the parameter of the point of intersection), and a second plane which is essentially arbitrary, and all trisecant planes represented by points of such a plane meet ∞^1 unisecants of C_4. The plane being $\overset{4}{\underset{1}{\Sigma}} A_i\alpha_i = 0$ it is seen that these lines generate the normal cubic scroll

$$\| \; x_1 \quad x_2 \quad \boldsymbol{A}_1 \; \| = 0, \qquad \ldots\ldots(27)$$

which is cut in conics by the planes in question.

Hence through any unisecant of C_4 can be drawn a cubic scroll containing C_4. We propose to indicate briefly how we may determine the generator through an arbitrary point in terms of the given one, etc. In the first place the arbitrary plane (A) contains a class cubic of lines representing systems of planes through the single points of C_4, namely the section by the plane (22). This cubic has a bitangent (the line of the congruence (3, 1) of lines in two osculating planes of a cubic curve), which must represent trisecant planes through a certain chord of C_4. This chord must be the directrix line of (23), and is therefore determinate. Its points of intersection are given by the solutions of

$$\| \; \theta^{i-1} \quad \phi^{i-1} \quad A_i \; \| = 0, \qquad i = 1, \ldots 4.$$

Now if a generator of (23) joins the point (θ) of C_4 to a point (η), the plane representing the system of trisecant planes which it meets is seen to be

$$\Sigma \, \alpha_i \, (\eta_{i+1} - \theta\eta_i) = 0,$$

and identifying this with the assumed plane we get (finally)

$$\eta_i = \theta^{i-1} \eta_1 + \kappa \, (\theta^{i-2} A_1 + \theta^{i-3} A_2 + \ldots + A_{i-1}),$$

giving explicitly the generator g_θ of (23) through a point of (C_4). To express these rationally in terms of a fixed generator, say at the point (ϕ), we have merely to write $A_i = \xi_{i+1} - \phi\xi_i$. The trisecant planes through the single points of g_θ are represented by points of a line of a pencil in $(A_i\alpha_i) = 0$. The centre of this pencil[*] is found to be

$$\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 = \begin{Vmatrix} 1 & \theta & \theta^2 & \theta^3 \\ A_1 & A_2 & A_3 & A_4 \\ A_1 & A_2 + \theta A_1 & A_3 + \theta^2 A_1 + \theta A_2 & \end{Vmatrix},$$

which moves on the bitangent of the class cubic, and puts that

---

[*] This point corresponds to the trisecant plane which passes through the generator.

line into projective correspondence with C_4, these values of (α) being, in fact, linear in θ. This enables us finally to obtain explicitly the point in which each plane of the system cuts each generator.

Contrary to what would be expected no great simplification occurs if the initial unisecant l lie in a tangent plane.

The theory of this paragraph may be extended to any number of dimensions. Through any unisecant of a normal C_n in $[n]$ will pass a normal $(n+1)$-tic scroll of lines, which may be determined explicitly in terms of any one of its generators by this method.

[NOTE. As alternative to §§ 2, 4 of Mr James' paper preceding, we may argue as follows: The trisecant planes of a rational quartic curve in four dimensions drawn from a point are the planes of a quadric point-cone, the equation of the cone drawn from a general point (θ) of a line l being of the form $U + 2V\theta + W\theta^2 = 0$. There are three points of l for which this reduces to a line-cone, namely the points where l is met by the three chords of the curve which meet l, the cone for the point in which l is met by a chord m consisting of the planes through m which otherwise meet the quartic curve. A point common to these three line-cones will be common to the point-cones for all points of l. The three line-cones meet in an octavic curve, of which the given quartic is one part. The remaining part is thus another quartic curve. Clearly, as a line (l) can be drawn to meet three lines, the theorem involved is as follows: Given three chords m, m', m'' of the original quartic, and the planes through each meeting the curve again in a varying point, the locus of intersection of three such planes, one from each (∞^1) series, is another quartic curve. In fact this is rational, and contains the extremities of the three given chords of the original quartic. Taking a particular plane, ϖ, through m, whose extremities are A, B, meeting the quartic again in C, the planes through m' meet ϖ in a conic passing through A, B, C. The planes through m'' meet ϖ in a further conic also passing through A, B, C. The remaining intersection of these conics, as ϖ varies, describes the second quartic curve. The reciprocity of relation of the two curves is thence obvious.

If the first quartic be given by

$$x(\theta - a) = y(\theta - b) = z(\theta - c) = t(\theta - d) = u\theta,$$

and the chords m, m', m'' are

$$x = y = u = 0, \quad z = 0 = t = u, \quad x - t = y - t = z - t = 0,$$

the second quartic is found to be given by

$$x(\sigma - b^{-1}) = y(\sigma - a^{-1}) = z(\sigma - d^{-1}) = t(\sigma - c^{-1}) = u\sigma,$$

where, in fact, $\sigma = \theta a^{-1} b^{-1}$. The points θ, and σ, $= \theta^{-1}$, of the two quartics, lie on a line meeting the common transversal line, l, of their three common chords, there being four such lines through every point of this transversal. In other words, if (x, y, z, t, u) be any point of the first quartic curve, a point of the second curve is $(u - y, u - x, u - t, u - z, u)$. The three-fold

$$z + t - u = \lambda(x + y - u),$$

through the point $(1, 1, \lambda, \lambda, 0)$, of the line l, meets the curves in two tetrads which are in perspective from this point. This three-fold is the common tangent three-fold of the two three-folds $\lambda U + V = 0$, $\lambda V + W = 0$, at the point (λ) of the line l. All lines from this point, in this tangent three-fold, to the points of the quartic surface $\lambda U + V = 0$, $\lambda V + W = 0$, lie entirely on the three-fold $UW - V^2 = 0$. The cones of trisecant planes of the original quartic drawn from the points of l, touch the F_3^4 given by $UW - V^2 = 0$ on such quartic surfaces. Cf. § 12 of Mr James' paper.]

Note on the twelve points of intersection of a quadri-quadric curve with a cubic surface. By Mr WILLIAM P. MILNE, Clare College.

The geometrical properties of the eight points of intersection of a quadri-quadric curve with a quadric surface are well known, the principal property being that the straight line joining any two of the eight points is a chord of the twisted cubic through the other six. The present note obtains the corresponding property in the case of the twelve points of intersection of the quadri-quadric with a cubic surface, namely:

The twisted cubic through any six of the twelve points of intersection of a quadri-quadric curve with a cubic surface meets in five points the twisted cubic through the other six points of intersection.

Let the quadri-quadric be the curve of intersection of the two quadrics U_2, V_2 and let the cubic surface be Φ_3. Consider the sextic curve $U_2\Phi_3$ and let T denote the twisted cubic through six of the points of intersection of $U_2\Phi_3$ with V_2. Plainly a unique quadric W_2 of the pencil (U_2, V_2) contains T. Also a unique cubic surface

$$\Psi_3 \equiv \Phi_3 + (\lambda x + \mu y + \nu z + \pi t) U_2 = 0$$

contains T, since we have four parameters λ, μ, ν, π at our disposal and hence can make Ψ_3 pass through four points of T in addition to the original six defining T, (i.e. Ψ_3 cuts T in $6 + 4 = 10$ points and hence must contain T). Plainly also W_2 and Ψ_3 each contain the twelve points of intersection of U_2V_2 and Φ_3. Hence the twelve points of intersection of U_2V_2 and Φ_3 lie on the complete curve of intersection of W_2 and Ψ_3 which in this case consists of the twisted cubic T and another twisted cubic T'. Since six of the twelve points of intersection of U_2V_2 and Φ_3 are known to lie on T, the other six must lie on T'. Also it is known that if a quadric cut a cubic surface in two twisted cubics, these two twisted cubics must intersect each other in five points. Thus T intersects T' in five points, which proves the theorem.

[NOTE. In the same way, if a surface of order n be drawn meeting the quadri-quadric curve in $4n$ points, and a cubic curve, T, be put through six of these, curves of order $2n - 3$, and genus $(n - 2)(n - 3)$, can be put through the remaining $4n - 6$ points, which meet the cubic curve T in $3n - 4$ points.]

The effect of deviations from the inverse square law on the scattering of α-particles. By E. S. Bieler, Ph.D. (Communicated by Mr R. H. Fowler.)

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§ 1. *Introduction.*

In order to explain the occasional large angle deflections, when a beam of α-particles passes through a thin film of matter, Sir Ernest Rutherford[*] put forward the nuclear theory of the atom in 1911. He showed that almost the whole of the scattering is due to the inverse square law electrostatic field of the heavy central nucleus, and that the scattering due to the electrons may be neglected.

This theory received ample confirmation from the exhaustive series of experiments on α-particle scattering carried out by Geiger and Marsden[†] in 1913, and again, in 1920, from the accurate determination by J. Chadwick[‡] of the nuclear charges of copper, silver, and platinum by a scattering method.

A series of experiments carried out by the author in the past two years, and soon to be published, indicates that, when the distance of closest approach of the α-particle to the deflecting nucleus becomes very short, the scattering is less than one would expect on the inverse square law. It has been found possible to explain the decrease approximately on the assumption that, in addition to the inverse square law repulsive force between the nucleus and the α-particle, there is an attractive force varying inversely as a higher power of the distance than the second.

In these investigations it has been necessary to carry out certain calculations for such combined laws of force. As these are somewhat laborious, and as the methods and more particularly the results may prove of value to other investigators in this field, it seems desirable that they should be put on record here.

§ 2. *The Scattering under any Law of Force.*

It is quite simple to obtain the scattering under any law of force for which the orbit of the α-particle can be calculated. We must first reduce the scattering nucleus to rest. The effect of its motion can be taken into account afterwards.

In Fig. 1, let O be the scattering nucleus, ABC be the path of the α-particle, p be the distance of the nucleus from its initial line

[*] Rutherford, *Phil. Mag.* 21, p. 669 (1911).
[†] Geiger and Marsden, *ibid.* 25, p. 604 (1913).
[‡] Chadwick, *ibid.* 40, p. 734 (1920).

of motion, and ϕ the angle through which the α-particle is deflected by the field of the nucleus.

If the field of the nucleus is a pure central field, the angle ϕ is a function of p only. The α-particles whose p lies between p and $p + dp$ will be deflected through angles lying between ϕ and $\phi + d\phi$, that is, within a solid angle $2\pi \sin \phi \, d\phi$.
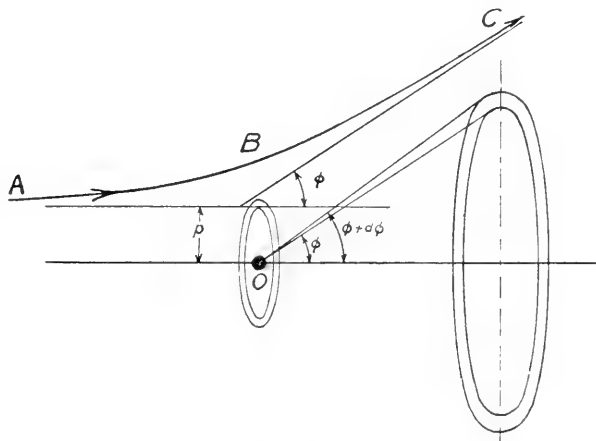
Fig. 1

If the scattering foil has thickness t and contains n atoms per unit volume, the probability that an α-particle passing through it will be deflected through an angle between ϕ and $\phi + d\phi$ is

$$2\pi \, nt p \, dp,$$

and the probability that it will be deflected within a solid angle $d\omega$ at ϕ is

$$2\pi \, nt p \, dp \cdot \frac{d\omega}{2\pi \sin \phi \, d\phi}$$

$$= nt \, d\omega \, \operatorname{cosec} \phi \, p \, \frac{dp}{d\phi}.$$

If Q particles strike the foil per second, the number scattered within $d\omega$ is

$$Qnt \, d\omega \, \operatorname{cosec} \phi \, p \, \frac{dp}{d\phi}$$

$$= Qnt \, d\omega \, \operatorname{cosec} \phi \, f(\phi) \, \frac{df}{d\phi} \qquad \qquad \ldots\ldots(1),$$

when $p = f(\phi)$.

We can therefore obtain the scattering if we can express p as a function of ϕ.

§ 3. *The Orbit of the α-Particle in the Combined* *Field of Force.*

Let M be the mass and V the velocity of the α-particle. Let ϕ (Fig. 2) be the angle through which it is deflected in a collision.
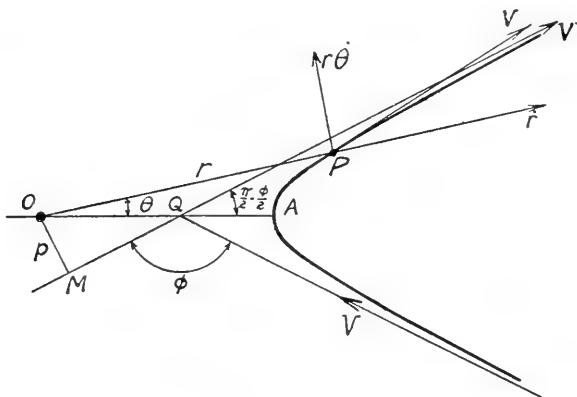
Fig. 2

Let the potential energy of the α-particle in the field of the atomic nucleus be

$$\frac{\lambda}{r} - \frac{\mu}{r^s}.$$

Let p be the distance of the nucleus from the original direction of the α-particle.

Let r, θ be the polar coordinates of the α-particle at any time, referred to the nucleus supposed at rest, and the radius vector to the apse of the orbit, and let v be the velocity of the α-particle at any time. Then we have equations

$$\tfrac{1}{2} M V^2 = \tfrac{1}{2} M v^2 + \frac{\lambda}{r} - \frac{\mu}{r^s} \qquad \ldots\ldots(1),$$

$$pV = r^2 \dot{\theta} \qquad \ldots\ldots(2).$$

Eliminating t in the usual manner, we obtain

$$d\theta = \frac{du}{\left\{ \dfrac{1}{p^2} - u^2 - \dfrac{2}{MpV^2} (\lambda u - \mu u^s) \right\}^{\frac{1}{2}}},$$

where $u = 1/r$. From this we have for the total deflection ϕ

$$\pi - \phi = 2 \int_0^{root} \frac{du}{\left\{ \dfrac{1}{p^2} - u^2 - \dfrac{2}{MpV^2} (\lambda u - \mu u^s) \right\}^{\frac{1}{2}}}.$$

The upper limit is the smallest positive root of the denominator of the integrand.

If we write σ for $\mu/p^{s-1}\lambda$, the ratio of the terms in the potential energy at distance p from the nucleus; τ for λ/MpV^2, the ratio of the inverse square law term in the potential energy to twice the initial kinetic energy of the α-particle, and ξ for pu, we obtain

$$\pi - \phi = 2\int_0^{root} \frac{d\xi}{\{1 - \xi^2 - 2\tau\xi(1 - \sigma\xi^{s-1})\}^{\frac{1}{2}}}.$$

If we can integrate this, we can obtain ϕ as a function of τ and therefore p as a function of ϕ, and can thus calculate the scattering by the use of equation (1), § 2.

The observed scattering was in all cases less than the theoretical. Only positive values of μ corresponding to attractive forces have therefore been considered. Two cases have been investigated, those of the inverse cube and inverse fourth power laws.

§ 4. *The Inverse Cube Law.*

In this case, corresponding to $s = 2$, the quantity under the *root* in equation (2) § 3 is a quadratic, and the expression on the right-hand side is completely integrable. The result is

$$\pi - \phi = \frac{2}{(1 - 2\tau\sigma)^{\frac{1}{2}}} \tan \frac{(1 - 2\tau\sigma)^{\frac{1}{2}}}{\tau}$$

$$= 2\sqrt{\frac{\tau'}{\tau' - 2\sigma}} \tan^{-1}\sqrt{\tau'(\tau' - 2\sigma)} \ldots\ldots(1),$$

where $\tau' = 1/\tau = pMV^2/\lambda$. This expresses ϕ in terms of τ' and therefore of p. On substituting in equation (1) § 2 we find that the number ν of particles scattered into the small solid angle $d\omega$ at angle ϕ to the original beam is

$$\nu = Qnt\,d\omega\left(\frac{\lambda}{MV^2}\right)^2 \operatorname{cosec} \phi\, F(\phi) \frac{dF}{d\phi} \quad \ldots\ldots(2),$$

where $\tau' = F(\phi)$. Using (1) we can tabulate ϕ and $d\phi/d\tau'$ for a series of values of τ' and, by substituting in (2), obtain values of ν for a number of different values of ϕ.

In equation (1), both σ and τ' contain p. Before ϕ can be tabulated as a function of τ', it is necessary to express σ in terms of τ'. Now

$$\sigma = \frac{\mu}{p^{s-1}\lambda} = \frac{\mu}{\lambda}\left(\frac{MV^2}{\lambda\tau'}\right)^{s-1}.$$

In the case we are considering,

$$\sigma = \frac{\mu MV^2}{\lambda^2\tau'} = \frac{x}{\tau'},$$

where x is a new parameter. This parameter x expresses the amount of deviation from inverse square law scattering. It is proportional to the constant determining the additional inverse cube law field, and to the kinetic energy of the α-particle.

When $\sigma = 0$, the usual case of the simple inverse square law, (1) reduces to

$$\pi - \phi = 2 \tan^{-1} \tau',$$

or $$\cot \tfrac{1}{2} \phi = \tau' = \frac{MV^2}{\lambda} \cdot p \qquad \ldots\ldots(3),$$

so that we obtain from (2)

$$\nu = Qnt\,d\omega \left(\frac{\lambda}{MV^2}\right)^2 \cdot \tfrac{1}{4} \operatorname{cosec}^4 \tfrac{1}{2} \phi \qquad \ldots\ldots(4).$$

By dividing equation (2) by equation (4) we obtain the ratio of the scattering under the combined law to that under the inverse square law.
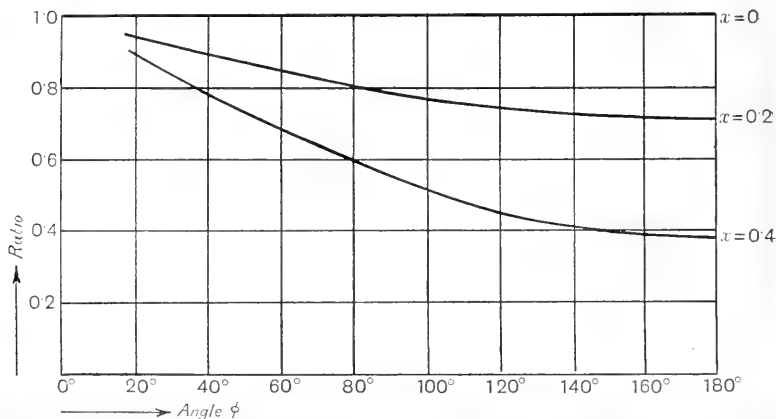
Fig. 3. Ratio of Number scattered under Combined Inverse Square Repulsive and Inverse Cube Attractive Force to Number scattered under Inverse Square Law.

Calculations were carried out for two values of x, namely $0 \cdot 2$ and $0 \cdot 4$. An idea of the relation between these constants and atomic distances may be obtained from the following consideration. The force acting on an α-particle at a distance r from the nucleus is

$$-\frac{d}{dr}\left(\frac{\lambda}{r} - \frac{\mu}{r^2}\right) = \frac{\lambda}{r^2} - \frac{2\mu}{r^3}.$$

This force vanishes when $r = r_0$, where

$$r_0 = 2\mu/\lambda.$$

Now $$x = \frac{\mu MV^2}{\lambda^2} = \frac{r_0 \cdot \tfrac{1}{2} MV^2}{\lambda}.$$

Thus x is the ratio of the kinetic energy of the α-particle to the

inverse square law term in the potential energy at the distance where the force vanishes.

In particular, if $x = 0.2$ for an α-particle of 7 cm. range scattered by aluminium,

$$r_0 = \frac{x\lambda}{\frac{1}{2}MV^2} = 10^{-13} \text{ cm.}$$

We have taken here

$$\lambda = 2Ne^2 = 2 \times 13 \times (4.77 \times 10^{-10})^2, \quad M = 6.5 \times 10^{-24} \text{ gm.,}$$

and $$V = 1.92 \times 10^9 \text{ cm./sec.}$$

If $$x = 0.4, \quad r_0 = 2 \times 10^{-13} \text{ cm.}$$

The final stages in the calculations are set forth in Table I, and the results plotted in Fig. 3.

TABLE I. *The Scattering under a Combined Inverse Square and Inverse Cube Law.*

x	$\tau' = F(\phi)$	ϕ	$\dfrac{d\phi}{d\tau'}$	$\operatorname{cosec} \phi$	$\operatorname{cosec} \phi F(\phi)\dfrac{dF}{d\phi}$	$\frac{1}{4}\operatorname{cosec}^4 \frac{1}{2}\phi$	$\dfrac{\operatorname{cosec}\phi F(\phi)\dfrac{dF}{d\phi}}{\frac{1}{4}\operatorname{cosec}^4 \frac{1}{2}\phi}$
0·2	0·0	180°	−2·36	—	0·1795	0·2500	0·718
	0·2	153° 30′	−2·22	2·241	0·2020	0·2780	0·720
	0·4	129° 52′	−1·893	1·3041	0·2753	0·3715	0·740
	0·6	110° 42′	−1·575	1·0650	0·4055	0·5285	0·767
	0·8	94° 48′	−1·20	1·0035	0·669	0·852	0·785
	1·0	82° 30′	−0·950	1·0086	1·060	1·322	0·802
	1·5	61° 30′	−0·555	1·1380	3·070	3·650	0·841
	2·0	48° 48′	−0·356	1·329	7·460	8·590	0·869
	3·0	34° 24′	−0·179	1·770	29·65	32·60	0·910
	4·0	26° 30′	−0·1064	2·241	84·3	90·6	0·930
	5·0	21° 30′	−0·0704	2·728	194·0	206·0	0·942
0·4	0·0	180°	−3·22	—	0·096	0·2500	0·384
	0·2	144° 48′	−2·80	1·735	0·124	0·303	0·409
	0·6	97° 12′	−1·445	1·008	0·419	0·789	0·531
	1·0	72° 12′	−0·797	1·050	1·318	2·070	0·636
	1·5	55°	−0·421	1·221	3·95	6·66	0·720
	2·0	44°	−0·298	1·440	9·66	12·70	0·760
	3·0	31° 48′	−0·1545	1·898	36·8	44·3	0·832
	5·0	20° 24′	−0·0635	2·869	226·0	254·0	0·890

It is easily verified by approximation that the ratio curve has a finite slope at $\phi = 0$, which is given by

$$\frac{dr}{d\phi} = -\frac{\pi}{2}\, x,$$

where ϕ is expressed in radians. Thus, an inverse cube term in the law of force will make itself felt in the scattering, no matter how small the scattering angle or how large the minimum distance between the deflecting nucleus and the α-particle.

§ 5. *The Inverse Fourth Power Law.*

In this case $s = 3$, and equation (1) § 3 reduces to

$$\pi - \phi = 2 \int_0^{root} \frac{d\xi}{\{1 - \xi^2 - 2\tau\xi (1 - \sigma\xi^2)\}^{\frac{1}{2}}} \quad \ldots\ldots(1),$$

$$= 2 \int_0^{root} \frac{d\xi}{\{1 - \xi^2 - 2\tau\xi (1 - \tau^2 y\xi^2)\}^{\frac{1}{2}}} \quad \ldots\ldots(2),$$

where y is a new parameter, independent of p or τ, and depending only on the kinetic energy of the α-particle, and on the constants λ and μ, which determine the inverse square and inverse fourth power law potentials. Its value is given by,

$$y = \frac{\sigma}{\tau^2} = \frac{\mu}{p^2\lambda} \left(\frac{MpV^2}{\lambda} \right)^2 = \frac{\mu (MV^2)^2}{\lambda^3}.$$

Writing τ' for $1/\tau$, we obtain

$$\pi - \phi = 2 \int_0^{root} \frac{d\xi}{\left\{1 - \xi^2 - 2\frac{\xi}{\tau'} \left(1 - \frac{y}{\tau'^2}\xi^2\right)\right\}^{\frac{1}{2}}} \quad \ldots\ldots(3).$$

These expressions for $\pi - \phi$ might be integrated by the use of elliptic functions, but it was found simpler to use an approximation to (2) in the case where τ, and consequently σ, were fairly small, and to integrate (3) by quadratures in the case of the larger values of τ.

When σ is small, we may derive an approximate expression for (1) in the following way. We have

$$1 - \xi^2 - 2\tau\xi (1 - \sigma\xi^2) = 1 + \tau^2 (1 - \sigma\xi^2)^2 - \{\xi + \tau (1 - \sigma\xi^2)\}^2.$$

If we put $\quad \xi + \tau (1 - \sigma\xi^2) = \{1 + \tau^2 (1 - \sigma\xi^2)^2\}^{\frac{1}{2}} \sin \alpha,$

we obtain after reduction

$$\int_0^{root} \frac{d\xi}{\{1 - \xi^2 - 2\tau\xi (1 - \sigma\xi^2)\}^{\frac{1}{2}}}$$

$$= \int_{\tan^{-1}\tau}^{\frac{\pi}{2}} d\alpha + 2\tau\sigma \int_0^{root} \frac{\left[1 - \frac{\tau (1 - \sigma\xi^2) \sin \alpha}{\{1 + \tau^2 (1 - \sigma\xi^2)^2\}^{\frac{1}{2}}}\right] \xi d\xi}{\{1 - \xi^2 - 2\tau\xi (1 - \sigma\xi^2)\}^{\frac{1}{2}}}.$$

The contribution due to the last integral is small, since it is multiplied by the small coefficient $2\tau\sigma$. We may therefore approximate by putting $\sigma = 0$ in the integrand, and obtain

$$\int_0^{root} \frac{d\xi}{\{1 - \xi^2 - 2\tau\xi (1 - \sigma\xi^2)\}^{\frac{1}{2}}}$$

$$= \frac{\pi}{2} - \tan^{-1}\tau + \frac{2\tau\sigma}{1 + \tau^2} \int_0^{root} \frac{(1 - \tau\xi) \xi d\xi}{\{1 - \xi^2 - 2\tau\xi\}^{\frac{1}{2}}}.$$

If we now put $\qquad \xi + \tau = (1 + \tau^2)^{\frac{1}{2}} \sin \alpha,$

equation (1) reduces to

$$\pi - \phi = \pi - 2 \tan^{-1}\tau + \frac{4\tau\sigma}{1 + \tau^2} \int_{\tan^{-1}\tau}^{\frac{\pi}{2}} \{[(1 + \tau^2)^{\frac{1}{2}} \sin \alpha - \tau] - \tau [(1 + \tau^2)^{\frac{1}{2}} \sin \alpha - \tau]^2\} \, d\alpha,$$

and therefore

$$\phi = 2 \tan^{-1}\tau - \frac{4\tau^3 y}{1 + \tau^2} \left\{ 1 + \frac{3}{2}\tau^2 - \frac{3}{2}\tau (1 + \tau^2) \left(\frac{\pi}{2} - \tan^{-1}\tau \right) \right\} \ldots (4),$$

the required approximate expression for ϕ.

This was used in obtaining ϕ for values of τ less than 0·3. For larger values of τ, the right-hand side of equation (3) was integrated by quadratures. This was done in the following manner.

The smaller positive root a of the equation

$$G(\xi) \equiv 1 - \xi^2 - 2\frac{\xi}{\tau'} \left(1 - \frac{y}{\tau'^2}\xi^2 \right) = 0$$

was first found. The value of the integrand was then computed for a number of equidistant values of ξ between 0 and some convenient rational value just less than the root. The integral between 0 and this value was obtained by successive applications of Simpson's rule for four or five equidistant ordinates. To this was added the value of the integral between the highest ordinates and the root, evaluated as follows.

Near a the function $G(\xi)$ takes the form

$$- \frac{dG(a)}{da}(a - \xi).$$

Therefore when ϵ is small

$$\int_{a-\epsilon}^{a} \frac{d\xi}{\{1 - \xi^2 - 2\tau\xi (1 - \sigma\xi^2)\}^{\frac{1}{2}}} = \frac{1}{\left[\frac{dG}{da} \right]^{\frac{1}{2}}} \int_{a-\epsilon}^{a} \frac{da}{(a - \xi)^{\frac{1}{2}}}$$

$$= \sqrt{\left/ \left\{ \frac{2\epsilon}{\frac{1}{\tau'} + a - \frac{3a^2 y}{\tau'^3}} \right\} \right.}.$$

By these two methods values of ϕ were computed for a number of values of τ ranging from 0·05 to 100. In order to obtain from these smooth values of the function $\tau' = F(\phi)$, and of $F'(\phi)$, both of which are required to calculate the distribution of scattered particles by means of equation (2), § 4, advantage was taken of the fact that, when $\sigma = 0$, $F(\phi)$ reduces to

$$\tau_0' = F(\phi) = \cot \tfrac{1}{2}\phi, \qquad (\S 4, \text{ eq. } (3))$$

and therefore $\qquad \dfrac{d\tau_0'}{d\phi} = F'(\phi) = -\tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2}\phi.$

When σ is not too large, τ' is not very different from τ_0', and $\tau_0' - \tau'$ remains finite for the entire range of ϕ from $0°$ to $180°$. $\tau_0' - \tau'$ was therefore plotted against ϕ, the curve was smoothed by the method of differences and the value of $d(\tau_0' - \tau')/d\phi$ was obtained by the usual difference formula

$$y_0'(\delta x) = \tfrac{1}{2}(y_1 - y_{-1}) - \tfrac{1}{12}(\Delta_2 y_1 - \Delta_2 y_{-1}) + \ldots$$

The values of $F(\phi)$, $F'(\phi)$ so obtained were substituted in equation (2), § 4, and the distribution of scattered particles deduced. The ratio of this to the distribution under a pure inverse square law, as given by equation (4), § 4, is plotted in Fig. 4.
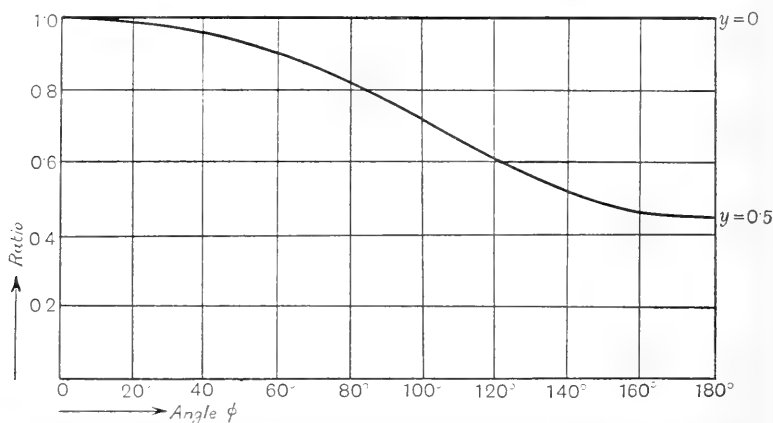
Fig. 4. Ratio of Number scattered under Combined Inverse Square Repulsive and Inverse Fourth Power Attractive Force to Number scattered under Inverse Square Law.

Owing to the labour involved, the calculation was only carried out and plotted for one value of y, namely 0·5. But from this one curve, the course of the corresponding curves for lower and slightly higher values of y may be deduced approximately. For $y = 0·5$, the ratio at $180°$ is 0·457. For $y = 0·2$ it is found to be 0·775. From this, and from the course of the inverse cube law curves for two values of x, it may be concluded that no great error will be made if it is assumed that the deficiency in scattering at any angle is approximately proportional to y, as long as y does not greatly exceed 0·5.

As in the case of the parameter x, it is convenient to express y in terms of the distance from the nucleus where the force vanishes. By differentiating the expression for the potential, we obtain, for the distance at which the force vanishes,

$$r_0{}^2 = 3\mu/\lambda,$$

and therefore $\quad y = \dfrac{\mu\,(MV^2)^2}{\lambda^3} = \dfrac{4}{3}\left(\dfrac{r_0}{\lambda}\cdot\dfrac{1}{2}\,MV^2\right)^2.$

The parameter y is therefore four-thirds of the square of the ratio of the initial kinetic energy of the α-particle to the inverse square law term in the potential energy at the distance where the force vanishes.

If $y = 0\cdot5$ for an α-particle of 7 cm. range scattered by aluminium,

$$r_0 = \frac{\lambda}{\frac{1}{2}\,MV^2}\,\sqrt{(\tfrac{3}{4}y)} = 3\cdot03 \times 10^{-13}\ \text{cm.}$$

It is important to notice that the curve of Fig. 4 has a horizontal tangent at $\phi = 0°$. Thus, an inverse fourth power term in the law of force has practically no effect on the scattering at very small angles.

TABLE II. *The Scattering under a Combined Inverse Square and Inverse Fourth Power Law.*

$y = 0\cdot5$

ϕ	$\tau' = F(\phi)$	$\dfrac{d\tau'}{d\phi}$	$\operatorname{cosec}\phi$	$\operatorname{cosec}\phi F(\phi)\dfrac{dF}{d\phi}$	$\frac{1}{4}\operatorname{cosec}^4\frac{1}{2}\phi$	$\dfrac{\operatorname{cosec}\phi F(\phi)\dfrac{dF}{d\phi}}{\frac{1}{4}\operatorname{cosec}^4\frac{1}{2}\phi}$
20°	5·548	16·82	2·924	270·0	275·0	0·982
40°	2·568	4·377	1·556	17·48	18·27	0·957
60°	1·530	2·034	1·1547	3·592	4·000	0·898
80°	0·9874	1·197	1·0154	1·198	1·464	0·818
100°	0·6511	0·7823	1·0154	0·5160	0·7264	0·711
120°	0·4209	0·5569	1·1547	0·2727	0·4444	0·609
140°	0·2519	0·4225	1·556	0·1665	0·3206	0·519
160°	0·1182	0·3527	2·924	0·1219	0·2660	0·458
180°	0·0000	0·3420	—	0·1142	0·2500	0·457

§ 6. *The Correction for the Motion of the Nucleus.*

The calculations up to now have been made on the assumption that the mass of the nucleus is very large, and that the motion imparted to it by the deflection of the α-particle may be neglected. C. G. Darwin[*] has shown how this may be taken into account.

If the mass M of the α-particle is replaced by M' where $1/M' = 1/M + 1/m$, the results apply to the relative motion of the α-particle and a nucleus of mass m. The actual deflection ϕ' may be found from ϕ by means of the two equations given by

[*] C. G. Darwin, *Phil. Mag.* 17, p. 499 (1914).

Darwin with slightly different notation

$$\tan \phi' = \frac{m \sin 2\theta}{M - m \cos 2\theta},$$

and

$$\theta = \tfrac{1}{2}\pi - \tfrac{1}{2}\phi,$$

where θ is the angle between the original direction of the α-particle, and the final direction of motion of the nucleus. Eliminating θ, we obtain

$$\cot \phi' = \cot \phi + \frac{M}{m} \operatorname{cosec} \phi \qquad \ldots\ldots(1),$$

from which the actual deflection ϕ' is easily obtained from the relative deflection ϕ.

Darwin shows, further, that the expression for the distribution of α-particles on the ordinary inverse square law which is given in § 4, equation (4), namely

$$\nu = Qnt\,d\omega \left(\frac{\lambda}{MV^2}\right)^2 . \tfrac{1}{4} \operatorname{cosec}^4 \tfrac{1}{2}\phi,$$

can be adapted to the case where the motion of the nucleus is not negligible by replacing the $\operatorname{cosec}^4 \tfrac{1}{2}\phi$ by

$$\left\{ \operatorname{cosec}^4 \tfrac{1}{2}\phi - 2 \left(\frac{M}{m}\right)^2 + (1 - \tfrac{3}{2}\sin^2 \phi)\left(\frac{M}{m}\right)^4 + \ldots \right\}.$$

In the experiments for which these calculations were carried out, the largest angle of scattering was $100°$. With aluminium as a scatterer, the correction at this angle amounts to about 1·5 per cent. Since this is considerably less than the experimental error involved, and since the correction, if carried out, would only apply to aluminium as a scatterer, the results have been given for the case in which the motion of the nucleus can be neglected.

The correction may be obtained to a first approximation by multiplying the ordinates of the curves of Figs. 3 and 4 by

$$\left\{ \operatorname{cosec}^4 \tfrac{1}{2}\phi - 2 \left(\frac{M}{m}\right)^2 \right\} \Big/ \operatorname{cosec}^4 \tfrac{1}{2}\phi.$$

The result will be the ratio of the number of particles scattered under the combined law, taking into account the motion of the nucleus, to the number scattered according to the elementary inverse square law theory (§ 4, eq. (4)), in which the motion of the nucleus is neglected.

To obtain the correction more accurately, it is necessary to tabulate ϕ' for a series of values of τ', determine $\dfrac{d\phi'}{d\tau'}$, and repeat the calculation of ν from equation (2), § 4, replacing ϕ by ϕ'.

§ 7. *The Apsidal Distance.*

From equation (1), § 3, we obtain

$$\frac{du}{d\theta} = \left\{ \frac{1}{p^2} - u^2 - \frac{2}{MpV^2}(\lambda u - \mu u^s) \right\}^{\frac{1}{2}}.$$

This vanishes, and the reciprocal of the radius vector has a stationary value when u is the root of the equation

$$\frac{1}{p^2} - u^2 - \frac{2}{MpV^2}(\lambda u - \mu u^s) = 0,$$

that is, when ξ is a root of the equation

$$1 - \xi^2 - 2\tau\xi(1 - \sigma\xi^{s-1}) = 0 \qquad \ldots\ldots(1).$$
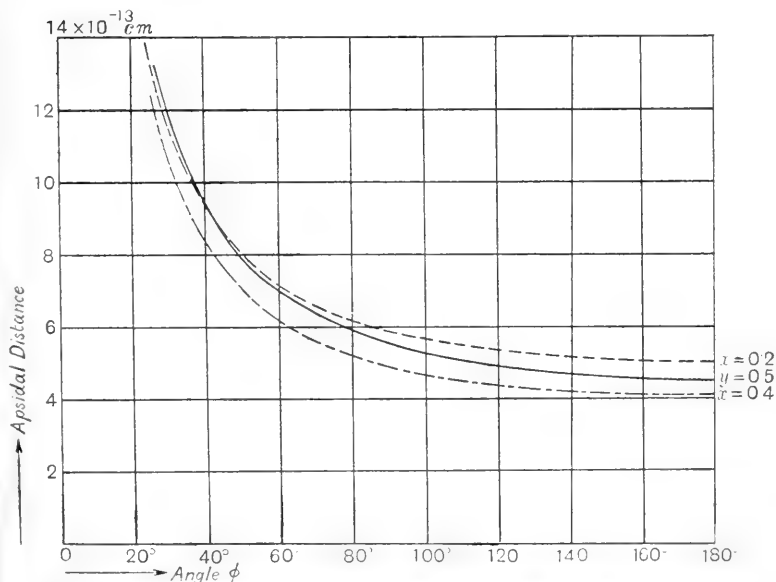
Fig. 5. Apsidal Distances. Dotted curves = Additional Inverse Cube Force; Full curve = Additional Inverse Fourth Power Law Force.

When the additional term in the law of force is an inverse cube term, *i.e.* when $s = 2$, the equation is a quadratic and is easily solved for ξ. The corresponding apsidal distance r is easily obtained from this, if we remember that

$$\xi = pu = p/r,$$

and

$$\tau = \lambda/MpV^2,$$

and therefore

$$r = \frac{p}{\xi} = \frac{1}{\tau\xi} \cdot \frac{\lambda}{MV^2} \qquad \ldots\ldots(2).$$

TABLE III. *Ápsidal Distances—Inverse Square and Inverse Cube.*
7 cm. α-particles scattered by Aluminium.

$\iota' = 0.2$—Force vanishes at 10^{-13} cm.

τ'	Relative Deflection ϕ	Actual Deflection ϕ'	Apsidal Distance r	Inverse Square Potential	Inverse Cube Potential	Resultant Potential
0·0	180°	180°	5.00×10^{-13} cm.	3.72×10^6 volts	-0.36×10^6 volts	3.36×10^6 volts
0·2	153°30'	149° 6'	5·08 ,,	3·66 ,,	-0·35 ,,	3·31 ,,
0·4	129°54'	122°42'	5·30 ,,	3·51 ,,	-0·32 ,,	3·19 ,,
0·6	110°24'	102° 6'	5·64 ,,	3·30 ,,	-0·29 ,,	3·01 ,,
0·8	94°48'	86°18'	5·98 ,,	3·11 ,,	-0·25 ,,	2·86 ,,
1·0	82°30'	74°18'	6·38 ,,	2·92 ,,	-0·22 ,,	2·70 ,,
1·5	61°30'	54°36'	7·58 ,,	2·45 ,,	-0·16 ,,	2·29 ,,
2·0	48°48'	43°	8·87 ,,	2·10 ,,	-0·12 ,,	1·98 ,,
3·0	34°24'	30° 6'	11·56 ,,	1·61 ,,	-0·07 ,,	1·54 ,,
4·0	26°30'	23°12'	14·32 ,,	1·30 ,,	-0·04 ,,	1·26 ,,
5·0	21°30'	18°48'	17·10 ,,	1·09 ,,	-0·03 ,,	1·06 ,,

$x = 0.4$—Force vanishes at 2×10^{-13} cm.

τ'	Relative Deflection ϕ	Actual Deflection ϕ'	Apsidal Distance r	Inverse Square Potential	Inverse Cube Potential	Resultant Potential
0·0	180°	180°	4.10×10^{-13} cm.	4.57×10^6 volts	-1.08×10^6 volts	3.49×10^6 volts
0·2	144°48'	139°18'	4·20 ,,	4·43 ,,	-1·03 ,,	3·40 ,,
0·6	97°12'	88°42'	4·91 ,,	3·79 ,,	-0·76 ,,	3·03 ,,
1·0	72°12'	64°30'	5·94 ,,	3·13 ,,	-0·52 ,,	2·61 ,,
1·5	55°	48°36'	7·24 ,,	2·57 ,,	-0·35 ,,	2·22 ,,
2·0	44°	38°42'	8·60 ,,	2·06 ,,	-0·25 ,,	1·81 ,,
3·0	31°48'	27°48'	11·39 ,,	1·63 ,,	-0·14 ,,	1·49 ,,
5·0	20°24'	17°48'	16·87 ,,	1·10 ,,	-0·06 ,,	1·04 ,,

TABLE IV. *Apsidal Distances—Inverse Square and Inverse Fourth Power.*
7 cm. α-particles scattered by Aluminium.

$q = 0·5$—Force vanishes at $3·03 \times 10^{-13}$ cm.

τ'	Relative Deflection ϕ	Actual Deflection ϕ'	Apsidal Distance r	Inverse Square Potential	Inverse Fourth Power Potential	Resultant Potential
0·1	162° 24'	159° 24'	$4·6 \times 10^{-13}$ cm.	$4·05 \times 10^6$ volts	$-0·57 \times 10^6$ volts	$3·48 \times 10^6$ volts
0·2	147° 12'	112	4·69 ,,	3·97 ,,	-0·54 ,,	3·43 ,,
0·3	130	122° 51'	4·88 ,,	3·82 ,,	-0·48 ,,	3·34 ,,
0·5	112	103° 42'	5·20 ,,	3·58 ,,	-0·40 ,,	3·18 ,,
1·0	79° 21'	71° 18'	6·34 ,,	2·94 ,,	-0·22 ,,	2·72 ,,
2·0	49	43° 12'	8·92 ,,	2·09 ,,	-0·08 ,,	2·01 ,,
3·0	32° 36'	28° 30'	12·55 ,,	1·48 ,,	-0·03 ,,	1·45 ,,
5·0	22° 36'	19° 42'	17·15 ,,	1·09 ,,	-0·01 ,,	1·08 ,,

This equation gives r in terms of τ, ξ, the root of (1), λ, the constant determining the inverse square potential energy, and the kinetic energy of the α-particle. Since corresponding values of τ and ϕ have been tabulated, we can determine r as a function of ϕ.

When the additional term is an inverse fourth power term, *i.e.* when $s = 3$, the equation is a cubic. The root of (1) which is required in this case is the root α, which has already been obtained for the calculation of ϕ (see § 5).

To take into account the motion of the nucleus, we replace $1/M$ in formula (2) by $1/M + 1/m$. This gives the true apsidal distance for a relative deflection ϕ. The actual deflection ϕ' is obtained from ϕ by the use of equation (1), § 6.

The apsidal distances in the cases considered in the last two sections are tabulated in Tables III and IV, for α-particles scattered by aluminium, as well as the potentials in volts, at those distances. They are plotted in Fig. 5.

On the derivation of the equations of transfer of radiation and their application to the interior of a star. By Mr E. A. MILNE, Trinity College.

§ 1. In Professor Eddington's theory of the radiative equilibrium of the stars a fundamental part is played by certain approximate relations connecting the pressure of radiation and the net flux of radiant energy with the temperature. Though the physical laws describing the interaction of radiation with macroscopic matter lead in general to integral equations, Eddington avoids these* by expanding the intensity of radiation at any point in a series of Legendre functions of $\cos \theta$, $\Sigma A_n P_n (\cos \theta)$, where θ is the angle made by the elementary pencil of radiation with the radius of the star through the point. (The radiation is not of course isotropic.) The deduction of the fundamental approximate relations then depends on showing that each coefficient A_n is numerically very small compared with the preceding one, so that in effect the expansion reduces to its first two terms. Now the A_n's are not determined explicitly, but satisfy certain recurrence relations which involve also their differential coefficients, and in consequence a slightly indirect procedure is required: it consists, in fact, in showing from the recurrence relations that if any one coefficient A_{n+2} can be neglected in comparison with a predecessor A_n, then A_{n+1} is a very small fraction of A_n; it then follows that A_n is a very small fraction of A_{n-1}, and so on.

The object of the present note is to offer a simpler alternative procedure. It consists in expanding the intensity of radiation in a power series in $\cos \theta$, in which the coefficients are found explicitly. The argument concerning the negligibility of the coefficients is then immediate, and further the precise form of the relations is apparent to any degree of approximation.

§ 2. *The approximate relations.* Let T be the temperature at any point of the star distant r from the centre, ρ the density, k the mass-coefficient of absorption. The material is assumed to be "grey," so that k is independent of wave-length and we may deal throughout with the integrated radiation. Let c be the velocity of light, a the radiation constant such that aT^4 is the energy density of isotropic black radiation at temperature T. Let F be the net outward flux of radiant energy across unit area perpendicular to the radius at any point.

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* *M.N., R.A.S.,* **77,** 16 (1916); *Zeits. für Phys.,* **7,** 351 (1921).

Eddington proves in the first place the approximate relation

$$F = - \frac{c}{k\rho} \frac{d\left(\frac{1}{3}aT^4\right)}{dr}, \qquad \ldots\ldots(1)$$

and in the second place he shows that the mechanical effect of the radiation on the matter it is traversing is approximately equivalent to that of a hydrostatic pressure p_r, the "pressure of radiation," given by

$$p_r = \tfrac{1}{3}aT^4, \qquad \ldots\ldots(2)$$

and thus the same as for isotropic black radiation.

§ 3. *The internal generation of energy.* Let J denote the intensity of black body radiation at temperature T. We have $J = bT^4$, where $b = ac/4\pi$. By Kirchhoff's law, the emission of energy per gram of the material per second is $k \times 4\pi bT^4$.

Now let $4\pi\epsilon$ denote the rate of internal generation of energy, per gram per second, at any point, due to any processes whatever—gravitational contraction, radio-active processes, synthesis of heavy atomic nuclei, etc. There are two ways of regarding this. We may suppose, as Eddington does, that the material emits $4\pi\epsilon$ *together with* the "thermal" emission $4\pi kbT^4$. Or we may suppose that it emits simply $4\pi kbT^4$, and that $4\pi\epsilon$ represents the balance between the amount emitted and the (smaller) amount absorbed. To the degree of approximation represented by equations (1) and (2) the two ways lead to the same results, though the details of the analysis differ slightly according as the one or the other is adopted. But it may be questioned whether Eddington's point of view is correct. Eddington regards the amount $4\pi\epsilon$ as simply added to the "ordinary temperature radiation," and thus implicitly assumes it to consist roughly of energy lying in the neighbourhood of λ_{max} for the temperature of the point, and to be immediately capable of absorption in the surrounding material with the same value of k. If it arises from intra-atomic transformations or inter-atomic reactions, it is more likely to be liberated in the first instance as very hard radiation[*], highly penetrating, which will ultimately become absorbed and degraded into thermal energy or energy of longer wave-length ; this then increases the ordinary temperature radiation, but simply by raising the matter to a temperature higher than it would otherwise possess. The effect is that as far as the radiation in the significant spectral region is concerned—which alone is really the subject of the analysis—the material emits its ordinary temperature radiation, but that this is higher than the amount absorbed. We have arrived at the second point of view.

The extreme generality of the concept of "ordinary temperature

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[*] Or as the kinetic energy of very high-speed electrons—the argument is substantially the same.

radiation" introduced in the thermodynamics of radiation may perhaps be emphasized. This radiation includes every kind of radiation the material can conceivably emit when existing in a uniform temperature enclosure, *i.e.* when the external radiation to which it is subject is black radiation corresponding to its own temperature. In such an enclosure, if $J_\lambda d\lambda$ is the intensity of black radiation for wave-length λ, the emission can only be $4\pi \int_0^\infty k_\lambda J_\lambda d\lambda$, where k_λ is the appropriate absorption coefficient. If at high temperatures intra-atomic reactions go on of themselves, they must be reversible, and the energy liberated is strictly taken into account in the above expression. In particular, if k_λ is independent of λ, the emission is $4\pi k \int_0^\infty J_\lambda d\lambda = 4\pi k b T^4$, and it cannot exceed this—there can be no question of an additional emission. Actually in the star, owing to leakage of energy at the surface, there is a drain of energy from certain parts of the interior, this representing the excess of the number of reactions generating energy over the number of those absorbing energy. But the excess is still accurately given by an expression of the form

$$4\pi \int_0^\infty k_\lambda \left(J_\lambda - I_\lambda \right) d\lambda,$$

where I_λ is the radiation to which the material at the point is subject. There is still no question of the material emitting its ordinary thermal radiation together with some extra radiation.

If the energy were contributed by an apparently non-reversible process such as radio-activity, then arguments derived from steady-state considerations do not apply. But by hypothesis the energy is not being contributed in the form of radiation characteristic of the temperature T, and therefore should not appear, in the first instance, in the calculation of this radiation. As remarked above, the energy when degraded helps to preserve the temperature constant by counterbalancing the excess of emission over absorption, but the emission of radiation characteristic of T is still the Kirchhoff emission. At temperatures so high that the wave-length of the thermal radiation is comparable with that of the γ-radiation, radio-active transformations are presumably reversible.

The second method of regarding the production of energy is perfectly general. If for instance energy is being generated by gravitational contraction, it will appear in the first instance as heat arising from the increased violence of the atomic encounters; this heat raises the temperature, and so raises the Kirchhoff emission, but in no way can the material be said to emit the Kirchhoff radiation characteristic of its own temperature *together with* the extra energy generated by contraction. Again, account can just as simply

be taken of energy appearing or disappearing by convection or conduction. Part of ϵ would consist of the (positive or negative) amounts gained by convection or conduction in the neighbourhood of the point in question, and these would contribute to the raising (or lowering) of the temperature at the point, and so enhance (or diminish) the previous temperature radiation until it corresponded to the new temperature. In the steady state there would be no change of temperature—the process would merely *sustain* the temperature and so the temperature radiation.

The term "radiative equilibrium" is in a sense a misnomer when there is any form of internal production of energy, for there is no equilibrium between the radiation absorbed and emitted at any point. We are simply determining the radiation field arising from a steady-state temperature distribution maintained by various agencies—atomic or gravitational generation of energy, conduction, convection, and of course radiation itself*. The steady rate of loss of energy which the radiation exchanges themselves are trying to set up is just balanced at any point by the rate of gain due to other agencies. The importance of discussing the radiation exchanges alone resides in the circumstance that they are very large compared with the amounts of energy involved in the other agencies. The small term $4\pi\epsilon$ is the cloak by which we conceal our ignorance of the other agencies.

In this paper the second point of view will be adopted.

§ 4. *Material stratified in parallel planes.* To illustrate the essentials of the procedure whilst avoiding complications due merely to geometry, we shall treat first the case of material stratified in parallel planes†. We give in § 5 the full treatment for Cartesian co-ordinates, from which the particular forms for spherical or other co-ordinates can be easily deduced.

Take an axis of x perpendicular to the planes of stratification, and let $I(x, \theta)$ be the intensity of radiation at the point in a direction at an angle θ with the direction of x. Let dS be a small element of area perpendicular to the direction θ. The amount of radiation crossing this per second in the pencil of solid angle $d\omega$ whose axis is along θ is $IdSd\omega$. During a short stretch of path $ds = \sec\theta dx$ the beam loses by absorption the amount $k\rho Id\omega dSds$, and gains by emission from the matter it has traversed the amount $k\rho Jd\omega dSds$. Hence

$$\cos\theta \frac{dI}{dx} = k\rho (J - I).$$

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* Cf. *Phil. Trans. Roy. Soc.*, A 223, 216 (1922).

† For strict radiative equilibrium ($\epsilon = 0$) the Legendre series and the power series of cosines both reduce strictly to their first two terms. See Milne, *M.N., R.A.S.*, 81, 361 (1921); Littlewood, *Proc. Camb. Phil. Soc.*, 21, 205 (1922).

For this one-dimensional case we can introduce the optical thickness τ instead of x, given by

$$\tau = \int^x k\rho\, dx,$$

and we have then

$$\cos\theta\,\frac{dI}{d\tau} = J - I. \qquad \ldots\ldots(3)$$

The rate of emission per unit mass is $4\pi kJ$, the absorption is $k\iint I\,d\omega$. Hence the equation expressing the rate of generation of energy is

$$k\iint (J - I)\,d\omega = 4\pi\epsilon, \qquad \ldots\ldots(4)$$

or

$$2\pi\int_0^\pi (J - I)\sin\theta\,d\theta = 4\pi\epsilon/k. \qquad \ldots\ldots(5)$$

The net flux of energy per unit area in the positive direction of x is F, given by

$$F = \iint I\cos\theta\,d\omega = 2\pi\int_0^\pi I\cos\theta\sin\theta\,d\theta. \qquad \ldots\ldots(6)$$

We now solve (3) as a linear differential equation for I, keeping θ constant; $J = bT^4$ is a function of position, and so a function of τ; we write it for this purpose $J(\tau)$. Assuming the point τ is so far in the interior that no radiation from the boundary regions effectively reaches it, we have

$$I(\tau, \theta) = e^{-\tau\sec\theta}\int_{-\infty}^\tau J(t)\,e^{t\sec\theta}\sec\theta\,dt$$

$$= \int_0^\infty J(\tau - t\cos\theta)\,e^{-t}\,dt.$$

Expand $J(\tau - t\cos\theta)$ in powers of $t\cos\theta$ by Taylor's theorem and integrate term by term. We find

$$I(\tau, \theta) = J(\tau) - \cos\theta\,J'(\tau) + \cos^2\theta\,J''(\tau) - \ldots \quad \ldots(7)$$

This is the expansion of I in powers of $\cos\theta$ replacing Eddington's expansion in $P_n(\cos\theta)$ functions.

Inserting in (5) and integrating we have

$$\frac{\epsilon}{k} = -\left[\frac{J''(\tau)}{3} + \frac{J^{(4)}(\tau)}{5} + \ldots\right]. \qquad \ldots\ldots(8)$$

The density of radiant energy E is given by

$$E = \frac{1}{c}\iint I\,d\omega$$

$$= \frac{4\pi}{c}\left[J(\tau) + \frac{J''(\tau)}{3} + \frac{J^{(4)}(\tau)}{5} + \ldots\right]. \qquad \ldots\ldots(9)$$

Again, inserting in (6) the net flux is given by

$$F = -4\pi \left[\frac{J'(\tau)}{3} + \frac{J'''(\tau)}{5} + \dots \right]. \qquad \dots\dots(10)$$

To calculate the mechanical force arising from radiation pressure, consider a thin slab, of area dS and thickness dx, normal to the axis of x. The energy incident on the slab contained in a pencil $d\omega$ is $I \cos\theta\, d\omega\, dS$; of this the fraction $k\rho\, dx \sec\theta$ is absorbed. Hence multiplying further by $(\cos\theta)/c$ to obtain the component of momentum, the force is

$$\frac{k\rho dx}{c} \iint I \cos\theta\, d\omega$$

$$= \frac{k\rho F}{c}\, dx.$$

If we regard this force as arising from the gradient of a pressure p_r, we have

$$\frac{dp_r}{dx} = -\frac{k\rho F}{c}, \qquad \dots\dots(11)$$

or

$$\frac{dp_r}{d\tau} = -\frac{F}{c}. \qquad \dots\dots(12)$$

Inserting expression (10) for F and integrating, we find

$$p_r = \frac{4\pi}{c} \left[\frac{J(\tau)}{3} + \frac{J''(\tau)}{5} + \dots \right]. \qquad \dots\dots(13)$$

Formulae (8), (9), (10) and (13) are the complete expansions of ϵ/k, E, F and p_r. Since

$$J(\tau) = bT^4, \quad J'(\tau) = \frac{1}{k\rho} \frac{d(bT^4)}{dx},$$

Eddington's approximations (1) and (2) are simply (10) and (13) each reduced to its first term. Eddington's argument from the numerical values shows at once, in fact, that each member of the sequence $J, J', J''\dots$ is small compared with its predecessor. For J', for instance, is the change in J in a distance such that $k\rho\, dx = 1$, which is equivalent to about 30 cms. of air at atmospheric density. Or, to quote Eddington again, if the star's radius is taken as the unit of length, $k\rho$ (which has the dimensions of a reciprocal length) is of the order of 10^{10}, whilst dJ/dx is in general now comparable with J, d^2J/dx^2 with dJ/dx, etc. Thus $J, J', J''\dots$ are of the relative orders of magnitude of $1, 10^{-10}, 10^{-20}\dots$.

It should be noted however that the argument only holds on the assumption that $4\pi\epsilon$, the generation of energy, is small com-

pared with $4\pi kbT^4$, the total emission[*]. For from (8) we have roughly

$$\left|\frac{J''}{J}\right| = \frac{3\epsilon}{kJ} = 3\cdot\frac{4\pi\epsilon}{4\pi kbT^4}.$$

In a giant star the mean rate of evolution of energy is of the order of 200 ergs gram.$^{-1}$ sec.$^{-1}$. For $T = 10^6$ degrees, which occurs, in a typical star, at about $\frac{1}{10}$ of the radius from the boundary, we have, with $k = 40$,

$$\left|\frac{J''}{J}\right| = 0\cdot 7 \times 10^{-19},$$

in substantial agreement with Eddington's 10^{-20}. Even where the temperature is only 10^4 degrees, $|J''/J|$ is of the order of 10^{-11}.

By comparison of (9) and (13) we see that the pressure of radiation is no longer, as in the isotropic case, exactly $\frac{1}{3}$ of the energy-density. The error is given approximately by

$$p_r - \tfrac{1}{3}E = \frac{4\pi}{c}\cdot\frac{4}{45}J''(\tau) = -\frac{4}{15}\cdot\frac{4\pi\epsilon}{ck}.$$

§ 5. *General case.* When the material is not stratified in parallel planes, we can no longer use a single variable τ. It will be seen too that the mechanical force due to radiation can no longer be expressed rigorously as the gradient of a pressure.

Take rectangular co-ordinates. Let $I(x, y, z; l: m: n)$ be the intensity at (x, y, z) in the direction $(l: m: n)$. Let ds denote a small element of length in the direction $(l: m: n)$. Then we have

$$\frac{dI}{ds} = k\rho\,(J - I). \qquad\qquad \ldots\ldots(14)$$

Put $$\tau = \int k\rho\,ds,$$

the integration being taken along the line from (x, y, z) in the direction $(l: m: n)$. It is convenient to let the suffix 0 refer temporarily to the point (x, y, z). Then solving (14) as before we have

$$I(x, y, z; l: m: n) = \int_{-\infty}^{0} J(\tau)\,e^{\tau}d\tau = \int_{0}^{\infty} J(-\tau)\,e^{-\tau}d\tau$$

$$= J_0 - J_0' + J_0'' - \ldots, \qquad\qquad \ldots\ldots(15)$$

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[*] This point hardly appears explicitly in Eddington's treatment as given in *Zeits. für Phys.*, *loc. cit.* Assuming any one of his coefficients can be neglected, he shows that his $\frac{1}{k\rho}\frac{dB}{dr}$ is of the second order of smallness compared with his A, but from his equations (6) and (15) $\frac{1}{k\rho}\frac{dB}{dr}$ is comparable with ϵ/k, which gives a contradiction unless ϵ/k is itself of the second order of smallness compared with $A = bT^4$.

on expanding J in powers of τ and integrating term by term. Dropping now the suffix, J', J'', ... are given by

$$J' = \left(\frac{l}{k\rho} \frac{\partial}{\partial x} + \frac{m}{k\rho} \frac{\partial}{\partial y} + \frac{n}{k\rho} \frac{\partial}{\partial z} \right) J,$$

$$J'' = \left(\frac{l}{k\rho} \frac{\partial}{\partial x} + \frac{m}{k\rho} \frac{\partial}{\partial y} + \frac{n}{k\rho} \frac{\partial}{\partial z} \right)^2 J,$$

and so on.

For brevity introduce the operators

$$D_x = \frac{1}{k\rho} \frac{\partial}{\partial x}, \quad D_y = \frac{1}{k\rho} \frac{\partial}{\partial y}, \quad D_z = \frac{1}{k\rho} \frac{\partial}{\partial z}.$$

They do not satisfy the commutative law, thus $D_x D_y \neq D_y D_x$. For example in the formal expansion of $(D_x + D_y + D_z)^2$, we must replace $2 D_x D_y$ by $D_x D_y + D_y D_x$.

Equation (4) expressing the rate of generation of energy holds as before. Inserting from (15) we have

$$\frac{4\pi\epsilon}{k} = \iint (l D_x + m D_y + n D_z) J d\omega - \iint (l D_x + m D_y + n D_z)^2 J d\omega + \dots.$$

The odd terms vanish identically. Using the result

$$\iint (la + mb + nc)^{2n} d\omega = \frac{4\pi (a^2 + b^2 + c^2)^n}{2n + 1},$$

we have

$$\frac{\epsilon}{k} = -\tfrac{1}{3} (\Sigma D_x^2) J - \tfrac{1}{5} (\Sigma D_x^2)^2 J - \tfrac{1}{7} (\Sigma D_x^2)^3 J - \dots, \quad \dots (16)$$

where

$$\Sigma D_x^2 \equiv D_x^2 + D_y^2 + D_z^2.$$

The energy density E is given by

$$E = \frac{1}{c} \iint I d\omega = \frac{4\pi}{c} \left[J + \tfrac{1}{3} (\Sigma D_x^2) J + \tfrac{1}{5} (\Sigma D_x^2)^2 J + \dots \right]. \quad (17)$$

If $F(l: m: n)$ denotes the net flux in the direction $(l: m: n)$ it is easily seen that

$$F(l: m: n) = l F_x + m F_y + n F_z,$$

where F_x, F_y, F_z are the net fluxes in the direction of the axes. Now

$$F_x = \iint I(x, y, z; l: m: n) l \, d\omega$$

$$= \sum_{r=0} (-1)^r \iint (l D_x + m D_y + n D_z)^r J l \, d\omega.$$

The even terms vanish identically. Using the result

$$\iint (la + mb + nc)^{2n+1} l \, d\omega = \frac{4\pi a (a^2 + b^2 + c^2)^n}{2n + 3},$$

we have

$$F_x = -4\pi \left[\tfrac{1}{3} D_x J + \tfrac{1}{5} D_x (\Sigma D_x^2) J + \tfrac{1}{7} D_x (\Sigma D_x^2)^2 J + \dots \right]. \quad (18)$$

The purely symbolic character of the notation must be recognised. For instance the expression

$$D_x(D_x^2 + D_y^2 + D_z^2)$$

does not mean the result of first operating with ΣD_x^2 and then operating with D_x; it means that each term in the formal product must be replaced by a sum of other terms whenever the order of the operators is relevant. Thus the term $D_x D_y^2$ in the formal product means

$$\tfrac{1}{3}[D_x D_y D_y + D_y D_x D_y + D_y D_y D_x].$$

Just as for the stratified case, the force on a unit-area slab of thickness ds with normal in the direction $(l: m: n)$ is

$$\frac{k\rho F(l: m: n)}{c} ds.$$

Thus the force on a unit-area slab perpendicular to the axis of x is

$$\frac{k\rho F_x}{c} dx.$$

If we attempted to represent the mechanical force as the gradient of a pressure p_r, we should have

$$\frac{\partial p_r}{\partial x} = -\frac{4\pi}{c}\left[\frac{1}{3}\frac{\partial J}{\partial x} + \frac{1}{5}k\rho\, D_x(\Sigma D_x^2)\,J + \ldots\right]. \quad \ldots\ldots(19)$$

But since the order of the operators D_x, etc. is relevant, the sum of the second and later terms on the right is not the partial derivative of a function of x, y, z, and therefore no pressure function exists unless the second and later terms can be neglected.

We may now approximate to formulae (16), (17), (18), (19) as in the stratified case, obtaining

$$\frac{\epsilon}{k} = -\frac{1}{3}(\Sigma D_x^2)\,J, \quad\quad\quad \ldots\ldots(20)$$

$$E = \frac{4\pi}{c}\,J, \quad\quad\quad \ldots\ldots(21)$$

$$F_x = -\frac{4\pi}{3}\,D_x J, \quad\quad\quad \ldots\ldots(22)$$

$$p_r = \frac{4\pi J}{3c}. \quad\quad\quad \ldots\ldots(23)$$

The terms neglected are in each instance of the same order of magnitude as those neglected in the stratified case, and the neglect of them for stellar applications is justified by the same argument.

Equations (20) and (22) can now be transformed into any co-ordinates required. For example, for spherical co-ordinates with central symmetry (20) becomes

$$\frac{\epsilon}{k} = -\frac{1}{3k\rho r^2}\frac{d}{dr}\left(\frac{r^2}{k\rho}\frac{dJ}{dr}\right).$$

The present method shows immediately that there is no failure of this particular approximation in the neighbourhood of $r = 0$. In Eddington's treatment this point requires separate consideration. The form of (20) for spherical co-ordinates with axial symmetry has been used by the writer in discussing the equilibrium of a rotating star.

Written out in full, equation (20) gives

$$4\pi\epsilon\rho = -\frac{4\pi}{3}\Sigma\frac{\partial}{\partial x}\left(\frac{1}{k\rho}\frac{\partial J}{\partial x}\right).$$

Integrate this through any volume and apply Green's theorem. We find

$$\iiint 4\pi\epsilon\rho\,dx\,dy\,dz = -\frac{4\pi}{3}\iint\left(\frac{l}{k\rho}\frac{\partial J}{\partial x} + \frac{m}{k\rho}\frac{\partial J}{\partial y} + \frac{n}{k\rho}\frac{\partial J}{\partial z}\right)dS$$
$$= \iint(lF_x + mF_y + nF_z)\,dS,$$

on using (22). This is the statement that the net emergent flux of radiation, totalled over a closed surface, is equal to the rate of generation of energy in the enclosed material. The result must of course be true exactly, but it is interesting to notice that the approximate formulae are consistent with it without further approximation.

§ 6. The reader will easily verify that if we adopt the first point of view concerning the mode of generation of energy instead of the second, the function J must be replaced wherever it occurs in § 4 and § 5 by the function j, given by

$$j = J + \epsilon/k.$$

The fundamental alteration occurs in formulae (3) and (5); the rest are consequentially altered. Since ϵk is of the order of magnitude of J'', the additional terms introduced are of the same order as those neglected in the approximations.

On the Solution of Difference Equations. By T. M. CHERRY, B.A., Trinity College, Isaac Newton Student.

Let $$f_r(x_1, x_2, \ldots x_n) \quad (r = 1, 2, \ldots n)$$

be n functions of $x_1, \ldots x_n$ which are expansible in convergent power series about the point $x_1 = x_2 = \ldots = x_n = 0$, at which their Jacobian is not zero; it is supposed that the coefficients in these series are real. This paper is concerned with difference equations of the form

$$\phi_r(s+1) = f_r\{\phi_1(s), \phi_2(s), \ldots \phi_n(s)\} \quad (r = 1, 2, \ldots n) \quad \ldots(1),$$

in which $\phi_1, \ldots \phi_n$ are functions to be determined, the treatment being restricted to real values of all the quantities which occur.

In §§ 1–4 it is shown that these equations have their most general solution of the form

$$\phi_r(s) = \psi_r(s + c_1, c_2, \ldots c_n) \quad (r = 1, 2, \ldots n),$$

in which $\psi_1, \ldots \psi_n$ are analytic functions of their n arguments, while $c_1, c_2, \ldots c_n$ are arbitrary periodic functions of s with period 1.

In § 5 it is shown that the theory can be extended in several directions, while § 6 is concerned with a special type of difference equation which possesses periodic solutions.

§ 1. *The One-Variable Case.*

We shall first consider the difference equation analogous to (1) with only one unknown function, viz.

$$\phi(s+1) = f\{\phi(s)\} \qquad \ldots\ldots(2),$$

where $f(x)$ is developable in the convergent series

$$f(x) \equiv a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots,$$

and df/dx is not zero for $x = 0$, so that $a_1 \neq 0$.

It will be shown, first that the general solution of (2) can be formally expressed by an infinite series, and secondly that this series is convergent.

§ 1·1.

If squares and higher powers of $\phi(s)$ are neglected on the right of (2) we obtain

$$\phi(s+1) = a_0 + a_1 \phi(s) \qquad \ldots\ldots(3),$$

of which it is easily verified that a solution is given as follows:

I. *If a_1 is positive and not equal to 1,*

$$\phi(s) = ca_1^s + \frac{a_0}{1 - a_1} \qquad \ldots\ldots(4).$$

II. *If* $a_1 = 1$, $\phi(s) = c + a_0 s$ ......(5).

III. *If* a_1 *is negative,*

$$\phi(s) = c \,|\, a_1 \,|^s \cos \pi s + \frac{a_0}{1 - a_1} \qquad ......(6),$$

where in each case c is an arbitrary constant.

Change the variable in (3) from ϕ to c by means of (4), (5) or (6), as the case may be, and we obtain

$$c(s+1) = c(s);$$

this shows that the most general solution of (3) is given by (4), (5) or (6), in which c is an arbitrary periodic function of s having the period 1.

§ 1·2. The equation (2) may now be solved by successive approximation. At each stage the equation to be solved will be of the form

$$\phi(s+1) - a_1 \phi(s) = \psi(s),$$

where $\psi(s)$ is a known function of s, an equation possessing considerable analogy with a linear differential equation. Its general solution is expressible as the sum of a "particular integral" and "complementary function," the latter being the most general solution of

$$\phi(s+1) - a_1 \phi(s) = 0,$$

which has already been investigated. For if $\phi'(s)$ be any particular solution and we write

$$\phi(s) = \phi'(s) + \phi''(s),$$

we have $$\phi'(s+1) - a_1 \phi'(s) = \psi(s),$$

whence $$\phi''(s+1) - a_1 \phi''(s) = 0.$$

Further, if $\psi(s)$ is the sum of a number of terms we may obtain a particular solution by adding the particular solutions belonging to the several terms of $\psi(s)$.

§ 1·3. The following list gives all the particular solutions that will be required. Their verification is immediate. It is supposed throughout that a_1 is positive; the corresponding forms for a_1 negative are obtained by writing

$$a_1^s = e^{s \log a_1}$$

and taking the real part of the resulting expressions.

In all these formulae, the quantity c may be any periodic function of s with period 1.

Type I. $a_1 \neq 1$; $\psi(s) = c a_1^{ns}$; $n \neq 1$:

$$\phi(s) = \frac{c a_1^{ns}}{a_1^n - a_1}.$$

Type II. $a_1 \neq 1$; $\psi(s) = c a_1^s$:

$$\phi(s) = c s a_1^{s-1}.$$

Type III. $a_1 \neq 1$; $\psi(s) = c s^m a_1^{ns}$; $n \neq 1$:

$$\phi(s) = c a_1^{ns}(A_0 s^m + A_1 s^{m-1} + \dots + A_m),$$

where the A's are constants determined by the equations

$$\left. \begin{aligned} A_0(a_1^n - a_1) &= 1 \\ A_1(a_1^n - a_1) + m a_1^n A_0 &= 0 \\ A_2(a_1^n - a_1) + (m-1) a_1^n A_1 + \frac{m(m-1)}{2!} a_1^n A_0 &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \right\}.$$

Type IV. $a_1 \neq 1$; $\psi(s) = c s^m a_1^s$:

$$\phi(s) = c a_1^s(B_0 s^{m+1} + B_1 s^m + \dots + B_m s),$$

where the B's are constants determined by the equations

$$\left. \begin{aligned} B_0 a_1(m+1) &= 1 \\ B_1 m + \frac{(m+1)m}{2!} B_0 &= 0 \\ B_2(m-1) + \frac{m(m-1)}{2!} B_1 + \frac{(m+1)m(m-1)}{3!} B_0 &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \right\}.$$

Type V. $a_1 = 1$; $\psi(s) = c(d_0 s^n + d_1 s^{n-1} + \dots + d_n)$:

$$\phi(s) = c(D_0 s^{n+1} + D_1 s^n + \dots + D_n s),$$

where the D's are constants determined by the equations

$$\left. \begin{aligned} (n+1) D_0 &= d_0 \\ n D_1 + \frac{(n+1)n}{2!} D_0 &= d_1 \\ (n-1) D_2 + \frac{n(n-1)}{2!} D_1 + \frac{(n+1)n(n-1)}{3!} D_0 &= d_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \right\}.$$

§ 1·4. *Formal solution of equation* (2).
Let the general solution be

$$\phi(s) = \phi_1(s) + \phi_2(s) + \phi_3(s) + \dots \qquad \dots\dots(7).$$

Substitute this in (2), and equate separately the sets of terms

whose order is the same, as determined by the sum of the suffices in the several factors. We thus obtain the series of equations:

$$\left.\begin{aligned}
\phi_1(s+1) &= a_1.\phi_1(s) + a_0 \\
\phi_2(s+1) &= a_1.\phi_2(s) + a_2.\phi_1^2(s) \\
\phi_3(s+1) &= a_1.\phi_3(s) + 2a_2.\phi_1(s).\phi_2(s) + a_3.\phi_1^3(s) \\
\phi_4(s+1) &= a_1.\phi_4(s) + a_2.\phi_2^2(s) + 2a_2.\phi_1(s).\phi_3(s) \\
&\quad + 3a_3.\phi_1^2(s).\phi_2(s) + a_4.\phi_1^4(s)
\end{aligned}\right\} \quad \dots\dots(8),$$

from which ϕ_1, ϕ_2, ... are to be determined in succession.

CASE I: *Suppose $a_1 \neq 1$, and positive.*

We have already found

$$\phi_1(s) = ca_1^s + \frac{a_0}{1-a_1}.$$

The equation for ϕ_2 is then

$$\phi_2(s+1) - a_1\phi_2(s) = a_2 \left\{ \left(\frac{a_0}{1-a_1}\right)^2 + \frac{2a_0}{1-a_1} ca_1^s + c^2 a_1^{2s} \right\},$$

whose most general solution is found from §§ 1·2, 1·3 to be

$$\phi_2(s) = \frac{a_2 a_0^2}{(1-a_1)^3} + \frac{2a_0 a_2}{a_1(1-a_1)} csa_1^s + \frac{a_2}{a_1^2 - a_1} c^2 a_1^{2s} + c' a_1^s,$$

where c is the arbitrary periodic function occurring in (4), and c' is another arbitrary periodic function of s with the period 1.

The equation for ϕ_3 is now

$$\begin{aligned}
\phi_3(s+1) - a_1\phi_3(s) =& \frac{a_3 a_0^3}{(1-a_1)^3} + \frac{2a_2^2 a_0^3}{(1-a_1)^4} + \left\{ \frac{3a_3 a_0^2}{(1-a_1)^2} + \frac{2a_0^2 a_2^2}{(1-a_1)^3} \right\} ca_1^s \\
&+ \frac{4a_0^2 a_2^2}{a_1(1-a_1)^2} sca_1^s + \left\{ \frac{3a_0 a_3}{1-a_1} - \frac{2a_0 a_2^2}{a_1(1-a_1)^2} \right\} c^2 a_1^{2s} \\
&+ \frac{4a_0^2 a_2^2}{a_1(1-a_1)} sc^2 a_1^{2s} + \left(a_3 + \frac{2a_2^2}{a_1^2 - a_1} \right) c^3 a_1^{3s} \\
&+ \frac{2a_0 a_2}{1-a_1} c' a_1^s + 2a_2 cc' a_1^{2s},
\end{aligned}$$

giving on use of the results of § 1·3

$$\begin{aligned}
\phi_3(s) =& \left(a_3 + \frac{2a_2^2}{1-a_1} \right) \frac{a_0^3}{(1-a_1)^4} + \left(3a_3 + \frac{2a_2^2}{1-a_1} \right) \frac{a_0^2}{(1-a_1)^2} \frac{csa_1^s}{a_1} + \frac{4a_0^2 a_2^2}{a_1(1-a_1)^2} \frac{c(s^2 - s}{2a_1} \\
&+ \frac{2a_0 a_2}{a_1(1-a_1)} sc' a_1^s + \frac{2a_2}{a_1^2 - a_1} cc' a_1^{2s} + \left\{ \frac{3a_0 a_3}{1-a_1} - \frac{2a_0 a_2^2}{a_1(1-a_1)^2} \right\} \frac{c^2 a_1^{2s}}{a_1^2 - a_1} \\
&+ \frac{4a_0^2 a_2^2}{a_1(1-a_1)} \left\{ \frac{s}{a_1^2 - a_1} - \frac{a_1^2}{(a_1^2 - a_1)^2} \right\} c^2 a_1^{2s} + \left(a_3 + \frac{2a_2^2}{a_1^2 - a_1} \right) \frac{c^3 a_1^{3s}}{a_1^3 - a_1} + c'
\end{aligned}$$

It is evident that $\phi_4(s)$ will be a polynomial of the fourth degree in s, $c'a_1{}^s$, $c''a_1{}^s$ with an added term $c'''a_1{}^s$, and similarly for $\phi_5(s)$, ... The quantities c, c', c'', ... are arbitrary periodic* functions of s, one being introduced in the solution of each of the equations (8).

In the case where $a_0 = 0$ the above expressions for ϕ_1, ϕ_2, ϕ_3, ... simplify considerably; the "mixed" terms such as $sca_1{}^s$ disappear, and $\phi_r(s)$ becomes a function of $ca_1{}^s$, $c'a_1{}^s$, $c''a_1{}^s$, ...

CASE II: *Suppose* $a_1 = 1$.

Then
$$\phi_1(s) = c + a_0 s.$$

The equation for $\phi_2(s)$ is
$$\phi_2(s+1) - \phi_2(s) = a_2(c^2 + 2ca_0 s + a_0{}^2 s^2),$$
giving
$$\phi_2(s) = \frac{a_0{}^2 a_2}{3} s^3 - \left(ca_0 a_2 + \frac{a_0{}^2 a_2}{3}\right) s^2 + \left(a_2 c^2 - ca_0 a_2 + \frac{a_0{}^2 a_2}{6}\right) s + c',$$
where c' is an arbitrary periodic function of s with period 1.

In general $\phi_n(s)$ is a polynomial in s of degree $(2n-1)$, containing n arbitrary periodic functions c, c', ...

CASE III: *Suppose* a_1 *negative*.

The expressions for ϕ_1, ϕ_2, ϕ_3, ... are derived from those of Case I by putting everywhere
$$a_1{}^s = e^{s \log a_1},$$
and taking the real part of the resulting expressions.

§ 1·5. *Form of the Solution of* (2).

Since s does not appear explicitly in the equation (2) we can derive from any particular solution
$$\phi = \psi(s),$$
a solution containing an arbitrary constant c:
$$\phi = \psi(s + c).$$

Change the variable in (2) from ϕ to c by means of the transformation
$$\phi(s) = \psi(s + c(s)),$$
and we obtain
$$\psi(s + 1 + c(s+1)) = \phi(s+1) = f\{\phi(s)\}$$
$$= f\{\psi(s + c(s))\} = \psi(s + 1 + c(s))$$
whence
$$c(s) = c(s+1).$$

Hence *the general solution of* (2) *is*
$$\phi = \psi(s + c),$$
where c *is an arbitrary periodic function of* s *with period* 1.

---

* Here and below all periodic functions have the period 1 unless the contrary is stated.

It follows that as regards the *form* of the series (7) the quantities $c, c', c'', \ldots$ can occur only in the additive combination

$$c + c' + c'' + \ldots;$$

as could also be shown directly by arranging that series in powers of $c, c', c'', \ldots$.

Thus to obtain the general solution of (2) it is sufficient to preserve c as an arbitrary periodic function of s, and the quantities $c', c'', \ldots$ can be disposed of in any convenient fashion as functions of s, c, periodic with respect to s; for example they may be equated to zero. We shall find later (§ 2·32) that by suitably adjusting these arbitraries we can ensure the convergence of the series (7) for any given value of s.

§ 2. *Convergence of the Series of* § 1.

This will be investigated by the method of "majorising functions." The series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

has a finite radius of convergence, R say. It will be majorised by

$$M(1 + \lambda x + \lambda^2 x^2 + \ldots),$$

provided $\lambda > 1/R$, and the positive quantity M be chosen sufficiently large. We have thus

$$|a_r| < M\lambda^r \quad (r = 1, 2, \ldots) \qquad \ldots\ldots(9),$$

and we shall suppose, as is obviously always legitimate,

$$M\lambda > 1 \qquad \ldots\ldots(10).$$

§ 2·1. Let $\qquad g(x) = M(\lambda x + \lambda^2 x^2 + \ldots);$

we shall take as our auxiliary equation

$$\chi(s+1) = g\{\chi(s)\} \qquad \ldots\ldots(11).$$

Let the auxiliary function $\chi(s)$ be determined from (11) in the same way that $\phi(s)$ was determined from the equation (2). Then writing

$$\chi(s) = \chi_1(s) + \chi_2(s) + \ldots \qquad \ldots\ldots(12),$$

we have for $\chi_1, \chi_2, \ldots$ the series of equations

$$\left.\begin{aligned}
\chi_1(s+1) &= \lambda M . \chi_1(s) \\
\chi_2(s+1) &= \lambda M . \chi_2(s) + \lambda^2 M . \chi_1{}^2(s) \\
\chi_3(s+1) &= \lambda M . \chi_3(s) + 2\lambda^2 M . \chi_1(s) . \chi_2(s) + \lambda^3 M . \chi_1{}^3(s) \\
\end{aligned}\right\} \quad \ldots(13).$$

The investigation of the convergence of the series (7) is now carried out in two stages. We must show

(i) that the series (12) is convergent for a certain range of values of s, and

(ii) that for the same range of s the series (12) is majorising for the series

$$\Sigma \, | \, \phi_r(s) \, |.$$

§ 2·2. *Convergence of the Series* (12).

If in the solution of $(13)_2$, $(13)_3$, ... we equate to zero the arbitrary periodic functions c', c'', ... which appear in the "complementary function" terms, we obtain in succession

$$\chi_1(s) = d'(M\lambda)^s,$$
$$\chi_2(s) = \frac{\lambda^2 M \cdot d'^2 (M\lambda)^{2s}}{M^2\lambda^2 - M\lambda},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots,$$

and, generally, $\chi_r(s)$ as a multiple of $d'^r(\lambda M)^{rs}$; d' is of course an arbitrary periodic function of s with period 1.

The region of convergence of the series (12) as thus constructed is found by obtaining the exact solution of the equation (11), an equation which can be written

$$\chi(s+1) = \frac{M\lambda \cdot \chi(s)}{1 - \lambda \cdot \chi(s)} \qquad \dots\dots(14).$$

On trying for $\chi(s)$ an expression of the form

$$\chi(s) = \frac{A \, de^{as}}{1 - de^{as}},$$

it is found that (14) is satisfied provided

$$M\lambda = e^a,$$
$$A = \frac{M\lambda - 1}{\lambda},$$

and d is any periodic function of s with period 1. The exact solution of (14) is therefore

$$\chi(s) = \frac{M\lambda - 1}{\lambda} \cdot \frac{d(M\lambda)^s}{1 - d(M\lambda)^s} \qquad \dots\dots(15).$$

Since (12) and (15) each give the general solution of (14)—each containing an arbitrary periodic function additively with s—the series (12) must be identical with the expansion of (15) in ascending powers of $d(M\lambda)^s$. The series is accordingly absolutely convergent provided

$$| \, d \, | \cdot (M\lambda)^s < 1,$$

where

$$d' = \frac{M\lambda - 1}{\lambda} \, d.$$

Its range of convergence is thus, since from (10) $M\lambda > 1$,

$$-\infty < s < -\frac{\log | \, d \, |}{\log M\lambda} \qquad \dots\dots(16).$$

§ 2·3. *Majorising property of the Series* (12).

We shall suppose that d has a constant positive value, so that every term in the series (12) is positive, and we shall consider only *constant* values of the arbitrary periodic function c which occurs in the series (7).

§ 2·31. *The inequality* $|\phi_1(s)| < \chi_1(s)$.

We have
$$\chi_1(s) = d \cdot \frac{M\lambda - 1}{\lambda} \cdot (M\lambda)^s,$$

while $\phi_1(s, c)^*$ is given by (4), (5) or (6). If d have any fixed value and the value $s = s_0$ is arbitrarily assigned, it is evident that we can always choose for c a real value c_0 such that

$$\phi_1(s_0, c_0) = \chi_1(s_0),$$

or since $\chi_1(s_0)$ is positive,

$$|\phi_1(s_0, c_0)| = \chi_1(s_0).$$

If now c be given a slightly smaller value c_1 we shall have

$$\phi_1(s_0, c_1) < \chi_1(s_0),$$

unless a_1 is negative and $\cos \pi s_0$ is negative, when c_1 must have a slightly larger value. Finally if the inequality

$$|\phi_1(s, c)| < \chi_1(s) \qquad \ldots\ldots(17)$$

be satisfied for $s = s_0$, $c = c_1$, it is also satisfied for all values of s and c in certain ranges

$$\begin{aligned} p \leqslant s \leqslant q \\ \alpha \leqslant c \leqslant \beta \end{aligned} \qquad \ldots\ldots(18),$$

which respectively contain these values.

Suppose now the value s_0 of s is pre-assigned. It is easy to see that we can find ranges (18) for s and c such that

(i)   the χ-series (12) is convergent for $p \leqslant s \leqslant q$;
(ii) the inequality (17) is satisfied for $p \leqslant s \leqslant q$ and $\alpha \leqslant c \leqslant \beta$;
(iii) $p < s_0 < q$.

For this purpose, first choose d positive and so small that $s = s_0$ falls within the range of convergence (16) of the χ-series; then, as has just been seen to be possible, find ranges for s and c :

$$p' \leqslant s \leqslant q', \quad \alpha \leqslant c \leqslant \beta,$$

such that $p' < s_0 < q'$ and such that (17) is satisfied for all values of s, c in these ranges; the required range (18) for s is then the common part of $p' \leqslant s \leqslant q'$ and (16), which certainly exists since each contains the point $s = s_0$.

---

* We take explicit notice of the fact that ϕ_1 depends on the two arguments s and c.

§ 2·32. *The inequalities* $|\phi_r(s)| < \chi_r(s)$ *for* $r > 1$.

It will be shown that the arbitrary periodic functions which occur in the solution of equations $(8)_2$, $(8)_3$, ... can be so chosen that the inequalities

$$\phi_r(s) < \chi_r(s) \quad (r = 2, 3, \ldots \infty)$$

are satisfied over the ranges (18) for s and c for which (17) is valid. The choice can moreover be so made that $\phi_r(s)$ is an analytic function of s and c.

We shall put both arguments in evidence in the functions which depend on s and c.

Let the arbitrary periodic function $c'(s, c)$ occurring in the solution of $(8)_2$ be so chosen that

$$|\phi_2(p, c)| < \chi_2(p) \qquad \ldots\ldots(19)$$

for $\alpha \leqslant c \leqslant \beta$. For definiteness we may choose

$$\phi_2(p, c) = 0 ;$$

this fixes the value of $c'(p, c)$.

Comparing the equations $(8)_2$, $(13)_2$ we then have

$$|a_r| < M\lambda^r \quad (r = 1, 2), \qquad \text{from (9),}$$

$$|\phi_1(p, c)| < \chi_1(p) \quad \text{for} \quad \alpha \leqslant c \leqslant \beta, \text{ from (17),}$$

$$|\phi_2(p, c)| < \chi_2(p) \quad \text{for} \quad \alpha \leqslant c \leqslant \beta, \text{ from (19),}$$

and the comparison gives

$$\phi_2(p + 1, c) < \chi_2(p + 1) \quad \text{for} \quad \alpha \leqslant c \leqslant \beta.$$

We have now, by fixing the value of $c'(p, c)$, ensured the satisfaction of the inequality

$$|\phi_2(s, c)| < \chi_2(s) \quad \text{for} \quad \alpha \leqslant c \leqslant \beta \qquad \ldots\ldots(20)$$

at *each end* of the interval

$$p \leqslant s \leqslant p + 1 \qquad \ldots\ldots(21).$$

It is evidently therefore possible to choose c' such an analytic function of s and c, periodic with respect to s, as to ensure the satisfaction of (20) at *all points* of the interval (21) for $\alpha \leqslant c \leqslant \beta$; the choice of c' is in fact within certain limits arbitrary, but we suppose some definite choice made, e.g. we may take for c' the sum of a finite number of terms of a Fourier series whose coefficients are polynomials in c.

Supposing $p + 1 \geqslant q$, we have shown that the inequality (20) is satisfied for the ranges (18) of s and c. If on the other hand $p + 1 < q$, we have

$$|\phi_1(s, c)| < \chi_1(s) \quad \text{for} \quad p \leqslant s \leqslant p + 1, \ \alpha \leqslant c \leqslant \beta,$$

and have just proved

$$|\phi_2(s, c)| < \chi_2(s) \quad \text{for} \quad p \leqslant s \leqslant p + 1, \ \alpha \leqslant c \leqslant \beta ;$$

a comparison of $(8)_2$ and $(13)_2$ then shows that (20) is satisfied for $p + 1 \leqslant s \leqslant p + 2$ and $\alpha \leqslant c \leqslant \beta$. The argument may be repeated step by step, the range of s for which it is established that (20) is satisfied being increased by unity at each step; after a certain number of steps the range of s will have been enlarged so far as to include the point $s = q$.

We have therefore chosen c' such an analytic function of s and c, periodic with respect to s with period 1, that (20) is satisfied for the ranges (18) of s and c.

We now compare equations $(8)_3$ and $(13)_3$, and an exact repetition of the argument shows that the arbitrary periodic function occurring in the general expression for $\phi_3(s)$ can be so chosen as an analytic function of s and c that

$$|\phi_3(s, c)| < \chi_3(s)$$

for the ranges (18) of s and c. Proceeding similarly we see that for these ranges the inequality

$$|\phi_r(s, c)| < \chi_r(s)$$

is satisfied for all values of the suffix r.

The series $\Sigma \phi_r(s)$ is therefore majorised by $\Sigma \chi_r(s)$ for the ranges (18) of s and c. The latter series is however convergent in this range of s. *Hence for*

$$p \leqslant s \leqslant q, \quad \alpha \leqslant c \leqslant \beta$$

the series $\Sigma \phi_r(s)$ is absolutely and uniformly convergent. Each term of the series is however an analytic function of s and c. *Hence the solution of the difference equation* (2) *is an analytic function of the variable s and of the arbitrary periodic function c.*

Of course, since (§ 1·5) in the general solution of (2) s and c occur additively in the combination $s + \log_{a_1} c$, it is unnecessary to establish that ϕ is an analytic function of *both* s and c in the above manner. The demonstration has been presented in this form with a view to the analogy with the many-variable case, which will now be treated.

§3. *The Many-Variable Case.*

We shall now proceed to consider the equations (1), viz.

$$\phi_r(s+1) = f_r\{\phi_1(s),\ \phi_2(s),\ \dots\ \phi_n(s)\}$$
$$= a_0^{(r)} + a_1^{(r)}\phi_1(s) + \dots + a_n^{(r)}\phi_n(s) + a_{11}^{(r)}\phi_1^2(s) + \dots$$
$$(r = 1, 2, \dots n)\ \dots\dots(22),$$

where the series on the right are convergent for sufficiently small values of $\phi_1(s),\ \dots\ \phi_n(s)$. As their treatment is merely an extension of that which has preceded, only the novel points will be dwelt on.

§3·1. *Transformation of the linear terms.*

The first stage is to make a linear change of the ϕ's so as to bring the linear terms on the right to a canonic form. The details of such transformations are too well known to need repetition. The canonic form depends on the roots of the characteristic equation

$$
\begin{vmatrix}
a_1^{(1)} - \lambda & a_2^{(1)} & \dots & a_n^{(1)} \\
a_1^{(2)} & a_2^{(2)} - \lambda & \dots & a_n^{(2)} \\
\dots & \dots & \dots & \dots \\
a_1^{(n)} & a_2^{(n)} & \dots & a_n^{(n)} - \lambda
\end{vmatrix} = 0,
$$

of which none is zero because the Jacobian of the right-hand sides of (22) is not zero for $\phi_1 = \dots = \phi_n = 0$.

Corresponding to each simple root λ we have a simple equation of the form

$$\phi(s+1) = b + \lambda \cdot \phi(s) \qquad \dots\dots(23);$$

corresponding to an r-fold root λ we have a group of r equations of the form

$$
\left.
\begin{aligned}
\phi_1(s+1) &= b_1 + \lambda \cdot \phi_1(s) \\
\phi_2(s+1) &= b_2 + b_{21} \cdot \phi_1(s) + \lambda \cdot \phi_2(s) \\
\phi_3(s+1) &= b_3 + b_{31} \cdot \phi_1(s) + b_{32} \cdot \phi_2(s) + \lambda \cdot \phi_3(s) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\right\} \quad \dots\dots(24),
$$

where any or all of the constants b_{rs} may be zero.

The solution of (23) has already been found in all cases, while the equations (24) can be solved in succession, giving (for the case of λ positive and not equal to 1)

$$\phi_1(s) = \frac{b_1}{1-\lambda} + c_1 \lambda^s \qquad \dots\dots(25),$$

$$\phi_2(s+1) - \lambda \phi_2(s) = b_2 + b_{21}\left(\frac{b_1}{1-\lambda} + c_1 \lambda^s \right),$$

$$\therefore \ \phi_2(s) = \frac{b_2}{1-\lambda} + \frac{b_1 b_{21}}{(1-\lambda)^2} + \frac{c_1 s b_{21} \lambda^s}{\lambda} + c_2 \lambda^s \quad \dots\dots(26),$$

$$\phi_3(s+1) - \lambda \phi_3(s) = b_3 + b_{31}\left(\frac{b_1}{1-\lambda} + c_1 \lambda^s \right)$$

$$+ b_{32}\left\{ \frac{b_2}{1-\lambda} + \frac{b_1 b_{21}}{(1-\lambda)^2} + \frac{c_1 b_{21} s \lambda^s}{\lambda} + c_2 \lambda^s \right\}.$$

$$\therefore \ \phi_3(s) = \frac{b_3}{1-\lambda} + \frac{b_1 b_{31}}{(1-\lambda)^2} + \frac{b_2 b_{32}}{(1-\lambda)^2} + \frac{b_1 b_{21} b_{32}}{(1-\lambda)^3} + (c_1 b_{31} + c_2 b_{32})\frac{s \lambda^s}{\lambda}$$

$$+ \frac{c_1 b_{21} b_{32}}{\lambda} \frac{(s^2 - s) \lambda^s}{\lambda} + c_3 \lambda^s \quad \dots\dots(27),$$

and so on for $\phi_4(s)$, ... The quantities c_1, c_2, ... are as usual arbitrary periodic functions of s with period 1, of which we see there are n introduced in solving the linear terms of (22).

Should any root λ be equal to 1, the corresponding equations (24) yield ϕ_1, ϕ_2, ... as polynomials in s and arbitrary periodic functions c_1, c_2, ...

Should any root λ be negative the values (25), (26), ... must be modified by putting $\lambda^s = e^{s \log \lambda}$, and taking the real part of the resulting complex expressions.

§ 3·2. *Formal solution of the equations.*

This is now carried out by successive approximation. We first make the linear change of variables which gives the linear terms in canonic form, and, changing the notation, suppose that (22) are the equations as thus transformed. We substitute in (22)

$$\phi_r(s) = \phi_r^{(1)}(s) + \phi_r^{(2)}(s) + \dots \qquad \dots\dots(28),$$

and, calling the "order" of any product of ϕ's the sum of their superscripts, we equate separately to zero the terms of first, second, ... orders. The first order equations are thus like (23), (24) whose solution has been seen to introduce n independent arbitrary periodic functions of s, c_1, c_2, ... c_n, say.

The second order equations are of the form

(i) for each simple root λ :

$$\phi_r^{(2)}(s+1) - \lambda . \phi_r^{(2)}(s) = F(\phi_1^{(1)}(s), \phi_2^{(1)}(s), \dots \phi_n^{(1)}(s));$$

(ii) for each multiple root λ :

$$\left.\begin{aligned}
\phi_1^{(2)}(s+1) - \lambda . \phi_1^{(2)}(s) &= F_1(\phi_1^{(1)}, \phi_2^{(1)}, \dots \phi_n^{(1)}) \\
\phi_2^{(2)}(s+1) - \lambda . \phi_2^{(2)}(s) &= b_{21} . \phi_1^{(2)}(s) + F_2(\phi_1^{(1)}, \phi_2^{(1)}, \dots \phi_n^{(1)}) \\
\phi_3^{(2)}(s+1) - \lambda . \phi_3^{(2)}(s) &= b_{31} . \phi_1^{(2)}(s) + b_{32} . \phi_2^{(2)}(s) + F_3(\phi_1^{(1)}, \dots \phi_n^{(1)}) \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\},$$

where F, F_1, F_2, ... are known functions. If these are solved in succession the right-hand side is in each case the sum of a number of terms of the form

$$A c s^m \lambda_1^{p_1 s} \lambda_2^{p_2 s} \dots \lambda_n^{p_n s},$$

where A is a constant; c an arbitrary periodic function of s with period 1; m, p_1, p_2, ... p_n are positive or zero integers; and λ_1, ... λ_n the roots of the characteristic equation.

The only new particular solutions required beyond those given in § 1·3 are (i) that of

$$\phi(s+1) - \lambda \phi(s) = A c s^m \lambda_1^{p_1 s} \lambda_2^{p_2 s} \dots \lambda_n^{p_n s} \quad [\lambda \neq 1],$$

which will be found to be

$$\phi(s) = \frac{A c \lambda_1^{p_1 s} \lambda_2^{p_2 s} \dots \lambda_n^{p_n s}}{\lambda_1^{p_1} \lambda_2^{p_2} \dots \lambda_n^{p_n} - \lambda} \cdot \{\text{polynomial in } s\},$$

unless $\lambda_1^{p_1} \ldots \lambda_n^{p_n} = \lambda$, when the equation reduces to

$$\phi(s+1) - \lambda\phi(s) = Acs^m\lambda^s,$$

which has already been solved in § 1·3 ;
    (ii) that of

$$\phi(s+1) - \phi(s) = Acs^m\lambda_1^{p_1s} \ldots \lambda_n^{p_ns},$$

which has a solution of the form

$$\phi(s) = Ac\lambda_1^{p_1s} \ldots \lambda_n^{p_ns}.\{\text{polynomial in } s\}.$$

The second order terms can thus be determined as polynomials in s, λ_1^s, λ_2^s, $\ldots \lambda_n^s$. The equations for the third order terms are then of the form

$$\phi^{(3)}(s+1) - \lambda\phi^{(3)}(s) = \text{known function of } s,$$

which can be solved similarly; and so on. The series (28) can thus be determined term by term.

In finding the second order terms we introduce n new arbitrary periodic functions of s, in finding the third order terms another n, and so on.

§ 3·3. *Form of the Solution.*

As in § 1·5 it may be shown that only the n periodic functions $c_1, c_2, \ldots c_n$ introduced in solving the first order equations need be left arbitrary to obtain the general solution. For suppose we have any solution containing n such arbitraries :

$$\phi_r(s) = \psi_r(s, c_1, c_2, \ldots c_n) \quad (r = 1, 2, \ldots n) \quad \ldots\ldots(29),$$

and such that $\dfrac{\partial(\psi_1, \psi_2, \ldots \psi_n)}{\partial(c_1, c_2, \ldots c_n)}$ is not identically zero. Regarding (29) as a transformation from $\phi_1, \ldots \phi_n$ to new variables $c_1, c_2, \ldots c_n$ it may be shown exactly as in § 1·5 that the equations (22) transform to

$$c_r(s+1) = c_r(s) \quad (r = 1, 2, \ldots n),$$

and this shows that (29) gives the general solution if we regard $c_1, \ldots c_n$ as arbitrary periodic functions of s with period 1. The condition that the Jacobian is not identically zero is easily established. For if this Jacobian, belonging to the equations (22), vanished identically, the same would have to be true of the similar Jacobian belonging to the equations obtained if quadratic and higher terms are omitted from the right of (22); but this latter Jacobian is, from (25), (26), (27), etc., $(\lambda_1\lambda_2\ldots\lambda_n)^s$, where the λ's are the roots of the characteristic equation, of which none is zero.

Thus to obtain the general solution we may leave $c_1, \ldots c_n$ as arbitrary periodic functions, and the remaining arbitraries can be disposed of in any convenient fashion as functions of s, $c_1, \ldots c_n$, periodic with period 1 with respect to s.

Since the equations (22) do not involve s explicitly, we may derive from any particular solution

$$\phi_r = \psi_r(s) \quad (r = 1, 2, \dots n),$$

a solution containing an arbitrary periodic function c:

$$\phi_r = \psi_r(s + c) \quad (r = 1, 2, \dots n).$$

Hence *the general solution of* (22) *is of the form*

$$\phi_r(s) = \psi_r(s + c_1, c_2, \dots c_n) \quad (r = 1, 2, \dots n),$$

where $c_1, c_2, \dots c_n$ *are arbitrary periodic functions of s with period* 1.

§ 4. *Convergence of the Series of* § 3.

§ 4·1. We shall continue to suppose as in § 3·2 that in the functions $f_1(x_1, x_2, \dots x_n), \dots f_n(x_1, x_2, \dots x_n)$ as expanded in powers of $x_1, \dots x_n$ the linear terms have their canonic form. Since these series are convergent we may obtain for them the same majorising function, viz. the expansion of

$$\frac{M}{1 - \alpha(x_1 + x_2 + \dots + x_n)}$$

in ascending powers of $x_1, \dots x_n$. We may always suppose M so large that $Mn\alpha > 1$.

As auxiliary equations to (22) we take

$$\chi_1(s+1) = \chi_2(s+1) = \dots = \chi_n(s+1) = \frac{M}{1 - \alpha\{\chi_1(s) + \dots + \chi_n(s)\}} - M$$
$$\dots\dots(30),$$

which give

$$\chi_1 = \chi_2 = \dots = \chi_n = \chi,$$

where

$$\chi(s+1) = \frac{Mn\alpha\chi(s)}{1 - n\alpha\chi(s)} \qquad \dots\dots(31).$$

The solution is (d being an arbitrary periodic function of s)

$$\chi(s) = \frac{Mn\alpha - 1}{n\alpha} \cdot \frac{d(Mn\alpha)^s}{1 - d(Mn\alpha)^s},$$

whose expansion in powers of $d(Mn\alpha)^s$, say

$$\chi(s) = \chi^{(1)}(s) + \chi^{(2)}(s) + \dots,$$

coincides with the solution of (31) found by the method of successive approximation, and is convergent provided

$$|d| \cdot (Mn\alpha)^s < 1.$$

§ 4·2. In establishing the majorising property of the series $\Sigma\chi^{(r)}(s)$ for the series

$$\Sigma \phi_m{}^{(r)}(s) \quad (m = 1, \dots n),$$

the only point that needs attention is the establishment of the inequalities

$$|\phi_m{}^{(1)}(s)| < \chi^{(1)}(s) \quad (m = 1, \dots n) \qquad \dots\dots(32).$$

Suppose we desire to prove the ϕ-series convergent for values of s near a given value s_0 when the arbitrary functions $c_1, \ldots c_n$ have values lying in suitable ranges. We first choose a positive value of d such that the range of convergence of the χ-series includes the point $s = s_0$. The values necessary for $c_1, \ldots c_n$ in order that the inequalities (32) may be satisfied for values of s near $s = s_0$ are then investigated as follows:

(i) For the first order equation which corresponds to a simple root λ

$$\phi_r^{(1)}(s+1) = a + \lambda\,\phi_r^{(1)}(s)$$

the comparison of the solution

$$\phi_r^{(1)}(s) = \frac{a}{1-\lambda} + c_r \lambda^s$$

with

$$\chi^{(1)}(s) = d \cdot \frac{Mn\alpha - 1}{n\alpha} \cdot (Mn\alpha)^s \qquad \ldots\ldots(33)$$

shows, exactly as in § 2·31, that we can find ranges for s and c_r:

$$p \leqslant s \leqslant q, \quad \beta_r \leqslant c_r \leqslant \gamma_r \qquad \ldots\ldots(34),$$

such that

$$|\phi_r^{(1)}(s)| < \chi^{(1)}(s),$$

provided s and c_r lie in their ranges (34), and such that $p < s_0 < q$.

(ii) Corresponding to a multiple root λ we have expressions of the form (25), (26), (27), $\ldots$ for $\phi_1^{(1)}, \phi_2^{(1)}, \phi_3^{(1)}, \ldots$. We first, as above, find ranges for s and c_1:

$$p_1 \leqslant s \leqslant q_1, \quad \beta_1 \leqslant c_1 \leqslant \gamma_1 \quad (\text{where } p_1 < s_0 < q_1) \quad \ldots\ldots(35),$$

such that $|\phi_1^{(1)}| < \chi^{(1)}$ for all values of s and c_1 in their ranges (35). We then compare the equations (26) and (33), and it may be shown as in § 2·31 that for any given values of s and c_1 we can give c_2 such a value that $|\phi_2^{(1)}| < \chi^{(1)}$; it then follows that there are ranges of s, c_1 and c_2:

$$p_2 \leqslant s \leqslant q_2 \ (\text{where } p_2 < s_0 < q_2), \quad \beta_1 \leqslant c_1 \leqslant \gamma_1, \quad \beta_2 \leqslant c_2 \leqslant \gamma_2,$$

over which this inequality is satisfied.

Proceeding in this way, we can find ranges for $c_1, \ldots c_n$:

$$\beta_r \leqslant c_r \leqslant \gamma_r \quad (r = 1, 2, \ldots n) \qquad \ldots\ldots(36),$$

and n ranges for s each containing the point $s = s_0$, which have a common part

$$p \leqslant s \leqslant q \qquad \ldots\ldots(37),$$

such that the inequalities

$$\phi_m^{(1)} | < \chi^{(1)} \quad (m = 1, 2, \ldots n)$$

are simultaneously satisfied when $s, c_1, \ldots c_n$ lie in their respective ranges.

The remainder of the proof that the series

$$\Sigma\phi_m^{(r)} \quad (m = 1, 2, \dots n) \qquad \dots\dots(38)$$

are all majorised by the χ-series proceeds exactly as in § 2·32. It is shown that the arbitrary periodic functions which occur in the solution of the ϕ-equations may be so chosen that the inequalities

$$|\phi_m^{(r)}| < \chi^{(r)} \quad \begin{pmatrix} m = 1, 2, \dots n \\ r = 2, 3, \dots \infty \end{pmatrix}$$

are all satisfied for the ranges (36), (37) of the variables; moreover the arbitraries may for this purpose be chosen as *analytic* functions of $s, c_1, \dots c_n$, periodic with period 1 with respect to s, e.g. as the sum of a finite number of terms of Fourier series whose coefficients are polynomials in $c_1, c_2, \dots c_n$.

Since the χ-series is convergent in the range (37) (or at any rate a portion thereof since $s = s_0$ is included in its range of convergence), it follows that the series (38) are absolutely and uniformly convergent. But these are series of analytic functions. *Hence the general solution of the difference equations* (22) *is of the form*

$$\phi_r(s) = \psi_r(s + c_1, c_2, \dots c_n) \quad (r = 1, 2, \dots n),$$

in which $\psi_1, \dots \psi_n$ *are analytic functions of their n arguments whose Jacobian is not identically zero, while* $c_1, c_2, \dots c_n$ *are arbitrary periodic functions of s with period* 1.

§ 5. *Extensions of the preceding Theory.*

§ 5·1. A system of difference equations containing the independent variable explicitly:

$$\phi_r(s + 1) = f_r\{s, \phi_1(s), \phi_2(s), \dots \phi_n(s)\} \quad (r = 1, 2, \dots n) \quad \dots(39)$$

is easily reduced to the form (1) by regarding $s, \phi_1, \dots \phi_n$ as functions of a new variable t, where

$$s(t + 1) = s(t) + 1 \qquad \dots\dots(40),$$

$$\phi_r(t + 1) = f_r\{s(t), \phi_1(t), \dots \phi_n(t)\} \quad (r = 1, 2, \dots n) \quad \dots\dots(41).$$

For a solution of (40), though not the most general solution, is $s = t$, so that all the solutions of (39) are included in those of the system (40), (41), which is of the form (1).

§ 5·2. A difference equation of the form

$$\phi(s + n) = f\{\phi(s + n - 1), \phi(s + n - 2), \dots \phi(s), s\}$$

is reduced to the form (39) by writing

$$\phi(s + 1) = \phi_1(s), \phi(s + 2) = \phi_2(s), \dots \phi(s + n - 1) = \phi_{n-1}(s),$$

for we then have

$$\begin{aligned}
\phi_{n-1}(s+1) &= f\{\phi_{n-1}(s),\ \phi_{n-2}(s),\ \dots\ \phi(s),\ s\} \\
\phi(s+1) &= \phi_1(s) \\
\phi_1(s+1) &= \phi_2(s) \\
&\dots\dots\dots\dots\dots
\end{aligned} \Bigg\}.$$

§ 5·3. *Difference equations depending on arbitrary parameters.*

A set of difference equations

$$\phi_r(s+h) = f_r\{\phi_1(s),\ \dots\ \phi_n(s),\ \mu_1,\ \mu_2,\ \dots\} \quad (r = 1, 2, \dots n)$$

where $f_1, \dots f_n$ are analytic functions of one or more parameters $\mu_1, \mu_2, \dots$ may be solved exactly as in § 3, but the terms of the series (28) will depend explicitly on $\mu_1, \mu_2, \dots$. It may be shown by the method of §§ 2, 4 that in the general solution $\phi_1, \dots \phi_n$ appear as analytic functions of $\mu_1, \mu_2, \dots$; for the arbitraries which occur in the terms of successive orders may be made to depend explicitly on $\mu_1, \mu_2, \dots$ in any manner we please, and the choice may be so made that the series are majorised by the χ-series over certain ranges of $\mu_1, \mu_2, \dots$. A comparison of the first order equations as in § 2·31 fixes the ranges of s, $c_1, \dots c_n$, $\mu_1, \mu_2, \dots$ over which the series will be majorised by the χ-series, and the complete proof of the majorising property follows as before.

The conclusion also holds in the more general case where the "difference" h is not constant, but is an analytic function of $\mu_1, \mu_2, \dots$.

§ 6. *A Difference Equation possessing a Periodic Solution.*

§ 6·1. In an investigation into the form of the solution of Dynamical equations, which it is hoped will shortly appear in the *Transactions* of this Society, it is necessary to show that *the difference equation*

$$\phi(z+\lambda) - \phi(z) = f\{z + \phi(z)\} \qquad \dots\dots(42),$$

where $f(x)$ *is a periodic function of* x *with period* 1 *and mean value zero, possesses periodic solutions of which the period is* 1. This result will now be investigated.

If $\phi = \psi(z)$ be any solution of (42), it is easily seen that a solution containing an arbitrary constant c is

$$\phi = \psi(z+c) + c \qquad \dots\dots(43),$$

and thence as in § 1·5 that (43) gives the general solution if c be an arbitrary periodic function of z with period λ. Now if z be changed into $(z+1)$ the equation (42) remains unaltered in virtue of the periodicity of $f(x)$. Hence

$$\phi = \psi(z+1+c') + c'$$

must also give the general solution of (42). We therefore have the identity

$$\psi(z + 1 + c') + c' \equiv \psi(z + c) + c,$$

or writing z in place of $(z + c)$

$$\psi(z + 1 + c' - c) + c' - c \equiv \psi(z) \qquad \ldots\ldots(44).$$

Since the right-hand side is independent of c and c', this shows that $(c' - c)$ must be a constant, characteristic of the function ψ, which we shall write

$$c' - c = \omega - 1,$$

so that (44) becomes

$$\psi(z + \omega) + \omega - 1 \equiv \psi(z) \qquad \ldots\ldots(45).$$

This gives on differentiating

$$\psi'(z + \omega) \equiv \psi'(z),$$

so that $\psi'(z)$ is periodic with period ω. On integration we deduce from (45)

$$\psi(z) \equiv \frac{1 - \omega}{\omega} z + \chi(z),$$

where χ is periodic with period ω. This determines the form of the function ψ in (43), so that the general solution of (42) can be written

$$\phi = \frac{1 - \omega}{\omega}(z + c) + \chi(z + c) + c \qquad \ldots\ldots(46).$$

We may now determine ω in terms of the mean value of the periodic function $f(x)$ by substituting in (42) the general solution (46). We obtain the identity

$$\frac{1 - \omega}{\omega} \lambda \equiv \chi(z + c) - \chi(z + c + \lambda) + f\left\{\frac{z + c}{\omega} + \chi(z + c)\right\},$$

in which, since $f(x)$ has the period 1, the right-hand side is a periodic function of $(z + c)$ with period ω. Equating the mean values of the two sides of this identity we obtain

$$\frac{1 - \omega}{\omega} \lambda = \bar{f},$$

where $\bar{f} = \int_0^1 f(x)\, dx$. Hence if this mean value is zero we have $\omega = 1$, the non-periodic term disappears from (46), and all those solutions of (42) for which the arbitrary c is constant are periodic with period 1.

§ 6·2. A system of difference equations analogous to (42) is

$$\phi_r(z_1 + \lambda) - \phi_r(z_1) = f_r\{z_1 + \phi_1(z_1),\ z_2 + \phi_2(z_1),\ \ldots\ z_n + \phi_n(z_1)\} - \bar{f}_r$$
$$(r = 1, 2, \ldots n)\ \ \ldots\ldots(47),$$

in which the functions $f_1(z_1, z_2, \ldots z_n),\ \ldots\ f_n(z_1, \ldots z_n)$ are periodic with period 1 with respect to z_1, so that $\bar{f}_r$, the mean value of f_r, is in general a function of $z_2, \ldots z_n$, while these latter quantities are arbitrary parameters. It may be shown by a method similar to § 6·1 that these equations possess a solution

$$\phi_r = \psi_r(z_1) \quad (r = 1, 2, \ldots n),$$

in which $\psi_1, \ldots \psi_n$ are functions independent of the parameters $z_2, \ldots z_n$, and periodic with period 1 with respect to z_1. This result is of importance in the investigation which has been referred to at the beginning of § 6·1.

Some Refinements of the Theory of Dissociation Equilibria.
By Mr C. G. DARWIN, and Mr R. H. FOWLER.

§ 1. *Introductory.* In a recent series of papers* certain ana-lytical methods have been introduced, which have lightened the mathematical calculus required for the proof of theorems of statistical mechanics. The starting-point of this calculus is the identity between expressions naturally occurring and the coefficients of a multinomial expansion. This enables us to modify to our satisfaction the whole development in all the simpler cases. For more complicated problems, however, such as gas-reactions, the multinomial theorem is not applicable, and use was made (in part of the problem) of older methods in which Stirling's theorem is used—a process justifiable enough but inelegant. At the time it was not perceived that a slight modification would enable us to use precisely the same methods for the general problem as for the simpler ones. The expressions we require are still coefficients in certain expansions in multiple power series; to sum them it is only necessary to use the exponential instead of the multinomial theorem. The required coefficients are then picked out by a *multiple* instead of a simple complex integral. The variables which are introduced for this purpose play such an important and characteristic part in our theory, that we venture to call them "selector variables" in this connection. The single selector variable used in our earlier method has a physical counterpart in the temperature suitably measured. In the new theory the extra selector variables, one of which corresponds to each type of atom present, have also a natural physical interpretation. They correspond to the partial potentials of thermodynamical theory.

In the present paper there will be little or no discussion of physical questions. The data (with a few improvements in notation) will be taken from paper 4, and we shall be concerned only with the new mathematical technique. The analysis determines naturally all types of averages and fluctuations; moreover it provides a good method of discussing the entropy of dissociating assemblies which is similar in form and directness to that of paper 2.

§ 2. *The new method for a simple case—a gaseous assembly with two types of atoms and one possible reaction* $A' + A^2 \rightleftarrows A'A^2$. Let us start by discussing for simplicity an assembly in which there are atoms of two types A' and A^2, X_1 and X_2 in number

* Darwin and Fowler, *Phil. Mag.*, Vol. 44, pp. 450, 823 (1922), papers 1 and 2; *Proc. Camb. Phil. Soc.*, Vol. 21, p. 262 (1922), paper 3; p. 392 (1923), paper 6; Fowler, *Phil. Mag.*, Vol. 45, pp. 1, 497 (1923), papers 4 and 5.

respectively, which can unite in a single way to form the molecule $A'A^2$. We take (paper 4) the phase space (classical or quantized) for an atom A' to consist of cells numbered $1, 2, \ldots, u, \ldots$ of weights $p_1', p_2', \ldots, p_u', \ldots$ and energies $\epsilon_1', \epsilon_2', \ldots, \epsilon_u', \ldots$. In any particular example of the assembly there are $a_1', a_2', \ldots, a_u', \ldots$ free atoms of type A' in the corresponding cells. We make a similar set of definitions for atoms of type A^2. For the molecules we take cells $1, 2, \ldots, v, \ldots$, which may for convenience be specified so as to include not only the classical translational motion of the molecule as a whole, but also its quantized rotations and internal vibrations. These cells have *effective** weights $q_1, q_2, \ldots, q_v, \ldots$, energies $\eta_1, \eta_2, \ldots, \eta_v, \ldots$, and contain $b_1, b_2, \ldots, b_v, \ldots$ molecules each.

Suppose now we consider as usual a particular example of the assembly in which there are M_1 free atoms A', M_2 free atoms A^2, and N molecules $A'A^2$. For such an example

$$\Sigma_u a_u' = M_1, \quad \Sigma_u a_u^2 = M_2, \quad \Sigma_v b_v = N, \qquad \ldots\ldots(1)$$
$$M_1 + N = X_1, \quad M_2 + N = X_2. \qquad \ldots\ldots(2)$$

Let E be the total energy of the assembly relative to some specified zero. The specification, which we adhere to, is that the assembly has zero energy when all the atoms are free (no molecules) and at relative rest outside one another's effective fields. The ϵ's and η's must then be specified to conform to this zero. In general the state of lowest energy of a molecule will then have a negative energy χ corresponding to the heat of dissociation at the absolute zero. The state of least energy of a free atom is necessarily of zero energy. The a's and b's are then subject to one further condition that

$$\Sigma_u a_u' \epsilon_u' + \Sigma_u a_u^2 \epsilon_u^2 + \Sigma_v b_v \eta_v = E. \qquad \ldots\ldots(3)$$

The enumeration of complexions is taken in two stages. For a particular example of the assembly for which a definite selection must be made as to which of the atoms are to be combined and which free, we are led to a number of weighted complexions

$$\frac{M_1!\, M_2!\, N!\, (p_1')^{a_1'} \ldots (p_1^2)^{a_1^2} \ldots (q_1)^{b_1} \ldots}{a_1'! \ldots a_1^2! \ldots b_1! \ldots}. \qquad \ldots\ldots(4)$$

This expression can be and was treated by our earlier methods. But the second stage consists in enumerating the examples of the assembly, that is in permuting the atoms between the molecules and the free form. This multiplies (4) by a factor

$$X_1!\, X_2! / (M_1!\, M_2!\, N!),$$

* By the effective weight of any cell for a molecule we mean the weight as ordinarily defined (papers 1 and 4) divided by the symmetry number σ. It is easy to see that this gives the proper effect to σ, for any such cell is counted σ times over when we take account of all the possible arrangements of the given atoms. For the particular molecule $A'A^2$, $\sigma = 1$.

so that we obtain an expression for the number of weighted complexions of the form

$$\frac{X_1! \, X_2! \, (p_1')^{a_1'} \dots (p_1^2)^{a_1^2} \dots (q_1)^{b_1} \dots}{a_1'! \dots a_1^2! \dots b_1! \dots}, \qquad \dots\dots(5)$$

representing all examples with given M_1, M_2, N. For the total number of weighted complexions this is to be summed over all zero and positive values of the M's and N's and therefore of the a's and b's subject to

$$\Sigma_u a_u' + \Sigma_v b_v = X_1, \qquad \dots\dots(6)$$

$$\Sigma_u a_u^2 + \Sigma_v b_v = X_2, \qquad \dots\dots(7)$$

$$\Sigma_u a_u' \epsilon_u' + \Sigma_u a_u^2 \epsilon_u^2 + \Sigma_v b_v \eta_v = E. \qquad \dots\dots(8)$$

To conform to the three equations (6)—(8), we introduce simultaneously three selector variables x_1, x_2, z and form the expression

$$X_1! \, X_2! \, \Sigma_{a,b} \frac{(p_1' x_1 z^{\epsilon_1'})^{a_1'} \dots (p_1^2 x_2 z^{\epsilon_1^2})^{a_1^2} \dots (q_1 x_1 x_2 z^{\eta_1})^{b_1} \dots}{a_1'! \dots \qquad a_1^2! \dots \qquad b_1! \dots},$$

a multiple series in which the a's and b's take *all* positive and zero values. Its sum is obviously

$$X_1! \, X_2! \, \exp\{x_1 f_1(z) + x_2 f_2(z) + x_1 x_2 g(z)\}. \qquad \dots\dots(9)$$

where f and g are the partition functions $f_1(z) = \Sigma_u p_u' z^{\epsilon_u'}$, etc. It is evident that the total number of weighted complexions subject to (6)—(8) is the coefficient of $x_1^{X_1} x_2^{X_2} z^E$ in (9).

The step just described is the essential feature in the new method. The total number of weighted complexions representing the assembly is therefore

$$C = \frac{X_1! \, X_2!}{(2\pi i)^3} \iiint \frac{dx_1 dx_2 dz}{x_1^{X_1+1} x_2^{X_2+1} z^{E+1}} \exp\{x_1 f_1(z) + x_2 f_2(z) + x_1 x_2 g(z)\}. \qquad \dots\dots(10)$$

The integral for each variable is taken round a circle in its own plane with its centre at the origin.

We must construct of course similar integrals for the average value of any quantity such as $\overline{a_u'}$. To obtain $C\overline{a_u'}$ we have to sum (5) with the addition of the factor a_u' under the summation sign. It follows therefore that

$$C\overline{a_u'} = p_u' \frac{\partial C}{\partial p_u'}. \qquad \dots\dots(11)$$

In (10) p_u' occurs only in $f_1(z)$, and $p_u' \partial f_1/\partial p_u' = p_u' z^{\epsilon_u'}$. Therefore

$$C\overline{a_u'} = \frac{X_1! X_2!}{(2\pi i)^3} \iiint \frac{dx_1 dx_2 dz}{x_1^{X_1+1} x_2^{X_2+1} z^{E+1}} (p_u' x_1 z^{\epsilon_u'}) \exp\{x_1 f_1(z) + x_2 f_2(z) + x_1 x_2 g(z)\}. \qquad \dots\dots(12)$$

Similarly

$$C\overline{M_1} = C\Sigma_u \overline{a_u'} = \Sigma_u p_u' \frac{\partial C}{\partial p_u'},$$

$$= \frac{X_1!\,X_2!}{(2\pi i)^3} \iiint \frac{dx_1 dx_2 dz}{x_1^{X_1+1} x_2^{X_2+1} z^{E+1}} (x_1 f_1(z)) \exp\{x_1 f_1(z) + x_2 f_2(z) + x_1 x_2 g(z)\}.$$
$$\ldots\ldots(13)$$

For $\overline{E_{A'}}$, the mean energy of the atoms of type A' we have

$$C\overline{E_{A'}} = C\Sigma_u \overline{a_u'} \epsilon_u' = \Sigma_u p_u' \epsilon_u' \frac{\partial C}{\partial p_u'}, \qquad \ldots\ldots(14)$$

and the extra factor in the integrand is

$$x_1 \Sigma_u p_u' \epsilon_u' z^{\epsilon_u'} \text{ or } x_1 z \partial f_1/\partial z.$$

For $C\overline{b_v}$ the extra factor in the integrand is $q_v x_1 x_2 z_{\eta_v}$ and for $C\overline{N}$, $x_1 x_2 g(z)$. For the mean energy $E_{A_1 A_2}$ of the molecules $A_1 A_2$ the extra factor is $x_1 x_2 z \partial g/\partial z$.

Let us assume for the moment (for a proof see § 4) that, when the assembly is sufficiently large, integrals such as these can be approximated to by the method of steepest descents, making the contours pass through a set of positive real values $\xi_1, \xi_2, \Im$ of x_1, x_2, z, at which the first partial differential coefficients of the integrand all vanish. We shall then find by extensions of the arguments of paper 1 that

$$\overline{a_u'} = \xi_1' p_u' \Im^{\epsilon_u'}, \qquad \overline{b_v} = \xi_1 \xi_2 q_v \Im^{\eta_v}, \qquad \ldots\ldots(15)$$

$$\overline{M_1} = \xi_1 f_1(\Im), \qquad N = \xi_1 \xi_2 g(\Im), \qquad \ldots\ldots(16)$$

$$\overline{E_{A'}} = \xi_1 \Im \frac{\partial f_1(\Im)}{\partial \Im}, \qquad \overline{E_{A'A^2}} = \xi_1 \xi_2 \Im \frac{\partial g(\Im)}{\partial \Im}, \qquad \ldots\ldots(17)$$

etc. The equations defining $\xi_1, \xi_2, \Im$ are

$$\left.\begin{array}{r}\xi_1 f_1(\Im) + \xi_1 \xi_2 g(\Im) = X_1, \\ \xi_2 f_2(\Im) + \xi_1 \xi_2 g(\Im) = X_2, \\ \xi_1 \Im \frac{\partial f_1(\Im)}{\partial \Im} + \xi_2 \Im \frac{\partial f_2(\Im)}{\partial \Im} + \xi_1 \xi_2 \Im \frac{\partial g(\Im)}{\partial \Im} = E.\end{array}\right\} \ldots\ldots(18)$$

Assuming that $\xi_1, \xi_2, \Im$, which define the concentrations and the temperature of the assembly are thus uniquely determined, the complete form of "the equation of mass-action" follows at once from (16) by eliminating ξ_1 and ξ_2. We find

$$\frac{\overline{M_1}\overline{M_2}}{N} = \frac{f_1(\Im)f_2(\Im)}{g(\Im)}; \qquad \ldots\ldots(19)$$

this is the familiar equation in a slightly modified notation.

Integrals can obviously also be given for such quantities as $\overline{CM_1^s}$ and $C(M_1 - \overline{M_1})^s$ which have immediate applications in

simplifying the discussion of the fluctuations of dissociating assemblies in paper 6. Writing $x_1 = e^{\lambda_1}$, $x_2 = e^{\lambda_2}$, $z = e^\mu$, $f_1(e^\nu) = \psi_1(\nu)$, etc., we find for example

$$C \overline{(M_1 - \overline{M_1})^s} = \frac{X_1! \, X_2!}{(2\pi i)^3} \int \int \int d\lambda_1 d\lambda_2 d\mu \exp\{-X_1\lambda_1 - X_2\lambda_2 - E\mu + e^{\lambda_2}\psi_2(\nu) + e^{\lambda_1+\lambda_2}\chi(\nu)\}$$
$$\times \left(\frac{\partial}{\partial\lambda_1}\right)^s \exp\{e^{\lambda_1}\psi_1(\nu) - \overline{M_1}\lambda_1\}. \quad \ldots\ldots(20)$$

§ 3. *Gaseous assemblies with any numbers of components and reactions.* In view of the preliminary formulation of the simple case of § 2, it is now only necessary to specify a notation suitable for the general gaseous assembly. Let the different types of atoms be denoted by the affix r, molecules by the affix s. Then the energy, weight and number of free atoms of type r associated with their uth cell can be written ϵ_u^r, p_u^r and a_u^r. For molecules of type s the corresponding quantities are ϵ_v^s, p_v^s and a_v^s, the weight p_v^s containing the symmetry number. If A^r is the atomic symbol for the atom of type r, the molecular symbol for the molecule of type s may be written in the chemical form

$$\Pi_r A^r q_s^r.$$

We have assumed that the molecule of type s contains q_s^r atoms of type r. All possible reactions may then be regarded as contained in the set

$$\Sigma_r q_s^r A^r \rightleftarrows \Pi_r A^r q_s^r, \quad (s = 1, 2, \ldots),$$

or constructed out of members of the set. The actual sequence of reactions by which equilibrium is attained is without effect on the dissociative *equilibrium*. Let the number of atoms of type r be X_r, the free atoms of type r, M_r, and the molecules of type s, N_s. Then

$$X_r = M_r + \Sigma_s q_s^r N_s, \quad (r = 1, 2, \ldots). \qquad \ldots\ldots(21)$$

To preserve correct the atomic and molecular total we require

$$\Sigma_u a_u^r = M_r, \quad \Sigma_v a_v^s = N_s,$$

and therefore in general

$$\Sigma_u a_u^r + \Sigma_s \Sigma_v q_s^r a_v^s = X_r, \quad (r = 1, 2, \ldots). \quad \ldots\ldots(22)$$

To satisfy the energy equation we require also

$$\Sigma_r \Sigma_u a_u^r \epsilon_u^r + \Sigma_s \Sigma_v a_v^s \epsilon_v^s = E. \qquad \ldots\ldots(23)$$

The total number of weighted complexions is therefore

$$(\Pi_r X_r!) \, \Sigma \, . \, \Pi_r \Pi_u \frac{(p_u^r)^{a_u^r}}{a_u^r!} \Pi_s \Pi_v \frac{(p_v^s)^{a_v^s}}{a_v^s!}, \quad \ldots\ldots(24)$$

where the summation is extended over all positive integral and zero values of the a's which satisfy (22) and (23). To sum (24)

subject to these conditions we introduce the appropriate selector variables and form the expression

$$(\Pi_r X_r!) \, \Pi_r \Pi_u \left[\Sigma_a \frac{(p_u{}^r x_r z^{\epsilon_u{}^r})^{a_u{}^r}}{a_u{}^r!} \right] \Pi_s \Pi_v \left[\Sigma_a \frac{(p_v{}^s x_1{}^{q_s'} x_2{}^{q_s{}^2} \dots z^{\epsilon_v{}^s})^{a_v{}^s}}{a_v{}^s!} \right],$$
$$\dots\dots(25)$$

where Σ_a has its usual meaning of summation over all positive and zero values of the corresponding a. To obtain (24) we must select from (25) the coefficient of $x_1{}^{X_1} x_2{}^{X_2} \dots z^E$. The expression (25) simplifies at once to

$$(\Pi_r X_r!) \exp\{\Sigma_s x_r f_r(z) + \Sigma_s x_1{}^{q_s'} x_2{}^{q_s{}^2} \dots g_s(z)\},$$

where $f_r(z)$ and $g_s(z)$ are the ordinary partition functions for the atom of type r and the molecule of type s. We observe that the exponential in this "selector function" has a single term corresponding to each type of atom or molecule present. We obtain at once, there being j types of atom in all,

$$C = \frac{(\Pi_r X_r!)}{(2\pi i)^{j+1}} \int^{(j+1)} \dots \int \frac{dz \, \Pi_r dx_r}{z^{E+1} \Pi_r x_r{}^{X_r+1}} \exp[\Sigma_r x_r f_r(z) + \Sigma_s x_1{}^{q_s'} x_2{}^{q_s{}^2} \dots g_s(z)].$$
$$\dots\dots(26)$$

For $C\overline{a_u{}^r}$ and similar expressions we can obtain similar integrals. For example

$$C\overline{a_u{}^r} = p_u{}^r \frac{\partial C}{\partial p_u{}^r} = p_u{}^r z^{\epsilon_u{}^r} \frac{\partial C}{\partial f_r}. \qquad \dots\dots(27)$$

The integrand therefore differs only from that for C in (26) by the extra factor $p_u{}^r x_r z^{\epsilon_u{}^r}$. In other cases the extra factors are as follows: for $C\overline{M_r}$, $x_r f_r(z)$; for $C\overline{a_v{}^s}$, $p_v{}^s x_1{}^{q_s'} x_2{}^{q_s{}^2} \dots z^{\epsilon_v{}^s}$; for $C\overline{N_s}$, $x_1{}^{q_s'} x_2{}^{q_s{}^2} \dots g_s(z)$; for $C\overline{E_r}, x_r z \partial f_r(z)/\partial z$; for $C\overline{E_s}, x_1{}^{q_s'} x_2{}^{q_s{}^2} \dots z \partial g_s(z)/\partial z$.

Anticipating the results of § 4, a set of parameters $\xi_1, \xi_2, \dots \vartheta$ may be defined as the unique real positive solution of the equations

$$\xi_r f_r(\vartheta) + \Sigma_s q_s{}^r \xi_1{}^{q_s'} \xi_2{}^{q_s{}^2} \dots g_s(\vartheta) = X_r, \quad (r = 1, 2, \dots), \quad (28)$$

$$\Sigma_r \xi_r \vartheta \frac{\partial f_r(\vartheta)}{\partial \vartheta} + \Sigma_s \xi_1{}^{q_s'} \xi_2{}^{q_s{}^2} \dots \vartheta \frac{\partial g_s(\vartheta)}{\partial \vartheta} = E. \quad \dots\dots(29)$$

It then follows by the method of steepest descents that

$$\overline{a_u{}^r} = p_u{}^r \xi_r \vartheta^{\epsilon_u{}^r}, \quad M_r = \xi_r f_r(\vartheta), \quad \overline{a_u{}^r} = \overline{M_r} p_u{}^r \vartheta^{\epsilon_u{}^r}/f_r(\vartheta), \quad \dots(30)$$

$$\overline{a_v{}^s} = p_v{}^s \xi_1{}^{q_s'} \xi_2{}^{q_s{}^2} \dots \vartheta^{\epsilon_v{}^s}, \quad \overline{N_s} = \xi_1{}^{q_s'} \xi_2{}^{q_s{}^2} \dots g_s(\vartheta), \quad \overline{a_v{}^s} = \overline{N_s} p_v{}^s \vartheta^{\epsilon_v{}^s}/g_s(\vartheta).$$
$$\dots\dots(31)$$

The laws of mass-action follow from (30) and (31) in the form

$$\frac{\overline{N_s}}{\Pi_r (\overline{M_r})^{q_s{}^r}} = \frac{g_s(\vartheta)}{\Pi_r \{f_r(\vartheta)\}^{q_s{}^r}}, \quad (s = 1, 2, \dots). \dots\dots(32)$$

We obtain also the energy distribution laws

$$\overline{E}_r = \xi_r \Im \frac{\partial f_r(\Im)}{\partial \Im} = \overline{M}_r \Im \frac{\partial}{\partial \Im} \log f_r(\Im), \qquad \ldots\ldots(33)$$

$$\overline{E}_s = \xi_1{}^{q_s'} \xi_2{}^{q_s{}^2} \ldots \Im \frac{\partial g_s(\Im)}{\partial \Im} = \overline{N}_s \Im \frac{\partial}{\partial \Im} \log g_s(\Im), \quad \ldots(34)$$

and similarly any other details of the equilibrium state.

§ 4. *Proof of the results of §§ 2 and 3.* We now give a proof of the results of §§ 2 and 3, parts of which for simplicity are written out for the case of three variables x, y, z. The proof however will be seen to be obviously quite general.

The form of the integrand of C and the analogous integrals is that of a triple (in general multiple) power series

$$\Phi = \Sigma_{abc} Q_{abc} x^a y^b z^c, \qquad \ldots\ldots(35)$$

in which the Q_{abc} are all positive and the a, b, c (integers) start at negative values and run to $+\infty$. The domain of convergence of the series (35) in our actual problem is all values of x and y and all z's such that $|z| < 1$, but its actual form is immaterial. For our proof we require certain properties of this function which are obtained in the following

Lemma. For real positive values of x, y, z the function Φ has an absolute minimum at ξ, η, $\Im$ which is the unique solution of the equations

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial z} = 0 \qquad \ldots\ldots(36)$$

in this domain.

(i) Since Φ is always positive, and since $\Phi \to +\infty$ as x, y, z tend to their boundary values (*i.e.* 0, ∞ or 0, 1) in any manner, Φ must have an absolute minimum value Φ_0 which it assumes at some points of the domain of real positive values x, y and z. At such a point ξ, η, $\Im$ equations (36) must of course be satisfied.

(ii) That ξ, η, $\Im$ is the *unique* solution of (36) in the domain will follow at once if it can be shown that *any* stationary value of Φ must be an absolute minimum—that is that, if Φ_0 is any stationary value,

$$\Phi - \Phi_0 \geqslant 0$$

for the whole domain, equality being only possible when $x = \xi$, $y = \eta$, $z = \Im$. If we write $x = e^\lambda$, $y = e^\mu$, $z = e^\nu$, then

$$\Phi = \Sigma_{abc} Q_{abc} e^{a\lambda + b\mu + c\nu},$$

and, by Taylor's theorem, for *any* stationary value Φ_0,

$$\Phi - \Phi_0 = \tfrac{1}{2} \left[(\lambda - \lambda_0)^2 \frac{\partial^2 \Phi}{\partial \lambda^2} + \ldots + 2(\lambda - \lambda_0)(\mu - \mu_0) \frac{\partial^2 \Phi}{\partial \lambda \partial \mu} \right] \quad (37)$$

an expression in which all the partial differential coefficients are to be evaluated for some particular set of values of λ, μ, ν. It is

therefore only necessary to prove that the expression on the right of (37) is a positive quadratic form.

(iii) The proof of the lemma reduces therefore to the proof of the essential inequalities

$$\frac{\partial^2 \Phi}{\partial \lambda^2} > 0, \quad \begin{vmatrix} \dfrac{\partial^2 \Phi}{\partial \lambda^2} & \dfrac{\partial^2 \Phi}{\partial \lambda \partial \mu} \\[2mm] \dfrac{\partial^2 \Phi}{\partial \lambda \partial \mu} & \dfrac{\partial^2 \Phi}{\partial \mu^2} \end{vmatrix} > 0, \quad \begin{vmatrix} \dfrac{\partial^2 \Phi}{\partial \lambda^2} & \dfrac{\partial^2 \Phi}{\partial \lambda \partial \mu} & \dfrac{\partial^2 \Phi}{\partial \lambda \partial \nu} \\[2mm] \dfrac{\partial^2 \Phi}{\partial \lambda \partial \mu} & \dfrac{\partial^2 \Phi}{\partial \mu^2} & \dfrac{\partial^2 \Phi}{\partial \mu \partial \nu} \\[2mm] \dfrac{\partial^2 \Phi}{\partial \lambda \partial \nu} & \dfrac{\partial^2 \Phi}{\partial \mu \partial \nu} & \dfrac{\partial^2 \Phi}{\partial \nu^2} \end{vmatrix} > 0.$$

[For more variables the series of inequalities is correspondingly extended.] Firstly

$$\frac{\partial^2 \Phi}{\partial \lambda^2} = \Sigma_{abc} Q_{abc} a^2 e^{a\lambda + b\mu + c\nu} > 0,$$

for every term is positive. Secondly

$$\begin{vmatrix} \dfrac{\partial^2 \Phi}{\partial \lambda^2} & \dfrac{\partial^2 \Phi}{\partial \lambda \partial \mu} \\[2mm] \dfrac{\partial^2 \Phi}{\partial \lambda \partial \mu} & \dfrac{\partial^2 \Phi}{\partial \mu^2} \end{vmatrix} = \begin{vmatrix} \Sigma_{abc} Q_{abc} a^2 e^{a\lambda + b\mu + c\nu} & \Sigma_{a'b'c'} Q_{a'b'c'} a'b' e^{a'\lambda + b'\mu + c'\nu} \\[2mm] \Sigma_{abc} Q_{abc} ab e^{a\lambda + b\mu + c\nu} & \Sigma_{a'b'c'} Q_{a'b'c'} (b')^2 e^{a'\lambda + b'\mu + c'\nu} \end{vmatrix}.$$

If we collect together all terms containing $Q_{abc} Q_{a'b'c'}$, we see that this determinant reduces to

$$\Sigma' Q_{abc} Q_{a'b'c'} e^{(a+a')\lambda + (b+b')\mu + (c+c')\nu} \left\{ \begin{vmatrix} a^2 & a'b' \\ ab & (b')^2 \end{vmatrix} + \begin{vmatrix} (a')^2 & ab \\ a'b' & b^2 \end{vmatrix} \right\}.$$

The terms { } are formed of all possible permutations of the dashed and plain letters and reduce to

$$ab' \begin{vmatrix} a & a' \\ b & b' \end{vmatrix} + a'b \begin{vmatrix} a' & a \\ b' & b \end{vmatrix} = \begin{vmatrix} a & a' \\ b & b' \end{vmatrix}^2.$$

The summation Σ' is over all possible values of a, b, c, a', b', c', the specified permutations being excluded. Since every term in Σ' is positive the second condition is fulfilled. Finally an exactly similar argument shows that

$$\begin{vmatrix} \dfrac{\partial^2 \Phi}{\partial \lambda^2} & \dfrac{\partial^2 \Phi}{\partial \lambda \partial \mu} & \dfrac{\partial^2 \Phi}{\partial \lambda \partial \nu} \\[2mm] \dfrac{\partial^2 \Phi}{\partial \lambda \partial \mu} & \dfrac{\partial^2 \Phi}{\partial \mu^2} & \dfrac{\partial^2 \Phi}{\partial \mu \partial \nu} \\[2mm] \dfrac{\partial^2 \Phi}{\partial \lambda \partial \nu} & \dfrac{\partial^2 \Phi}{\partial \mu \partial \nu} & \dfrac{\partial^2 \Phi}{\partial \nu^2} \end{vmatrix} \quad \left[= J\left(\dfrac{\dfrac{\partial \Phi}{\partial \lambda}, \ \dfrac{\partial \Phi}{\partial \mu}, \ \dfrac{\partial \Phi}{\partial \nu}}{\lambda, \ \mu, \ \nu} \right) \right]$$

$$= \Sigma' Q_{abc} Q_{a'b'c'} Q_{a''b''c''} e^{(a+a'+a'')\lambda + (b+b'+b'')\mu + (c+c'+c'')\nu} \begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix}^2,$$

$$> 0.$$

This completes the proof of the Lemma, which can obviously be extended to any number of variables.

The integrals which we desire to study asymptotically are all of the form

$$\frac{1}{(2\pi i)^3} \iiint \Phi \, \frac{dx\,dy\,dz}{xyz} \,,$$

or combinations of integrals of the type

$$\frac{1}{(2\pi i)^3} \iiint X\Phi \, \frac{dx\,dy\,dz}{xyz}$$

where X is one of the terms in Φ. On the contours of integration the maximum value of the modulus of the integrand occurs when all the variables are real and positive, and, as we shall see, if the contours are arranged to go through the real-value minimum of Φ, it is only the contribution from this neighbourhood which need be considered.

To see that only this neighbourhood makes an effective contribution needs somewhat careful consideration. If certain relations are satisfied between the a, b, c, there may be other points on the contours at which the amplitudes of all the terms are again equal so that the same maximum value is repeated. The same difficulty comes in when z is the only selector variable (paper 1), in which case we have shown that the repetitions of the maximum are without effect on the physical applications, and can in fact be avoided by proper choice of the unit of energy. In this case in which Φ takes the special form

$$\frac{\exp (x_1 f_1 + x_2 f_2 + x_1 x_2 g)}{x_1^{X_1} x_2^{X_2} z^E}$$

it is not difficult to see that no repetitions of the maximum can occur except those which are identical with the repetitions of paper 1. To attain the maximum every term in $x_1 f_1 + x_2 f_2 + x_1 x_2 g$ must be real and positive. This will occur and occur only at points at which the amplitudes θ_{x_1}, θ_{x_2}, θ_z of x_1, x_2, z satisfy the relations

$$\theta_{x_1} + \epsilon_u' \theta_z \equiv 0 \;(\mathrm{Mod}\,2\pi) \quad (all\ u),$$
$$\theta_{x_2} + \epsilon_u^2 \theta_z \equiv 0 \;(\mathrm{Mod}\,2\pi) \quad (all\ u),$$
$$\theta_{x_1} + \theta_{x_2} + \eta_v \theta_z \equiv 0 \;(\mathrm{Mod}\,2\pi) \quad (all\ v).$$

The first of these equations is equivalent to the assertions (1) that ϵ_u' is of the form $\epsilon_0' + n\zeta_u'$, where n and the ζ_u' are positive integers, and (2) that

$$\theta_z = \frac{2\pi r}{n} \,, \quad \theta_{x_1} = -\frac{2\pi r}{n}\,\epsilon_0', \quad (0 \leq r < n).$$

The remaining equations add the information that

$$\epsilon_u^2 = \epsilon_0^2 + n\zeta_u^2, \quad \eta_v = \eta_0 + n\zeta_v, \quad \theta_{x_2} = -\frac{2\pi r}{n}\,\epsilon_0^2,$$
$$\eta_0 - \epsilon_0' - \epsilon_0^2 \equiv 0 \;\mathrm{Mod}\,(n/r).$$

It is easy to see that if $n > 1$ these relations may permit of a number of subsidiary maxima. It must be recalled however that f_1 and f_2 are necessarily partition functions for free atoms and therefore start with a cell of zero energy, $\epsilon_0' = \epsilon_0^2 = 0$. Subsidiary maxima can therefore only occur for real values of the x's, that is to say can only be strictly analogous to the subsidiary maxima of paper 1, and can be removed by a change of the unit of energy.

It may be mentioned finally that, even if the most general type of subsidiary maximum could occur, it would not affect any physical result, for these depend only on the ratio of two of our integrals, and owing to the special forms of the integrands all such ratios are completely unaltered.

It is convenient at this point to recall the forms of the partition functions f and g. Each of them contains as a factor the partition function for the translatory motion of the atom or molecule as a whole. This function (paper 1 § 12) is

$$\Sigma_u \frac{(dq_1 \ldots dp_3)_u}{h^3} z^{\epsilon_u}.$$

The limiting form of this function is an integral which evaluates to

$$\frac{(2\pi m)^{\frac{3}{2}} V}{h^3 (\log 1/z)^{\frac{3}{2}}}, \qquad \ldots\ldots(37\,a)$$

in which m is the mass of the molecule or atom and V is the volume of the assembly available to this particular species. We can obviously however carry out the integrations with respect to the q's before beginning to proceed to the limit for the p's. We thus obtain for each partition function the factor V, the rest of the function depending only on z. It is convenient to recognize this structure explicitly, by writing $V_r F_r(z)$ for $f_r(z)$ and $V_s G_s(z)$ for $g_s(z)$. If there are semipermeable membranes to be considered the V_r and V_s need not all be equal. As we shall see directly, the asymptotic expansions which we require when E and the X's are large naturally involve also the corresponding largeness of the V_r and V_s.

Consider now the integral

$$C = \frac{X_1! \, X_2!}{(2\pi i)^3} \iiint \frac{dx_1 dx_2 dz}{x_1 x_2 z} \frac{\exp\{V_1 x_1 F_1 + V_2 x_2 F_2 + V x_1 x_2 G\}}{x_1^{X_1} x_2^{X_2} z^E},$$

whose integrand satisfies the conditions of the Lemma. The unique minimum on the positive real axis is at ξ_1, ξ_2, ϑ, where ξ_1, ξ_2, ϑ is the unique relevant solution of the equations

$$V_1 \xi_1 F_1 + V \xi_1 \xi_2 G = X_1, \qquad \ldots\ldots(38)$$

$$V_2 \xi_2 F_2 + V \xi_1 \xi_2 G = X_2, \qquad \ldots\ldots(39)$$

$$V_1 \xi_1 \vartheta F_1' + V_2 \xi_2 \vartheta F_2' + V \xi_1 \xi_2 \vartheta G' = E. \quad (F_1' = \partial F_1/\partial \vartheta, \text{ etc.}) \quad (40)$$

We observe that ξ_1, ξ_2, ϑ are *intensive* parameters. Their values are unaltered if E, the X's and the V's are made large in any fixed ratios. We may assume that, when the circles of integration are made to pass through $\xi_1 \xi_2 \vartheta$, this point provides the unique relevant maximum value of the modulus of the integrand on the contours of integration. To show that its neighbourhood contributes the dominant part of the whole integral we write

$$x_1 = \xi_1 e^{i\alpha_1}, \quad x_2 = \xi_2 e^{i\alpha_2}, \quad z = \vartheta e^{i\beta},$$

$$V_1 x_1 F_1 + V_2 x_2 F_2 + V x_1 x_2 G = V \Psi(i\alpha_1, i\alpha_2, i\beta),$$

so that if the ratios of the V's are fixed Ψ is independent of V. Then for small values of α_1, α_2 and β the integrand takes the form

$$\frac{\exp\{V_1\xi_1 F_1(\vartheta) + V_2\xi_2 F_2(\vartheta) + V\xi_1\xi_2 G(\vartheta)\}}{\xi_1{}^{X_1}\xi_2{}^{X_2}\vartheta^E} \exp\left[-\tfrac{1}{2}V\left\{\alpha_1{}^2\frac{\partial^2\Psi}{\partial i a_1{}^2} + \cdots \right.\right.$$
$$\left.\left. + 2a_1 a_2\frac{\partial^2\Psi}{\partial i a_1\,\partial i a_2}\right\}_0 + O\left(V a^3\right) + O\left(V a^4\right)\right], \quad \dots\dots(41)$$

in which the differential coefficients are to be evaluated at $\alpha_1 = \alpha_2 = \beta = 0$. We have already shown in the proof of the Lemma that this quadratic form is essentially positive. If V is large it follows by the arguments of paper 1 that the variables α_1, α_2, β in the quadratic terms may be supposed to range from $-\infty$ to $+\infty$ while all other terms remain small. By a linear transformation the quadratic form can be reduced to its principal axes, and the value of this exponential integrated in all variables from $-\infty$ to $+\infty$ can be shown to be

$$\left(\frac{2\pi}{V}\right)^{\frac{3}{2}}\left\{J\left(\begin{matrix}\dfrac{\partial\Psi}{\partial i\alpha_1}, & \dfrac{\partial\Psi}{\partial i\alpha_2}, & \dfrac{\partial\Psi}{\partial i\beta}\\ i\alpha_1, & i\alpha_2, & i\beta\end{matrix}\right)_0\right\}^{-\frac{1}{2}}. \quad \dots\dots(42)$$

The terms $O\left(V a^3\right)$ vanish on integration. The terms $O\left(V a^4\right)$ leave an error term $O\left(1/V\right)$. We have already shown in the Lemma that $J > 0$. We therefore find ultimately for C the asymptotic form

$$C = \frac{X_1!\,X_2!}{(2\pi V)}\frac{\exp\{V\Psi(0,0,0)\}}{\xi_1{}^{X_1}\xi_2{}^{X_2}\vartheta^E}\left[\left\{J\left(\begin{matrix}\dfrac{\partial\Psi}{\partial i\alpha_1}, & \dfrac{\partial\Psi}{\partial i\alpha_2}, & \dfrac{\partial\Psi}{\partial i\beta}\\ i\alpha_1, & i\alpha_2, & i\beta\end{matrix}\right)_0\right\}^{-\frac{1}{2}} + O\left(\frac{1}{V}\right)\right].$$
$$\dots\dots(43)$$

Any other integral containing an extra factor X in the integrand can be discussed in exactly the same way. The leading term differs from C simply in the extra factor $X(\xi_1, \xi_2, \vartheta)$ and there is still an error term $O\left(1/V\right)$ as before. We are led at once to all the results obtained formally in §§ 2 and 3.

In conclusion it should be mentioned that it may often happen in actual cases that certain theoretically possible species may be exceedingly rare in the assembly; some term such as $\xi_r f_r(\vartheta)$ may be excessively small. This will not in any way invalidate our formulae. If we suppose that a certain species is *completely* absent, we have merely to drop a single term from the main exponential. The arguments can then all be repeated unaltered, and it is clear from the form of the equations that the equilibrium laws which we so obtain differ only imperceptibly from their complete form. The general validity in fact of this proof of the laws of dissociative equilibrium is dependent solely on the sufficient size of the assembly as a whole and not on the effective presence of any particular possible species.

§ 5. *The existence and properties of entropy.* The new method allows of a satisfactory handling of the entropy of the assembly. We define the absolute temperature and the entropy by means of the usual equation (papers 2 and 5)

$$dQ = dE + \Sigma Y dy = T dS, \qquad \ldots\ldots(44)$$

in which $\Sigma Y dy$ represents the external work done by the assembly in a small variation. In the present notation the generalized force Y (of course a mean value), which is defined by the equation

$$Y = \Sigma_r \Sigma_u \overline{a_u{}^r} \left(- \frac{\partial \epsilon_u{}^r}{\partial y} \right) + \Sigma_s \Sigma_v \overline{a_v{}^s} \left(- \frac{\partial \epsilon_v{}^s}{\partial y} \right),$$

reduces with the help of (30) and (31) to

$$Y = \frac{1}{\log 1/\vartheta} \left\{ \Sigma_r \xi_r \frac{\partial f_r}{\partial y} + \Sigma_s \xi_1{}^{q_s'} \xi_2{}^{q_s^2} \ldots \frac{\partial g_s}{\partial y} \right\}. \quad \ldots\ldots(45)$$

On differentiating E (29), we find that

$$\log 1/\vartheta \, dQ = d\left[\Sigma_r \xi_r \left\{ f_r + \log 1/\vartheta \cdot \vartheta \frac{\partial f_r}{\partial \vartheta} \right\} + \Sigma_s \xi_1{}^{q_s'} \xi_2{}^{q_s^2} \ldots \left\{ g_s + \log 1/\vartheta \cdot \vartheta \frac{\partial g_s}{\partial \vartheta} \right\} \right]$$
$$- \Sigma_r f_r \, d\xi_r - \Sigma_s g_s \, d(\xi_1{}^{q_s'} \xi_2{}^{q_s^2} \ldots).$$

The last sets of terms reduce at once with the help of (28) to

$$- \Sigma_r X_r \, d\xi_r / \xi_r,$$

and we therefore find that $\log 1/\vartheta \, dQ$ is a perfect differential and that

$$1/kT = \log 1/\vartheta,$$

$$(S - S_0)/k = \Sigma_r \xi_r \left\{ f_r + \log 1/\vartheta \cdot \vartheta \frac{\partial f_r}{\partial \vartheta} \right\} + \Sigma_s \xi_1{}^{q_s'} \xi_2{}^{q_s^2} \ldots \left\{ g_s + \log 1/\vartheta \cdot \vartheta \frac{\partial g_s}{\partial \vartheta} \right\} - \Sigma_r X_r \log \xi_r,$$
$$\ldots\ldots(46)$$

$$= \Sigma_r \xi_r f_r + \Sigma_s \xi_1{}^{q_s'} \xi_2{}^{q_s^2} \ldots g_s - E \log \vartheta - \Sigma_r X_r \log \xi_r. \qquad \ldots\ldots(47)$$

This shows at once that the integrand of C is $\exp\{(S - S_0)/k\}$. The expression can be reduced when desired to the form given in paper 5.

We are now in a position to establish simply the "increasing property" of the entropy for the junction of two assemblies—that is, that with suitable conventions as to the arbitrary constants

$$S' + S'' \leqslant S,$$

where S' and S'' are the entropies of the two assemblies before junction and S the entropy of the combined assembly after junction. It is convenient for this purpose to exhibit explicitly the dependence of the partition functions on the volume available to each species of molecule or atom as in § 4. If then we distinguish all quantities

referring to the two separate assemblies by single or double primes, we have

$$S'/k = \Sigma_r \xi_r' V_r' F_r(\Im') + \Sigma_s \xi_1'^{q_s'} \xi_2'^{q_s^2} \ldots V_s' G_s(\Im') - E' \log \Im' - \Sigma_r X_r' \log \xi_r',$$

$$S''/k = \Sigma_r \xi_r'' V_r'' F_r(\Im'') + \Sigma_s \xi_1''^{q_s'} \xi_2''^{q_s^2} \ldots V_s'' G_s(\Im'') - E'' \log \Im' - \Sigma_r X_r'' \log \xi_r'',$$

$$S/k = \Sigma_r \xi_r(V_r' + V_r'') F_r(\Im) + \Sigma_s \xi_1^{q_s} \xi_2^{q_s^2} \ldots (V_s' + V_s'') G_s(\Im) - (E' + E'') \log \Im - \Sigma_r(X_r' + X_r'') \log \xi_r$$
$$= S'(\Im, \xi_1, \xi_2, \ldots)/k + S''(\Im, \xi_1, \xi_2, \ldots)/k.$$

It has been shown however in § 4 that $\Im'$, ξ_1', ξ_2', ... define the unique minimum of the function S', so that

$$S'(\Im', \xi_1', \xi_2', \ldots) \leqslant S'(\Im, \xi_1, \xi_2, \ldots),$$

equality being only possible when

$$\Im' = \Im, \quad \xi_1' = \xi_1, \quad \xi_2' = \xi_2, \ldots.$$

Similarly $\quad S''(\Im'', \xi_1'', \xi_2'', \ldots) \leqslant S''(\Im, \xi_1, \xi_2, \ldots),$

equality being only possible when

$$\Im'' = \Im, \quad \xi_1'' = \xi_1, \quad \xi_2'' = \xi_2, \ldots.$$

We see therefore that $\qquad S' + S'' \leqslant S,$

equality being only possible when

$$\Im' = \Im'' = \Im, \quad \xi_1' = \xi_1'' = \xi_1, \quad \xi_2' = \xi_2'' = \xi_2, \ldots.$$

On referring to equations (28)—(31) we see that the necessary and sufficient conditions are

$$\Im' = \Im'', \quad \xi_1' = \xi_1'', \quad \xi_2' = \xi_2'', \ldots, \qquad \ldots\ldots(48)$$

which are equivalent to asserting that the separate assemblies must have equal temperatures and concentrations.

§ 6. *The physical meaning of the ξ's.* We have been led to introduce these variables $\xi_1, \xi_2, \ldots$ by the nature of the mathematics, and to use them in defining the statistical equilibrium of an assembly. It is not without interest to point out in passing that they possess a natural physical interpretation in terms of *the partial potentials* of the various constituents in the assembly.

Partial potentials μ_r may be defined in thermodynamics by the equation[*]

$$dE + \Sigma Y dy = dQ = TdS + \Sigma_r \mu_r dM_r; \qquad \ldots\ldots(49)$$

in forming this variation we are to suppose that the quantities M_r of the various constituents in our assembly are varied as well as the temperature and the ordinary geometrical constraints. If we form the variation of (46) in this manner we obtain

$$TdS = dQ - \Sigma_r kT \log \xi_r dX_r. \qquad \ldots\ldots(50)$$

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[*] Cf. Bryan, *Thermodynamics*, p. 152.

Now X_r is the number of atoms of type r, and in (49) M_r is the quantity of the rth substance reckoned in gram-molecules. Therefore

$$X_r = R M_r / k,$$

and

$$\mu_r = RT \log \xi_r. \qquad \qquad \ldots\ldots(51)$$

Equation (51) shows then that ξ_r is simply related to the partial potential of the rth constituent of the assembly. If we evaluate ξ_r by means of the equation

$$\overline{M_r} = \xi_r V_r F_r(\vartheta)$$

for the concentration of free atoms of type r we find, on inserting the value of $F_r(\vartheta)$ from $(37\,a)$,

$$\log \xi_r = \log \nu_r - \tfrac{3}{2} \log T + \log \frac{h^3}{(2\pi m k)^{\frac{3}{2}}} \qquad \ldots\ldots(52)$$

which is consistent with the usual value of μ_r for a perfect gas*.

§ 7. *Crystals.* It is easy to adapt the present methods to include in the assembly crystals for which we can construct partition functions (paper 3). Consider first an assembly of X molecules (or atoms) of which P compose a single crystal (or a small number of such crystals) and N its molecular vapour, so that only questions of evaporation and condensation, not of dissociation, arise. To enumerate the total number of weighted complexions, we observe first that the number of weighted complexions of the vapour of N molecules with energy F is by the principles of § 2, the coefficient of $z^F x^N$ in

$$N! \, \Sigma_a \frac{(p_1 x z^{\epsilon_1})^{a_1} \cdots}{a_1! \cdots} = N! \exp \{x g(z)\},$$

where $g(z)$ is the partition function for the free molecules. The crystal (paper 3) has a partition function which is effectively of the form $[k(z)]^P$, represented by weighted complexions in number equal to the coefficient of z^U in $[k(z)]^P$ when U is its internal energy. If the energy zeros are suitably defined then $U + F = E$, the total energy of the assembly, and the total number of weighted complexions representing this example of the assembly, including all ways of dividing the energy between the crystal and the vapour, is the coefficient of $x^X z^E$ in

$$N! \, [x k(z)]^P \exp \{x g(z)\}.$$

There are $X!/N!$ such examples†. When therefore we include all

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* Bryan, *loc. cit.*, p. 120.
† The symmetry number of the molecule is already included in its weight.
  It is perhaps worth while to point out that the essential difference between solids and vapours appears just in this, that we divide here by $N!$ but not by $P!$ This is of course necessitated by the fact that a complete interchange of position among the molecules of the vapour has already been allowed for in calculating the

values of N, we find that the total number of weighted complexions C is given by

$$C = \frac{X!}{(2\pi i)^2} \iiint \frac{dx\,dz}{x^{X+1}z^{E+1}} \exp\{xg(z) - \log[1 - xk(z)]\}. \quad (53)$$

Equation (53) leads at once to the usual formulae. For example*

$$C\bar{P} = \frac{X!}{(2\pi i)^2} \iiint \frac{dx\,dz}{x^{X+1}z^{E+1}} \frac{xk(z)}{1 - xk(z)} \exp\{xg(z) - \log[1 - xk(z)]\},$$

$$\dots\dots(54)$$

leading to

$$\bar{P} = \frac{\xi k(\vartheta)}{1 - \xi k(\vartheta)}, \quad \dots\dots(55)$$

where

$$E = \xi\vartheta \frac{\partial g(\vartheta)}{\partial(\vartheta)} + \frac{\xi\vartheta \dfrac{\partial k(\vartheta)}{\partial\vartheta}}{1 - \xi k(\vartheta)}, \quad \dots\dots(56)$$

$$X = \xi g(\vartheta) + \frac{\xi k(\vartheta)}{1 - \xi k(\vartheta)}. \quad \dots\dots(57)$$

On evaluating (55) we find that

$$\xi k(\vartheta) = 1, \quad \dots\dots(58)$$

if the crystal phase is effectively present, so that $\bar{P}$ is large. From (55) and (57), or by the usual direct argument,

$$\bar{N} = \xi g(\vartheta),$$

which in virtue of (58) becomes

$$\bar{N} = g(\vartheta)/k(\vartheta). \quad \dots\dots(59)$$

This in the present notation is the usual formula for the vapour pressure of the crystal†.

It is now clear how to include crystals in the general assembly. In the expression (26) for C the exponential factor in the absence of crystals is

$$\exp[\Sigma_r x_r f_r(z) + \Sigma_s x_1{}^{q_s'} x_2{}^{q_s{}^2} \dots g_s(z)].$$

partition function for their translatory motion and would without the factor $1/N!$ be counted twice over, whereas so long as molecules remain in the solid they are incapable of changing their relative positions.

It should be observed that the form $[k(z)]^P$ for the partition function of the crystal is only valid for large values of P. For small values there is of course a partition function $k(z, P)$, but the deviation of this from $[k(z)]^P$ will be without effect on the final equilibrium, provided that the crystal phase is effectively present.

* A strict view of equations (53) and (54) leads one to include the extra factor $1/(1 - xk(z))$ in (54) in the exponential term, with an expectation that (54) may have to be evaluated at values of ξ and ϑ different from those for (53). It appears on a closer investigation that no effective difference is made by this inclusion, as a consequence of the particular values of ξ and ϑ concerned.

† Paper 4, equation 10·3.

If for example the molecule 1 is also present in crystal form, with partition function $k_1(z)$, the exponential factor becomes instead

$$\exp\left[\Sigma_r x_r f_r(z) + \Sigma_s x_1^{q_s'} x_2^{q_s^2} \dots g_s(z) - \log\left\{1 - x_1^{q_1'} x_2^{q_1^2} \dots k_1(z)\right\}\right].$$
$$\dots\dots(60)$$

The equations (28) become

$$X_r = \xi_r f_r(\vartheta) + \Sigma_s q_s^r \xi_1^{q_s'} \xi_2^{q_s^2} \dots g_s(\vartheta) + \frac{q_1^r \xi_1^{q_1'} \xi_2^{q_1^2} \dots k_1(\vartheta)}{1 - \xi_1^{q_1'} \xi_2^{q_1^2} \dots k_1(\vartheta)}, \ (61)$$

and the equation for $\bar{P}_1$, the molecules in the crystal,

$$\bar{P}_1 = \frac{\xi_1^{q_1'} \xi_2^{q_1^2} \dots k_1(\vartheta)}{1 - \xi_1^{q_1'} \xi_2^{q_1^2} \dots k_1(\vartheta)}, \qquad \dots\dots(62)$$

which leads ($\bar{P}_1$ large) to

$$\xi_1^{q_1'} \xi_2^{q_1^2} \dots k_1(\vartheta) = 1. \qquad \dots\dots(63)$$

The ordinary laws of mass action in the vapour phase (32), all follow without modification :

$$\frac{\overline{N}_s}{\Pi_r (\overline{M}_r)^{q_s^r}} = \frac{g_s(\vartheta)}{\Pi_r \{f_r(\vartheta)\}^{q_s^r}}.$$

To these must now be added the vapour-pressure equation

$$\overline{N}_1 = g_1(\vartheta)/k_1(\vartheta) \qquad \dots\dots(64)$$

which follows from (31) and (63) when this type of crystal is present. The analysis thus leads at once to the well known modifications of the laws of mass action for gas reactions due to the presence of a solid (crystalline) phase of any of the constituent molecules.

The method of evaluating any other mean values connected with assemblies containing crystals, is now sufficiently clear to need no further remarks.

On the Correction for Non-Uniformity of Field in Experiments on the Magnetic Deflection of β Rays. By D. R. HARTREE, St John's College. (Communicated by Mr R. H. FOWLER.)

§ 1. *Introduction.*

In experiments on the magnetic deflection of β rays, the rays produced by a source O (Fig. 1) are passed through a slit S and, after being bent by an approximately constant magnetic field, fall upon a photographic plate placed in different positions in different experimental arrangements. If the field were constant, the trajectory of a β particle would be a circle, and the one passing through the centre of the slit would strike the plate in some point P; the object of this note is to investigate the effect of deviations from uniformity of the field. The corrections for these effects can be expressed as variations of position of P, or by specifying the constant variation of the field which would produce the same displacement on the plate as the actual variation, which may be different in different parts of the field. The latter is done by means of a 'weighting function' expressing the relative importance of variations of the field in different parts of the trajectory.

The problem is slightly complicated by the fact that in general we want to know the displacement on the plate, not for particles which leave the source O in the same direction in the normal and varied field, but for those which leave the source in such directions as to pass through the centre of the slit in the two cases. These directions will generally be different, and the effect of this change of direction need not be small compared to the direct effect of variations of the field.

Fig. 1

Some general equations will be established first, and then applied to two particular experimental arrangements. The argument is of course a special case of a well known method of calculating the first order variations of the solutions of any system of differential equations.

It is convenient to use the value of $(H\rho)$ for a β particle rather than its velocity. $(H\rho)$ is the quantity actually measured in the laboratory, and is often easier to deal with; at low velocities it is
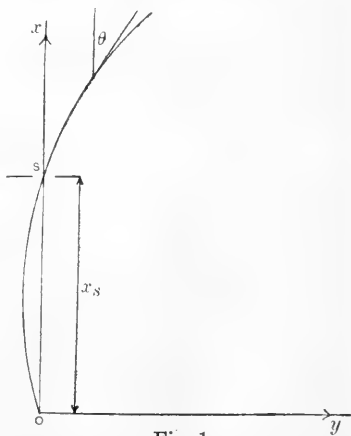
equal to mv/e. The problem then is purely geometrical, not mechanical. It is convenient to have one symbol for this quantity, which will be written B, so that in a field H the radius of curvature of the trajectory is

$$\rho = B/H. \qquad \ldots\ldots(1)$$

The analysis will apply equally well to the magnetic deviation of α particles.

Only first order variations will be considered.

§ 2. *Equations of Variation of the Trajectory of a β Particle.*

Take the axis of x through source and slit (see Fig. 1), and the axis of y perpendicular to it and to the field, and write θ for the angle between the direction of motion of the β particle and the x axis, and s for the distance travelled from the origin. Then the equations of the trajectory of a β particle are

$$\frac{d\theta}{ds} = \frac{H}{B}, \quad \frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta. \quad \ldots(2)\,(a, b, c)$$

If the induced electric field on the electron due to its motion through a variable magnetic field is neglected, these equations do not depend on H or B being constant along a trajectory.

Let the prefix D before any quantity indicate the difference between the values of that quantity, *for the same value of s,* in the normal trajectory and in a trajectory in a slightly different field $H + DH$ (where DH may vary along the trajectory) and with slightly different initial conditions. Then since only first order variations are being considered, D can be used as a differential operator subject to the condition $Ds = 0$, so from equations (2) we obtain the equations of variation

$$\frac{d\,(D\theta)}{ds} = \frac{DH}{B}, \quad \frac{d\,(Dx)}{ds} = -\sin\theta\,.\,D\theta, \quad \frac{d\,(Dy)}{ds} = \cos\theta\,.\,D\theta,$$
$$\ldots\ldots(3)\,(a, b, c)$$

where the quantities occurring without the prefix D are to be given the values they have in the normal trajectory.

The displacement on the plate is (to the first order) of the general form

$$DR = \xi Dx_P + \eta Dy_P, \qquad \ldots\ldots(4)$$

where the suffix P indicates the values of the quantities Dx and Dy for the value of s at the point P on the normal trajectory where it strikes the plate, and R is the distance of P from an origin fixed on the plate.

Equations (3) could be solved by quadratures for any given distribution of DH along the trajectory, and given variations $D\theta_0$, Dx_0 and Dy_0 of the initial conditions, but a trial and error method would have to be used to find the value of $D\theta_0$ so that the varied

trajectory should go through the same point of the slit as the normal trajectory, and also the results would not be useful for any other set of values of DH. Also in the course of the solution, values of $D\theta_0$ would have to be calculated for a set of points on the trajectory; this is superfluous as the only quantity whose value is required is DR defined by (4).

A neater, more useful and more interesting solution is obtained by the use of a new set of equations associated with equations (3), called the adjoint system*, viz.

$$\frac{d\alpha}{ds} = \beta \sin\theta - \gamma\cos\theta, \quad \frac{d\beta}{ds} = 0, \quad \frac{d\gamma}{ds} = 0. \quad (5) \ (a, b, c)$$

By multiplying equations (5) by $D\theta$, Dx, Dy respectively, and equations (3) by α, β, γ and adding, we get

$$\frac{d}{ds}(\alpha D\theta + \beta Dx + \gamma Dy) = \frac{\alpha DH}{B}, \qquad \ldots\ldots(6)$$

or, integrating from a variable point p to the fixed point P already defined,

$$[\alpha D\theta + \beta Dx + \gamma Dy]_P = [\alpha D\theta + \beta Dx + \gamma Dy]_p + \int_p^P \frac{\alpha DH ds}{B}.$$
$$\ldots\ldots(7)$$

It will be seen that equations (5) demand that in this problem β and γ must be constants independent of s. It is however advisable for the sake of general symmetry to carry through the solution formally as if β and γ were functions which vary with s. Thus when we speak of arbitrary initial values of β and γ these are naturally β and γ themselves.

Suppose the arbitrary constants in the solution of equations (5) are chosen to satisfy the conditions

$$\alpha_P = 0, \quad \beta_P = \xi, \quad \gamma_P = \eta. \qquad \ldots\ldots(8)$$

Then using (4) and taking the origin O for the point p, (7) becomes

$$DR = [\alpha D\theta + \beta Dx + \gamma Dy]_0 + \int_O^P \frac{\alpha DH ds}{B}. \quad \ldots\ldots(9)$$

Here we have a formula for the direct calculation of the quantity DR, whose value is required, given the variations in conditions $D\theta$, Dx, Dy at the origin, and the variation of the field DH along the trajectory. This formula depends on the solution of the adjoint equations (5), with arbitrary constants determined by conditions (8).

Taking a general lower limit p instead of the special one O in (9) for a moment, and putting $Dx_p = Dy_p = 0$, and $DH = 0$ over the whole trajectory, then

$$DR = \alpha_p D\theta_p,$$

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* See E. Goursat, *Cours d'Analyse*, Tome II, p. 481.

so that α_p can be interpreted as *the function which specifies the first order variation in R for a variation of conditions consisting of a change of θ at the point p only, i.e.*

$$\alpha_p = \frac{\partial R}{\partial \theta_p}.$$

Similar interpretations can be given to β and γ.

The determination of the value of $D\theta_0$ so that the varied trajectory goes through the same point of the slit as the normal trajectory is easily made using the adjoint system.

If suffix S denotes values at the slit, then the variation in the value of y at which a trajectory cuts the line $x = x_S$ is

$$DR_S = Dy_S - Dx_S \tan \theta_S$$

which is of the form of (4) with $\xi = -\tan \theta_S$, $\eta = 1$.

Let α', β', γ' be the solution of (5) with arbitrary constants determined by

$$\alpha_S = 0, \quad \beta_S = \xi' = -\tan \theta_S, \quad \gamma_S = \eta' = 1. \quad \ldots\ldots(10)$$

Then using the integral of (6) between limits O and S we get

$$DR_S = [\alpha' D\theta + \beta' Dx + \gamma' Dy]_0 + \int_O^S \frac{\alpha' DH\, ds}{B}, \quad \ldots\ldots(11)$$

corresponding to (9). For the trajectory under the varied conditions to go through the same point of the slit as the normal trajectory, $DR_S = 0$; hence (11) gives $D\theta_0$, which can then be substituted in (9) to give DR.

When $Dx_0 = Dy_0 = 0$, which is the most important case, the result is

$$DR = \frac{1}{B}\left[\int_O^S \left(\alpha - \frac{\alpha_0}{\alpha_0'}\alpha'\right) dH\, ds + \int_S^P \alpha\, DH\, ds\right]. \quad (12)$$

This gives the displacement on the plate due to the variation DH of the field from normal, for particles going through the same point of the slit in the two trajectories.

The effects of finite width of slit and source can be found by eliminating $D\theta_0$ between (9) and (11), and giving DR_S, Dx_0 and Dy_0 non-zero values.

As to the solution of the equations (5), the second and third, with conditions (8), give

$$\beta = \xi, \quad \gamma = \eta,$$

and putting these in and using (2 a), (5 a) becomes

$$\frac{d\alpha}{d\theta} = \frac{B}{H}[\xi \sin \theta - \eta \cos \theta]. \quad \ldots\ldots(13)$$

In practical cases B and H are constant for the normal trajectory and this equation can be solved exactly, but in the special cases it simplifies further, and the general solution is not needed.

§ 3. *Semicircular Normal Trajectory.*

In the experimental arrangement used by Rutherford and Robinson[*], and later by Ellis[†], the normal trajectory of the β particle is a semicircle (Fig. 2).

If $-\epsilon$ is the initial value of θ for this trajectory, then

$$\tan \epsilon = \frac{l}{R}, \quad \theta_0 = -\epsilon, \quad \theta_S = \epsilon, \quad \theta_P = \pi - \epsilon,$$

and the change of range DR is (to first order)

$$DR = Dy_P + Dx_P \tan \epsilon. \qquad \qquad \ldots\ldots(14)$$

Hence, comparing with (4),

$$\xi = \tan \epsilon, \quad \eta = 1, \qquad \qquad \ldots\ldots(15)$$
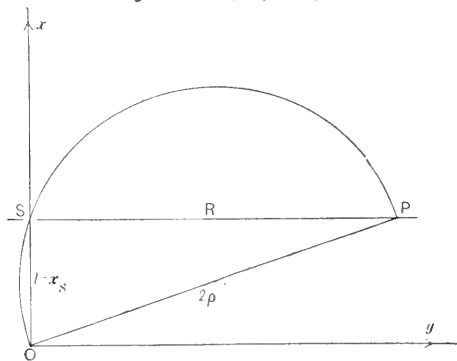
Fig. 2

so from (13)

$$\frac{d\alpha}{d\theta} = \frac{B}{H}[\tan \epsilon \sin \theta - \cos \theta] = -\frac{B}{H}\frac{\cos (\theta + \epsilon)}{\cos \epsilon},$$

and the arbitrary constant is given by $\alpha_P = 0$ (equation 8), *i.e.* $\alpha = 0$ for $\theta = \pi - \epsilon$. Hence

$$\alpha = -\frac{B}{H}\frac{\sin (\theta + \epsilon)}{\cos \epsilon}, \qquad \qquad \ldots\ldots(16)$$

and in particular $\qquad \qquad \alpha_0 = 0. \qquad \qquad \ldots\ldots(17)$

From equation (9), $\alpha_0 D\theta_0$ is the first order change of range DR for a change $D\theta_0$ in the initial direction of motion; $\alpha_0 = 0$ means that to the first order the range R is not affected by small variations in the initial direction of motion. This of course is an expression of the well known focussing effect obtained with this

* E. Rutherford and H. Robinson, *Phil. Mag.* 26, p. 717 (1913).
† C. D. Ellis, *P.R.S.* 99 A, p. 261 (1921). I am indebted to Mr Ellis for suggesting to me the problem of the relative importance of variations of the field in this arrangement.

arrangement. One result of this equation (17) is that the term in α' in (12) vanishes, so

$$DR = \frac{-1}{H \cos \epsilon} \int_O^P \sin \phi \, DH \, ds$$

$$= \frac{-B}{H^2 \cos \epsilon} \int_O^P \sin \phi \, DH \, d\phi, \qquad \ldots\ldots(18)$$

where $\phi = \theta + \epsilon$.

The constant value DH_0 of DH which would give the same DR is

$$DH_0 = \int_O^P \tfrac{1}{2} \sin \phi \, DH \, d\phi, \qquad \ldots\ldots(19)$$

so that the 'weighting function' for variations of H from normal over equal arcs in different parts of the trajectory is $\tfrac{1}{2} \sin \phi$, and so is a maximum half-way along the trajectory.

§ 4. *Trajectory with Small Deviation.*

The arrangement shown in Fig. 3 is another which is used, though more for α than β particle work. Here

$$DR = Dy_P - Dx_P \tan \omega. \qquad \ldots\ldots(20)$$

Let ϵ be the initial value of θ for the normal trajectory which goes through the centre of the slit. Then

$$\theta_0 = -\epsilon, \quad \theta_S = \epsilon, \quad \theta_P = \omega.$$

Comparing (20) with (4),

$$\xi = -\tan \omega, \quad \eta = 1,$$

and the solution of (13) required is that for which $\alpha = 0$ for $\theta = \omega$ (by equation 8), and is

$$\alpha = \frac{B}{H} \frac{\sin(\omega - \theta)}{\cos \omega}, \qquad \ldots\ldots(21)$$

Fig. 3

while α' is the same with $\epsilon \, (= \theta_S)$ substituted for $\omega \, (= \theta_P)$, *i.e.*

$$\alpha' = \frac{B}{H} \frac{\sin(\epsilon - \theta)}{\cos \epsilon}.$$

So from (12), substituting for ds from (2 a),

$$DR = \frac{B}{H^2 \cos \omega} \left[\int_O^S \left\{ \sin(\omega - \theta) - \frac{\sin(\omega + \epsilon)}{\sin 2\epsilon} \sin(\epsilon - \theta) \right\} DH \, d\theta \right.$$
$$\left. + \int_S^P \sin(\omega - \theta) \, DH \, d\theta \right]. \qquad \ldots\ldots(22)$$

The constant value DH_0 of DH which would give the same DR is

$$DH_0 = \int_O^P f(\theta) \, DH \, d\theta,$$
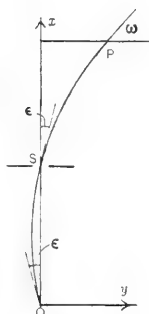
where the 'weighting function' for equal arcs, $f(\theta)$, is given by

$$f(\theta) = \left[\sin(\omega - \theta) - \frac{\sin(\omega + \epsilon)}{\sin 2\epsilon} \sin(\epsilon - \theta) \right] \bigg/ \left[1 - \frac{\cos \omega}{\cos \epsilon} \right],$$
$$\left. \begin{array}{c} \\ (\theta < \epsilon) \end{array} \right\} \quad (23)$$
$$f(\theta) = \sin(\omega - \theta) \bigg/ \left[1 - \frac{\cos \omega}{\cos \epsilon} \right]. \quad (\theta > \epsilon)$$

In Fig. 4 the function $f(\theta)$ is shown for $\epsilon = 5°$, $\omega = 30°$. Both branches are very nearly straight, and the field at the slit is the most important. For comparison, the curve of the function $f_1(\theta)$, which would be obtained if the change in the initial angle were neglected, is also shown.
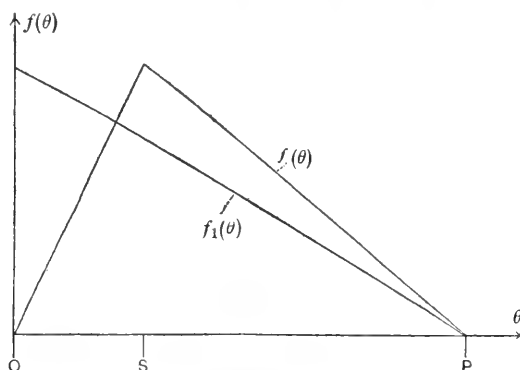
Fig. 4

If the variation in some other quantity than R is required at P, the adjoint equations must be solved for some other set of values of α, β, γ at P. For instance, if the change in the angle ω (Fig. 3) at which the particle strikes the plate is required, we have

$$D\omega = D\theta_P + Dx_P \frac{d\theta}{dx}$$
$$= D\theta_P + Dx_P \frac{H}{B \cos \omega},$$

so that in order that the left-hand side of (7) shall be $D\omega$, the arbitrary constants in the solutions of equations (5) must be chosen to satisfy the conditions

$$\alpha_P = 1, \quad \beta_P = \frac{H}{B \cos \omega}, \quad \gamma_P = 0.$$

The examples already discussed in full show how the general theory of first order variations can be applied to any such problem.

On the Problem of Three Bodies. By Mr J. BRILL, M.A.

The purpose of the present communication is to give the results of an investigation of Laplace's problem of three particles, considered from the point of view of classical dynamics, with the object of expressing Hamilton's Characteristic Equation as far as possible in terms of intrinsic coordinates. The masses of the particles were taken as l, m, n and λ, μ, ν were the sides of the triangle formed by them, λ being the length of the side joining the particles whose masses are m and n, and so on. In the first place the motion was referred to a right-handed set of rectangular moving axes, passing through the centre of gravity, O, of the three particles, the axis of z being perpendicular to the plane of the three particles, and those of x and y coinciding with the axes of their momental ellipse. P and Q were taken to represent the moments of inertia of the system about the axes of x and y respectively. It was found that we have the two relations

$$(l + m + n)(P + Q) = mn\lambda^2 + nl\mu^2 + lm\nu^2,$$

and

$$PQ = \frac{4lmn}{l + m + n}\Delta^2,$$

where Δ is the area of the triangle formed by the three particles. Thus P and Q may be considered as the roots of the quadratic equation

$$4(l + m + n)R^2 - 4(mn\lambda^2 + nl\mu^2 + lm\nu^2)R$$
$$+ lmn\{2(\mu^2\nu^2 + \nu^2\lambda^2 + \lambda^2\mu^2) - (\lambda^4 + \mu^4 + \nu^4)\} = 0.$$

Next, the normal through O to the invariable plane of the system was taken as making an angle θ with the axis of z, ϕ being the angle made by the plane, containing the axis of z and the normal through O to the invariable plane, with the plane of (xz). Finally, ψ was taken as denoting the angle made by the plane, containing the axis of z and the normal through O to the invariable plane, with some fixed plane through the said normal. Of the six coordinates λ, μ, ν, θ, ϕ, ψ involved in the final equation, ψ is the only extrinsic one.

The angular velocities of the axes about themselves are given by the equations

$$\theta_1 = \dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi, \quad \theta_2 = -\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi,$$
$$\theta_3 = \dot{\psi}\cos\theta - \dot{\phi};$$

and the energy integral takes the form

$$2E + P\theta_1^2 + Q\theta_2^2 + (P + Q)\theta_3^2 + 2M\theta_3 = 2\left(\frac{mn}{\lambda} + \frac{nl}{\mu} + \frac{lm}{\nu} + h\right),$$

where $\quad 2E = l(\dot{x}_1^2 + \dot{y}_1^2) + m(\dot{x}_2^2 + \dot{y}_2^2) + n(\dot{x}_3^2 + \dot{y}_3^2),$

and $\quad M = l(x_1\dot{y}_1 - \dot{x}_1 y_1) + m(x_2\dot{y}_2 - \dot{x}_2 y_2) + n(x_3\dot{y}_3 - \dot{x}_3 y_3).$

The coordinates of the particles (x_1, y_1), (x_2, y_2), (x_3, y_3) are further, according to the specifications above, connected by the equations

$$lx_1 + mx_2 + nx_3 = 0, \quad ly_1 + my_2 + ny_3 = 0, \quad lx_1 y_1 + mx_2 y_2 + nx_3 y_3 = 0.$$

The obtaining of the final equation from the above equations, combined with those expressing the lengths of the sides of the triangle in terms of the coordinates of its angular points, was found to be a somewhat lengthy and involved process; but the result obtained, taking V as the characteristic function, was

$$\left(\frac{1}{m} + \frac{1}{n}\right)\left(\frac{\partial V}{\partial \lambda}\right)^2 + \left(\frac{1}{n} + \frac{1}{l}\right)\left(\frac{\partial V}{\partial \mu}\right)^2 + \left(\frac{1}{l} + \frac{1}{m}\right)\left(\frac{\partial V}{\partial \nu}\right)^2 + \frac{1}{l}\frac{\mu^2 + \nu^2 - \lambda^2}{\mu\nu}\frac{\partial V}{\partial \mu}\frac{\partial V}{\partial \nu}$$

$$+ \frac{1}{m}\frac{\nu^2 + \lambda^2 - \mu^2}{\nu\lambda}\frac{\partial V}{\partial \nu}\frac{\partial V}{\partial \lambda} + \frac{1}{n}\frac{\lambda^2 + \mu^2 - \nu^2}{\lambda\mu}\frac{\partial V}{\partial \lambda}\frac{\partial V}{\partial \mu}$$

$$+ \frac{4\Delta}{(P-Q)^2}\frac{\partial V}{\partial \phi}\left\{(B+C)\frac{1}{\lambda}\frac{\partial V}{\partial \lambda} + (C+A)\frac{1}{\mu}\frac{\partial V}{\partial \mu} + (A+B)\frac{1}{\nu}\frac{\partial V}{\partial \nu}\right\}$$

$$+ P^{-1}\left[\frac{\partial V}{\partial \theta}\sin\phi + \frac{\cos\phi}{\sin\theta}\left(\frac{\partial V}{\partial \phi}\cos\theta + \frac{\partial V}{\partial \psi}\right)\right]^2 + Q^{-1}\left[\frac{\partial V}{\partial \theta}\cos\phi - \frac{\sin\phi}{\sin\theta}\left(\frac{\partial V}{\partial \phi}\cos\theta + \frac{\partial V}{\partial \psi}\right)\right]^2$$

$$+ \frac{P+Q}{(P-Q)^2}\left(\frac{\partial V}{\partial \phi}\right)^2 = 2\left(\frac{mn}{\lambda} + \frac{nl}{\mu} + \frac{lm}{\nu} + h\right),$$

where $\quad (l+m+n)A = n(l+m)\mu^2 - m(n+l)\nu^2,$

$$(l+m+n)B = l(m+n)\nu^2 - n(l+m)\lambda^2,$$

$$(l+m+n)C = m(n+l)\lambda^2 - l(m+n)\mu^2.$$

The quantities A, B, C would seem to be the components of a tensor that should play an important part in the theory. They are connected with P and Q by the formula

$$BC + CA + AB + (P-Q)^2 = 0;$$

and, we have also

$$2mn\lambda^2 = (P+Q)(m+n) + Cn - Bm,$$

$$2nl\mu^2 = (P+Q)(n+l) + Al - Cn,$$

$$2lm\nu^2 = (P+Q)(l+m) + Bm - Al.$$

The first nine terms of our equation may be reduced to five by replacing the variables λ, μ, ν by the three new variables

$$\alpha = f_1(P+Q, PQ), \quad \beta = f_2(P+Q, PQ), \quad \gamma = f(B/A)^*,$$

* It is to be remarked that, since B/A and C/A are linearly connected, it is not possible to obtain more than one independent ratio from the three quantities A, B, C.

the terms that disappear being those involving

$$\frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial \gamma},\quad \frac{\partial V}{\partial \beta}\frac{\partial V}{\partial \gamma},\quad \frac{\partial V}{\partial \alpha}\frac{\partial V}{\partial \phi},\quad \frac{\partial V}{\partial \beta}\frac{\partial V}{\partial \phi}.$$

Further, if we write $P + Q = u$ and $PQ = v$, we may secure the disappearance of the term involving $(\partial V/\partial \alpha)(\partial V/\partial \beta)$, as well as the equality of the coefficients of $(\partial V/\partial \alpha)^2$ and $(\partial V/\partial \beta)^2$, by the assumptions

$$u\left(\frac{\partial f_1}{\partial u}\right)^2 + 4v\frac{\partial f_1}{\partial u}\frac{\partial f_1}{\partial v} + uv\left(\frac{\partial f_1}{\partial v}\right)^2 = u\left(\frac{\partial f_2}{\partial u}\right)^2 + 4v\frac{\partial f_2}{\partial u}\frac{\partial f_2}{\partial v} + uv\left(\frac{\partial f_2}{\partial v}\right)^2,$$

and

$$u\frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial u} + 2v\left(\frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} + \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}\right) + uv\frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial v} = 0.$$

By writing $F = f_1 + if_2$, we may include these two conditions in the single one

$$u\left(\frac{\partial F}{\partial u}\right)^2 + 4v\frac{\partial F}{\partial u}\frac{\partial F}{\partial v} + uv\left(\frac{\partial F}{\partial v}\right)^2 = 0.$$

A complete primitive of this equation is

$$F = a\left(\sin^{-1} 2\frac{\sqrt{v}}{u} + i\log u\right) + b$$

$$= 2ia\log(\sqrt{P} + i\sqrt{Q}) + b;$$

and, consequently, we may write

$$\alpha + i\beta = \Phi(\sqrt{P} + i\sqrt{Q}).$$

The most convenient assumption to make is, $\alpha = \sqrt{P}$, $\beta = \sqrt{Q}$, in which case the terms involving $\partial V/\partial \alpha$ and $\partial V/\partial \beta$ take the simple form

$$\left(\frac{\partial V}{\partial \alpha}\right)^2 + \left(\frac{\partial V}{\partial \beta}\right)^2.$$

If we now write $B/A = w$, and

$$\frac{1}{\{lmn(l+m+n)\}^{\frac{1}{2}}}\frac{d\gamma}{1+\gamma^2} = \frac{dw}{l(m+n) + 2lmw + m(n+l)w^2},$$

or

$$\{lmn(l+m+n)\}^{\frac{1}{2}}\gamma = m(n+l)w + lm,$$

the terms involving $\partial V/\partial \gamma$ become

$$\frac{4(1+\gamma^2)}{(P-Q)^2}\left\{(P+Q)(1+\gamma^2)\left(\frac{\partial V}{\partial \gamma}\right)^2 + 2\sqrt{PQ}\frac{\partial V}{\partial \gamma}\frac{\partial V}{\partial \phi}\right\}.$$

It is further to be noted that, with our assumed form $f(B/A)$ for the third variable, the gravitational potential U will satisfy the equation

$$\alpha\frac{\partial U}{\partial \alpha} + \beta\frac{\partial U}{\partial \beta} + U = 0.$$

On a partial recalculation of the expression for $2T$, with the new coordinates α, β, γ, the significant terms were found to be of the requisite form to furnish the corresponding terms of the Characteristic Equation in the form given above as obtained by transformation.

It may be noted that one of the intermediate results obtained was

$$M = \frac{\alpha\beta\dot{\gamma}}{1 + \gamma^2}.$$

The device of using two of the principal moments of inertia for coordinates was adopted by Scheibner (*Crelle*, t. LXVIII).

On the Formulae of One-dimensional Kinematics.   By Dr W. Burnside, Honorary Fellow of Pembroke College.

In the following note an attempt is made to describe the only kind of observations which can justify a set of observers in saying that a point is moving uniformly relatively to them. Assumptions are introduced as to the connection between the observations of two sets of observers, each observing the other's motion; and from these assumptions equations connecting the time and place of an event as observed by one set with those as observed by the other set are deduced. On the assumptions made it is shown that there are just three possible forms for the equations, and a reason is given for disregarding one of the three.

The theories formed in connection with physical science are built on a foundation of observation. Each such theory is embodied in the form of a set of exact statements, or in the analytical form of a set of equations. The statements or the equations connect certain numbers which are given by observation. Now all observation is fallible, so that when these numbers, given by observation, are entered in the equations, it is generally found that the numbers on the two sides of any equation are not exactly the same. The symbols used in the equations then are not symbols for the results of actual observation. They are symbols for the results of observations assumed to be made without error. Another way of expressing this is to say that an exact physical theory postulates the existence of a set of accurate observers.

It is natural to ask: What conditions must be satisfied by the statements of two or more accurate observers?

Is the fact that they give different numerical estimates of what they describe as the same quantity admissible or not? This question can only be answered when the results of a theory have been completely worked out. It may then appear that, the conditions under which the two observers make the observations being unlike, there is no reason to expect the estimate of one to agree with that of the other. On the other hand for certain quantities it may be found that the conditions under which the observations are made do not affect the estimates. If, however, one accurate observer says that the numerical values of two quantities of the same kind are equal, all other accurate observers must make the same statement: for no theory can be built upon the statement that a is both equal to b and is not equal to b.

For instance the first observer may say that the velocity of a

point P is u, while the second may equally truly say it is v. But if the first says that the velocities of P and Q, moving in the same direction, are equal the second *must* say so too.

To make time-observations a clock is necessary. Suppose an accurate observer and his clock are at a place A, and that the observer is watching a moving point P. If in its motion P passes through A, the observer can see the clock and the moving point simultaneously, and note the time of their coincidence. If in its further motion P passes through B it is not possible for the observer to determine directly when P coincides with B. With a suitable arrangement of light-signals and a theory of the propagation of light, the observer might deduce when the coincidence took place, if he knew the distance AB; but he certainly could not observe it accurately.

The only direct time-observations the accurate observer can make are those of events at his own position. If there are a number of observers at A, each with his clock, they can all see the clocks simultaneously, and unless all the clocks show the same time, "the time at A" has no definite meaning for any of them. Further, if some of the observers leave A and travel to B in any way and then return to A, their clocks after return must show the same time as those that have stayed at A, or else "the time at A" again has no definite meaning for any of them. Once again if some or all of the observers leave A and travel in any way to B, their clocks must all show the same time, or else "the time at B" will have no definite meaning for them.

Now consider a set of observers in a one-dimensional world S_1. This implies that each observer can change his position only in two ways, viz. by going forward or by going backward. Suppose the observers have clocks such that "the time at P," if P is any marked point in the one-dimensional world, has a definite meaning. Suppose also that the observers have a measuring rod. Let one observer, at A_0, keep one end of the rod in coincidence with A_0, while another proceeds to the other end of the rod, and marks the point A_1 with which the other end of the rod coincides. If this second observer found that the end of the rod did not continue to coincide with A_1 he would conclude that the distance from A_0 to A_1, as determined by the measuring rod, varied with the time, and therefore that the distance between any two points of his world had no definite meaning. He would find that this made any science of motion impossible.

If the far end of the rod always coincided with A_1, he might speak of A_0 and A_1 as being at unit distance apart. By shifting the near end of the rod to A_1 he determines A_2 and so on. In this way a scale of marked points in S_1 is formed, each two adjacent ones being at unit distance apart,

$$\ldots A_{-i}, A_{-i+1}, \ldots, A_{-1}, A_0, A_1, \ldots, A_i, \ldots.$$

Let S_2 be another one-dimensional world, adjacent to S_1, and indistinguishable from S_1 at first. Some of the observers on S_1 pass to S_2 with their clocks and their measuring rod. They proceed to mark the points $\ldots B_{-i}, \ldots, B_{-1}, B_0, B_1, \ldots, B_i, \ldots$ of S_2 which coincide with the marked points of S_1, B_i coinciding with A_i for each i. So long as this coincidence lasts, S_1 and S_2 are said to be relatively at rest. While so at rest, let observers take up their positions at the marked points on S_1 and S_2. The clocks at A_0 and B_0 show the same time. Suppose that when this time is zero, S_2 and S_1 are set in relative motion so that A_i and B_i no longer coincide.

Denote by t_{ij} the time shown by the clock of the observer at A_i on S_1 when B_j coincides with A_i; and suppose that the values of t_{ij}, for a considerable series of sets of suffixes, are noted by the observers on S_1 and that they then gather at, say, A_0 to discuss the results. The nature of the set of numbers t_{ij} depends on the nature of the relative motion of S_1 and S_2. Apart from multiplying the number of observations, these numbers give the only possible way by which observers on S_1 can determine the relative motion of S_2 and S_1. Consider the difference $t_{i+1, j} - t_{i,j}$ between the time shown by the clock at A_i when B_j is passing and that shown by the clock at A_{i+1} when B_j is passing. The observers on S_1 must either accept this difference as the time that B_j takes to go from A_i to A_{i+1}, or they must accept the impossibility of determining this time.

If this difference is the same for a long series of values of i and is equal to $1/u$, the observers on S_1 say that B_j passes over equal spaces in equal times, and that its velocity is u. Now observers on S_2 say that B_j and B_k are both at rest, i.e. they have the same (zero) velocity. Hence the (accurate) observers on S_1 say that B_j and B_k have the same velocity, and therefore for a series of values of i $t_{i+1, k} - t_{i, k} = \dfrac{1}{u}$: and this must be true for a long series of values of k.

Observers on S_1 determine the distance between two marked points on S_1 by noting how many times the interval between them contains the measuring rod. It is impossible for them, when S_1 and S_2 are in relative motion, to use this method to determine the distance between B_j and B_k, two marked points on S_2. The observers on S_1 can only form a rule for deducing the distance between B_j and B_k from the observed numbers t_{ij}, and test it by determining whether the rule is free from contradiction. A natural form of the rule is to take $u\,(t_{i,j} - t_{i,k})$ as this distance. From the fact that $t_{i+1, j} - t_{i,j}$ is independent of i and j, it follows that $t_{i,j} - t_{i,k}$ is independent of i, while the fact that

$$u\,(t_{i,j} - t_{i,k}) + u\,(t_{i,k} - t_{i,m}) = u\,(t_{i,k} - t_{i,m})$$

makes the distance from B_i to B_m the sum of the distances from B_i to B_k and from B_k to B_m. If

$$u\,(t_{i,\,j} - t_{i,\,j+1}) = l$$

the observers on S_1 say that l is the distance from B_j to B_{j+1}. Now observers on S_2 say that the distance from B_j to B_{j+1} is the same as that from B_k to B_{k+1}, and therefore

$$t_{i,\,j} - t_{i,\,j+1} = \frac{l}{u}$$

must be true for all (observed) values of the suffixes i and j. The two relations so obtained give

$$t_{i,\,j} = \frac{i}{u} - \frac{jl}{u},$$

since it has been assumed that $t_{0,0}$ is zero.

The observers on S_2 make a similar set of observations which, when the relative motion of S_1 and S_2 has been stopped, can be compared with the observations of those on S_1. It is assumed that when the observers on S_1 find $t_{i+1,\,j} - t_{i,\,j}$ independent of i, those on S_2 find $t'_{j+1,\,i} - t'_{j,\,i}$ independent of j, where $t'_{j,\,i}$ is the time shown by the clock of the observer at B_j on S_2, when A_i is coincident with B_j. It is further assumed that the quantities corresponding to u and l, which the observers on S_2 deduce from their observations, are $-u$ and l. From this it follows at once that

$$t'_{j,\,i} = \frac{il}{u} - \frac{j}{u}.$$

Now $t_{i,\,j}$ and $t'_{j,\,i}$ are the times given by the observers on S_1 and S_2 respectively for the coincidence of A_i and B_j. Also, if the observers on S_1 and S_2 reckon distances from A_0 and B_0, i and j are the places given by the two sets of observers for the same coincidence. Hence (changing the notation) if x_1, t_1 are the place and time of the coincidence for S_1 and x_2, t_2 for S_2

$$t_1 = \frac{x_1}{u} - \frac{x_2 l}{u},$$

$$t_2 = \frac{x_1 l}{u} - \frac{x_2}{u},$$

or

$$x_2 = \frac{1}{l}\,x_1 - \frac{u}{l}\,t_1,$$

$$t_2 = \frac{l^2 - 1}{lu}\,x_1 + \frac{1}{l}\,t_1.$$

The observers on S_1 and S_2 have only obtained their results for the coincidence of one of the set of points $A_0, A_1, A_2, \ldots$ with one of

the set B_0, B_1, ...; or in other words when x_1 and x_2 are integers. To build up a theory they assume its truth for any two marked points on S_1 and S_2 respectively.

Let S_3 be a third one-dimensional world adjacent to S_1 and S_2; and suppose that the observers on S_3 are some of those that were originally, with their clocks and rods, on S_1. Suppose also that the observers on S_2 observe the times at which a marked point on S_3 passes B_0, B_1, B_2, ... and that they find the time differences are constant. Then if u', l' are the quantities for observers on S_2 and S_3 that correspond to u and l for those on S_1 and S_2; and if x_3, t_3 are the place and time of a coincidence for S_3 corresponding to x_2, t_2 for S_2,

$$x_3 = \frac{1}{l'} x_2 - \frac{u'}{l'} t_2,$$

$$t_3 = \frac{l'^2 - 1}{l'u'} x_2 + \frac{1}{l'} t_2.$$

The relation between x_3, t_3 and x_1, t_1 is therefore

$$x_3 = \frac{1}{ll'} \left\{ 1 - \frac{u'}{u} (l^2 - 1) \right\} x_1 - \frac{u + u'}{ll'} t_1,$$

$$t_3 = \frac{1}{ll'} \left\{ \frac{1}{u} (l^2 - 1) + \frac{1}{u'} (l'^2 - 1) \right\} x_1 + \frac{1}{ll'} \left\{ 1 - \frac{u}{u'} (l'^2 - 1) \right\} t_1.$$

If $x_3 = 0$, x_1/t_1 is constant, so that the observers on S_1 see the marked point C_0 on S_3 describing equal spaces in equal times. With the assumptions already made this implies that there are quantities U and L for S_1 and S_3 corresponding to u and l for S_1 and S_2, and that

$$x_3 = \frac{1}{L} x_1 - \frac{U}{L} t_1,$$

$$t_3 = \frac{L^2 - 1}{LU} x_1 + \frac{1}{L} t_1.$$

It is easily verified that the conditions that the last two pairs of equations should be the same are

$$\frac{l^2 - 1}{u^2} = \frac{l'^2 - 1}{u'^2} = k,$$

$$U = \frac{u + u'}{1 - kuu'},$$

where k is some constant.

Hence $$x_2 = \frac{1}{\sqrt{1 + ku^2}} x_1 - \frac{u}{\sqrt{1 + ku^2}} t_1,$$

$$t_2 = \frac{ku}{\sqrt{1 + ku^2}} x_1 + \frac{1}{\sqrt{1 + ku^2}} t_1:$$

while if the observers on S_1 say that S_2 is moving with velocity u and the observers on S_2 say that S_3 is moving with velocity u', then observers on S_1 say that S_3 is moving with velocity $\dfrac{u + u'}{1 - kuu'}$. If k is not zero its numerical value clearly depends on the unit of time; and hence, when the unit of time is suitably chosen, there are just three distinct possibilities, which may be taken to be $k = -1, 0, 1$.

If $k = -1$ the formulae imply that no velocity can exceed unity numerically. In the other two cases there is no limit to the magnitude of the velocity.

When $k = -1$, $U = \dfrac{u + u'}{1 + uu'}$; so that if u^2 and u'^2 are both less than unity, U^2 is also less than unity.

When $k = 0$, $U = u + u'$: so that if u and u' are both finite, U also is finite.

When $k = 1$, $U = \dfrac{u + u'}{1 - uu'}$. An arbitrary assumption in this case that the magnitude of a velocity has an upper limit leads to a contradiction in consequence of the formula for U. Now if $u' = \dfrac{1}{u}$, U is infinite; so that if k is unity the possibility of a point, which has a finite velocity for one observer, having an infinite velocity for another must be admitted. This seems to be a sufficient reason for regarding the other two cases as the only ones of real importance.

The formulae connecting x_2, t_2 with x_1, t_1 lead to

$$\frac{\dfrac{d^2x_2}{dt_2{}^2}}{\left\{1 - k\left(\dfrac{dx_2}{dt_2}\right)^2\right\}^{\frac{3}{2}}} = \frac{\dfrac{d^2x_1}{dt_1{}^2}}{\left\{1 + k\left(\dfrac{dx_1}{dt_1}\right)^2\right\}^{\frac{3}{2}}},$$

and this affords an example of a quantity to which all accurate observers give the same value.

On the Pedal Locus in Non-Euclidean Hyperspace. By J. P. GABBATT.

Introduction. The pedal property of the circumcircle of the euclidean plane triangle was discovered in 1799 by Dr William Wallace*. It was known to Steiner† that in euclidean space of three dimensions the pedal locus (i.e. the locus of a point from which the feet of the normals to the faces of a given tetrahedron lie in a plane) is a four-nodal cubic surface containing the edges of the tetrahedron of reference. Steiner's pedal locus (otherwise known as Cayley's cubic surface) is the isogonal transformation, q.‡ the tetrahedron of reference, of the plane at infinity; Beltrami§ and W. J. C. Sharp‖ have discussed the analogous theorem in euclidean space of n dimensions.

The locus, in a non-euclidean plane, of a point from which the feet of the normals to the sides of a given triangle are in line has been considered by several writers*. This locus is a cubic anallagmatic for the isogonal transformation q. the triangle; and the line containing the feet of the normals to the sides of the triangle from any point of the locus is the absolute polar of the isogonal conjugate of that point.

The subject of the present paper is the locus, in non-euclidean space of n dimensions, of a point from which the feet of the normals to the walls of a given simplex lie in a hyperplane. This locus is named the Wallacian, in honour of the discoverer of the pedal property of the euclidean circumcircle; and is an $(n + 1)$-ic hypersurface anallagmatic for the isogonal transformation (§ 1) q. the given simplex. It is shown that the pedal (3·22) and other quasi-circular or spherical properties (3·21, 4·14) of the Wallacian follow naturally from its specification (3·11) as the Jacobian of $n + 1$ point-hyperspheres. When the given simplex is orthocentric, it appears (§ 5) that the Wallacian and a related locus of the same type coalesce in the Hessian of a certain cubic hypersurface. Finally, it is proved (§ 6) that in the euclidean case the Wallacian of any simplex degenerates into the Infinite and its isogonal transformation q. the given simplex, while the Wallacian of the absolute polar

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* For references v. Gabbatt, *Proc. Camb. Phil. Soc.* 21 (1922–3), pp. 336–7.

† Letter to Jacobi, 25 March, 1845. *V. Arch. d. Math. u. Phys.* (3), 4 (1903), pp. 275–7. For further references v. Neuberg, *ibid.* (3), 11 (1907), p. 228. Some interesting properties of the locus have recently been given by Servais, *Bull. Sc. Acad. r. de Belgique* (5), 8 (1922), pp. 50–66, 103–123, where additional references may be found.

‡ The abbreviation q. signifies *with respect to.*

§ *Mem. Acc. Bologna* (3), 7 (1876), p. 241 = *Op. Mat.* III. p. 53.

‖ *Proc. Lond. Math. Soc.* 19 (1888), p. 446.

simplex degenerates into the hypersphere circumscribing the given simplex and the Infinite counted $n - 1$ times.

1. *Isogonal Conjugates in Non-Euclidean Hyperspace.* 1·1. Let S_m denote a linear space of m dimensions. Then $n + 1$ points A_i $(i = 0, 1, ..., n)$ which do not lie in any S_{n-1} determine a *simplex* in S_n, of which the points A_i are the *vertices*; this simplex will be referred to as *the simplex* $[A]$. The lines, planes, ..., hyperplanes determined by two, three, ..., n of the points A_i will be termed the *side-spaces*, of dimensions $1, 2, ..., n - 1$ respectively, of the simplex $[A]$. The side-spaces of dimensions $1, n - 2, n - 1$ will be termed respectively the *edges*, *bastions*, *walls* of the simplex. A side-space of dimensions m will be referred to as a s_m of the simplex. A side-space determined by any m vertices of the simplex, and that determined by the remaining $n - m + 1$ vertices, will be termed *opposite side-spaces* of the simplex. The wall of the simplex $[A]$ opposite any vertex A_i will be denoted by a_i. The side-space determined by any m vertices A_i, A_j, A_k ... will be denoted by $A_{ijk...}$, and the opposite side-space by $a_{ijk...}$.

1·2. Let any quadric (V^2_{n-1}) Q be regarded as Absolute. Then we may give the following definition: The order-quadrics (V^2_{n-1}) to which the simplex $[A]$ and the Absolute (regarded as a class-quadric) are apolar constitute an $(n - 1)$-fold linear system; any two points which are apolar to every quadric of the system are *isogonal conjugates* q. the simplex $[A]$.

In general one and only one point M' is the isogonal conjugate, q. $[A]$, of a given point M, the point M being also the isogonal conjugate of M'; but A_i is the isogonal conjugate of every point of a_i $(i = 0, 1, ..., n)$.   (1·21)

Let the angles determined by pairs of walls of the simplex $[A]$ be termed the *ditopic angles* of the simplex; then clearly the $\frac{1}{2}n(n + 1)$ pairs of hyperplanes bisecting the ditopic angles are degenerate quadrics of the $(n - 1)$-fold system. Thus the 2^n incentres of $[A]$ are common to all the quadrics of the system (which may therefore be termed the *incentric system* of quadrics), and are the only self-isogonal conjugate points.   (1·22)

More generally, each s_{m-1} of $[A]$ is the vertex of an $(n - m - 1)$-fold linear system of quadric cones (of species m) of the incentric system; any such cone may be termed an *incentric cone* of species m.   (1·23)

Now let $A_{ij...l}$ denote any side-space of dimensions $m - 1$ $(m < n)$ of $[A]$. Then the polar hyperplanes of any point M (of general position) q. the incentric cones with vertex $A_{ij...l}$ are the $(n - m - 1)$-fold linear system of hyperplanes containing $A_{ij...l}$ and the isogonal conjugate (M') of M; every hyperplane of the system therefore contains the S_m determined by M' and $A_{ij...l}$. Also if N denote

any point of the S_m determined by M and $A_{ij...l}$, then the isogonal conjugate N' of N is a point of the S_m determined by M' and $A_{ij...l}$. Thus: *If the line joining two points meet any side-space of the simplex* $[A]$; *then the line joining the isogonal conjugates, q.* $[A]$, *of those points also meets the given side-space.* (1·24)

In particular, let M, N be such that the line MN meets a bastion a_{ij} of the simplex $[A]$, and let us consider the one-fold linear system of incentric cones of which the vertex is a given $s_{n-3}\,(a_{ijk})$ which lies in that bastion. The polar hyperplane of M q. any cone of this system (1·21) contains M'; there is one cone (C, say) of the system such that the polar hyperplane, q. C, of M also contains N, and is therefore the hyperplane determined by M', N, a_{ijk}. The polar hyperplane, q. C, of N is the hyperplane determined by M, N', a_{ijk}; also the polar hyperplane, q. C, of any point of a_{ij} is the wall a_k determined by the edge A_{ij} and a_{ijk}. Thus: *If the line joining two points* M, N *meet a bastion of the simplex* $[A]$, *and if* M', N' *denote the isogonal conjugates, q.* $[A]$, *of* M, N *respectively; then the space of dimensions* $n-2$ *which contains any* s_{n-3} *of that bastion and meets the lines* MN', $M'N$ *also meets the edge of* $[A]$ *opposite the given bastion.* (1·25)

It follows from (1·24, 5) that: *If* M, N *be such that the line* MN *meets a bastion of the simplex* $[A]$, *and the line* MN' *meets the opposite edge; then the lines* MN, $M'N'$ *meet at a point of the given bastion, and the lines* MN', $M'N$ *meet at a point of the given edge.* Exceptions occur only in the anomalous cases when the points M, N lie in a wall of $[A]$. (1·26)

2. *A Group of Three Simplices.* 2·1. Let B_i denote the absolute pole of a_i, and b_i the absolute polar of $A_i\,(i = 0, 1, ..., n)$; then the points B_i are the vertices, and the hyperplanes b_i the walls respectively opposite those vertices, of a simplex $[B]$, the *absolute polar simplex* of $[A]$. A_i, B_i will be termed *associated vertices*, and a_i, b_i *associated walls*, of the two simplices.

2·2. Let A_i denote a (fixed) vertex of $[A]$, A_j any (variable) vertex of $[A]$ other than A_i, a_{ij} the bastion of $[A]$ opposite the edge A_{ij}, and B_j that vertex of $[B]$ which is associated with A_j. Then (for each of the n possible values of j) B_j, a_{ij} determine a hyperplane, and there is one and only one point (B_i', say) common to the n hyperplanes so determined. Thus with each vertex A_i of the simplex $[A]$ may be associated a vertex B_i' of another simplex $[B']$. The wall (*associated with* a_i) of $[B']$ opposite B_i' will be denoted by $b_i'\,(i = 0, 1, ..., n)$. (2·21)

Now let the hyperplanes bisecting the ditopic angles determined by the walls a_i, a_j of the simplex $[A]$ be denoted by α_{ij}, α_{ji}; then the hyperplanes a_i, a_j are harmonic q. α_{ij}, α_{ji}. The absolute poles of a_i, a_j are respectively B_i, B_j, and the absolute pole of each of the

hyperplanes α_{ij}, α_{ji} lies in the other. Projecting from a_{ij}, we see that the hyperplanes (B_i, a_{ij}), (B_j, a_{ij}) are harmonic q. α_{ij}, α_{ji}; also the hyperplane (B_j, a_{ij}) contains B_i'. Thus B_i, B_i' are apolar to each of the n pairs of hyperplanes α_{ij}, α_{ji} ($j \neq i$), and therefore: *The points B_i, B_i' are isogonal conjugates q. the simplex $[A]$* ($i = 0, 1, \ldots, n$). (2·22)

2·3. $\frac{1}{2}n(n+3)$ independent linear conditions are necessary and sufficient for the complete determination of a class-quadric in S_n, and n independent linear conditions for the apolarity of a class-quadric to all the order-quadrics of the incentric system; one and only one such class-quadric will therefore satisfy $\frac{1}{2}n(n+1)$ further independent linear conditions. In particular there is one and only one class-quadric which is apolar to all the order-quadrics of the incentric system and touches $\frac{1}{2}n(n+1)$ hyperplanes (otherwise arbitrary) each containing one of the $\frac{1}{2}n(n+1)$ bastions of the simplex $[A]$. Now any class-quadric apolar to all the order-quadrics of the incentric system is in particular apolar to the pair of hyperplanes α_{ij}, α_{ji} which bisect the ditopic angles determined by any two walls a_i, a_j of the simplex $[A]$; the hyperplanes α_{ij}, α_{ji} therefore bisect the angles determined by the tangent hyperplanes to such a class-quadric from the bastion a_{ij} ($i \neq j = 0, 1, \ldots, n$). Considering the case in which the $\frac{1}{2}n(n+1)$ pairs of tangent hyperplanes are themselves at right angles, we obtain the theorem: *There is one and only one class-quadric (Q') which touches each hyperplane of the $\frac{1}{2}n(n+1)$ pairs of hyperplanes bisecting the angles determined by the pairs of bisectors of the ditopic angles of the simplex $[A]$; and Q' is apolar to all the order-quadrics of the incentric system.* The quadric Q' will be termed the *Secondary* q. $[A]$. (2·31)

Let us now consider the four hyperplanes determined by the bastion a_{ij} of $[A]$ and the four points A_i, A_j, B_i, B_j respectively. From (2·22), the bisectors α_{ij}, α_{ji} are the double hyperplanes of the involution specified by $a_{ij}(A_i, A_j; B_i, B_j)$; and (since B_i is the absolute pole of the hyperplane a_i determined by A_j, a_{ij}) the tangent hyperplanes τ_{ij}, τ_{ji} from a_{ij} to the Absolute are the double hyperplanes of the involution specified by $a_{ij}(A_i, B_j; A_j, B_i)$. Thus, since the tangent hyperplanes (τ'_{ij}, τ'_{ji}) from a_{ij} to the Secondary are harmonic q. both the pairs of hyperplanes α_{ij}, α_{ji}; τ_{ij}, τ_{ji}; therefore τ'_{ij}, τ'_{ji} are the double hyperplanes of the involution specified by $a_{ij}(A_i, B_i; A_j, B_j)$, i.e. by $a_{ij}(A_i, B_j'; A_j, B_i')$. Hence a_i is the polar hyperplane of B_i' q. the hyperplane-pair τ'_{ij}, τ'_{ji}; a theorem in which the suffix i may be regarded as fixed, while j assumes in succession all the values $0, 1, \ldots, n$ except i. It follows that *B_i', a_i are pole and polar q. the Secondary; and the simplices $[A]$, $[B']$ are polar q. the Secondary.* (2·32)

From the specification (2·21) it appears that if the quadric Q' were taken as Absolute, then the rôles of the simplices $[B]$, $[B']$

would be reversed, and the quadric Q would become the Secondary q. $[A]$. Moreover (2·31) any pair of isogonal conjugates (Q Absolute) remain isogonal conjugates (Q' Absolute). The simplices $[B]$, $[B']$ and the quadrics Q, Q' are thus symmetrically related to the simplex $[A]$. $\qquad (2·33)$

2·4. From the specification (2·21) the line $B_i B_j'$ meets the bastion a_{ij} of $[A]$ ($i \neq j = 0, 1, ..., n$). In the special case where (for *one* pair of unequal values of i, j) the line $B_i B_j$ meets the edge $A_i A_j$ (opposite a_{ij}) of $[A]$, then (1·26, 2·22) the lines $A_i A_j$, $B_i B_j$, $B_i' B_j'$ meet at a point. $\qquad (2·41)$

Now if the simplex $[A]$ be orthocentric (Q Absolute), then the simplices $[A]$, $[B]$ are in perspective, the lines $A_i A_j$, $B_i B_j$ meet at a point for *all* unequal values of i, j from 0 to n, and the $\frac{1}{2}n\,(n + 1)$ meets lie in a hyperplane, the *orthaxis* of $[A]$, $[B]$. It follows from (2·41) that: *If the simplex $[A]$ be orthocentric, then the simplices $[A]$, $[B]$, $[B']$ are in perspective in pairs, with a common axial hyperplane.* $\qquad (2·42)$

3. *The Wallacian.* 3·1. Returning to the general case, let us consider the tangent cones (K_i) to the Absolute (Q) from the points B_i; the cone K_i is the *point-hypersphere* determined by B_i ($i = 0, 1, ..., n$). The cones $[K_i]$ determine an n-fold linear system of order-quadrics, $[K]$, say. The Jacobian of the system is known to be an $(n + 1)$-ic hypersurface containing the vertices of all the cones (of the first species) of the system, and in particular containing the points B_i ($i = 0, 1, ..., n$). This hypersurface will be termed the (absolute) *Wallacian* of the simplex $[A]$, and denoted by W. $\qquad (3·11)$

The Jacobian of such a system of quadrics is the locus of a pair of points apolar to every quadric of the system, the points of the pair being *correspondents* on the Jacobian. Now it is well known that the common points of the tangent cones from any two points (M, N) to a quadric (i.e. a V^2_{n-1}, say V) lie in a pair of hyperplanes, and that those hyperplanes are apolar to V and harmonic q. the polar hyperplanes (q. V) of M, N. In particular, the common points of the tangent cones from the points B_i, B_j to the Absolute lie in the pair of perpendicular hyperplanes which contain the bastion a_{ij} of the simplex $[A]$ and are harmonic q. a_i, a_j; these hyperplanes are the bisectors α_{ij}, α_{ji} of the ditopic angle a_{ij} of $[A]$ ($i \neq j = 0, 1, ..., n$). Thus the hyperplane-pairs α_{ij}, α_{ji} are degenerate quadrics of the system $[K]$; it follows from (1·22) that the system $[K]$ includes all the quadrics of the incentric system. Thus: *Correspondents on the Wallacian of any simplex are isogonal conjugates q. that simplex, and the Wallacian is anallagmatic for the isogonal transformation q. the simplex; in particular, the Wallacian of the simplex $[A]$ contains the points B_i' ($i = 0, 1, ..., n$).* $\qquad (3·12)$

Again, the Jacobian of the system [K] contains the intersections of all the hyperplane-pairs of [K], and therefore: *The Wallacian of any simplex contains all the bastions of that simplex.*
(3·13)

We have seen (1·21) that any point of a_i may be regarded as the isogonal conjugate, q. [A], of A_i. The indeterminacy of the correspondent, on W, of A_i is more restricted. For (if $i \neq j$) the polar, q. the cone K_j, of A_i is easily b_i; and the polar, q. the cone K_i, of A_i is the hyperplane determined by A_i and the intersection of a_i, b_i. Thus every point of this intersection may be regarded as correspondent to A_i, and: *The Wallacian of any simplex contains the intersection of each wall of that simplex with the associated wall of the polar simplex.* (3·14)

Thus $\frac{1}{2}(n+1)(n+2)$ linear spaces of dimensions $n-2$ are contained by the Wallacian of a given simplex; $n+1$ of those spaces lying in each wall of the simplex. (3·15)

3·2. The polar hyperplane (m_i) of any point M q. the cone K_i is the hyperplane normal, at the vertex B_i of that cone, to the line MB_i $(i = 0, 1, ..., n)$. The Wallacian is (3·11) the locus of a point M such that the $n+1$ hyperplanes m_i meet at a point M'; and (3·12) the points M, M' are then isogonal conjugates q. the simplex [A]. Thus: *The Wallacian of a given simplex is the locus of two points which subtend right angles at all the vertices of the absolute polar simplex; and the two points are isogonal conjugates q. the given simplex.* (3·21)

Now let P_i denote the point at which the line MB_i meets the wall a_i of [A]; then P_i is the absolute pole of the plane m_i, and therefore lies in the absolute polar hyperplane of M' $(i = 0, 1, ..., n)$. Thus: *The Wallacian of a given simplex is the locus of a point such that the feet of the normals from that point to the walls of the given simplex lie in a hyperplane; and that hyperplane is the absolute polar of the isogonal conjugate, q. the given simplex, of the given point.*
(3·22)

The hyperplane just specified may be termed the *pedal* (or *Wallace*) hyperplane, q. the given simplex, of the given point. Then: *The pedal hyperplanes of the points of the Wallacian envelope a hypersurface of class $n+1$.* (3·23)

Further, given any $n+1$ points in line on the Jacobian of an n-fold linear system of order-quadrics, it is easy to show that each of the points lies in the hyperplane determined by the correspondents of the remaining n points. Thus: *Given any $n+1$ points in line on the Wallacian; the meet of the pedal hyperplanes of any n of the points lies in the pedal hyperplane of the isogonal conjugate of the remaining point.* (3·24)

4. *Further Loci of Wallacian Type.* 4·1. The (absolute) Wallacian ($\overline{W}$) of the simplex [B] may be specified (3·11) as the Jacobian

of the n-fold linear system of order-quadrics determined by the tangent cones (point-hyperspheres) from the vertices of the simplex $[A]$ to the Absolute; or again (3·21) as the locus of two points which subtend right angles at all the vertices of the simplex $[A]$; or (3·22) as the locus of a point M such that the $n + 1$ points (MA_i, b_i) lie in a hyperplane, the *pedal hyperplane* of M q. the simplex $[B]$. It is convenient to term the locus $\overline{W}$ the *polar Wallacian* of the simplex $[A]$. (4·11)

It should be observed (cf. 3·12) that the $(n - 1)$-fold linear system of order-quadrics containing the 2^n circumcentres of $[A]$ forms part of the n-fold system of which $\overline{W}$ is the Jacobian; the pedal hyperplanes, q. $[B]$, of any two correspondents on $\overline{W}$ therefore meet each edge of $[A]$ at points harmonic q. the middle points of that edge. (4·12)

$\overline{W}$ is an $(n + 1)$-ic hypersurface containing (4·11) the vertices of the simplex $[A]$, and also (3·13) the bastions of the simplex $[B]$ and (3·14) the $n + 1$ intersections of associated walls of the two simplices. (4·13)

Thus part of the section of $\overline{W}$ by a wall (say a_n) of $[A]$ is the absolute polar S_{n-2} of a line $(A_n B_n)$. It is easy to determine the remainder of the section.

For let M denote any point which is common to $\overline{W}$ and a_n but does not lie in b_n. Then the $n + 1$ points (MA_i, b_i) $(i = 0, 1, ..., n)$ lie in a hyperplane (m'), the pedal hyperplane, q. $[B]$, of M. The n points (MA_i, b_i) $(i = 0, 1, ..., n - 1)$ also lie in the hyperplane a_n, and therefore in the $S_{n-2}(m', a_n)$. Also the section by a_n of b_i is the polar of A_i q. the section by a_n of the Absolute Q; i.e. q. the Absolute in the space of $n - 1$ dimensions a_n $(i = 0, 1, ..., n - 1)$. Thus: *If $[A^i]$ denote the simplex in a_i determined by all the vertices except A_i of the simplex $[A]$; then the section by a_i of the polar Wallacian of $[A]$ consists of the absolute polar S_{n-2} of the line $A_i B_i$, and the polar Wallacian (in a_i) of $[A^i]$ $(i = 0, 1, ..., n)$.* (4·14)

4·2. The Jacobian of the n-fold linear system of order-quadrics determined by the tangent cones $[K_i']$ from the vertices $[B_i']$ of the simplex $[B]$ to the quadric Q' (2·31) may be termed the *secondary Wallacian* of the simplex $[A]$, and denoted by W'. This locus, like W, is (2·31, 3·12) anallagmatic for the isogonal transformation q. $[A]$; also W' contains the bastions of $[A]$, the vertices of $[B]$, $[B']$ and the intersections of associated walls of $[A]$, $[B']$.

4·3. If $n = 2$, then the simplices (triangles) $[A]$, $[B]$ are in perspective; B_i' is the meet of the lines $A_j B_k$, $A_k B_j$ $(i, j, k = 0, 1, 2)$; the three lines a_i, b_i, b_i' meet at a point C_i $(i = 0, 1, 2)$, and the three points C_i are in line. Each of the plane cubics W, $\overline{W}$, W' contains the twelve points A_i, B_i, B_i', C_i; the three cubics are therefore identical. (4·31)

If $n > 2$, then the three $(n + 1)$-ics, W, $\overline{W}$, W' are in general

all distinct. For W, W' contain the bastions of $[A]$, while (4·14) $\overline{W}$ does not; and W, $\overline{W}$ contain the intersections of pairs of associated walls of $[A]$, $[B]$, while in general W' does not, since the intersection of a_i, b_i' is not generally the same as that of a_i, b_i (*v.*, however, 5·14 below). (4·32)

5. *The Case of the Orthocentric Simplex.* 5·1. If the simplex $[A]$ be orthocentric (q. the Absolute Q), let c denote its orthaxial hyperplane. Then c and the $n + 1$ hyperplanes a_i determine an $(n + 2)$-hedroid, $[\mathbf{A}]$ say. Any n of the $n + 2$ walls of $[\mathbf{A}]$ determine a vertex, and the remaining two walls a bastion, of $[\mathbf{A}]$; such a vertex and bastion may be termed *opposite*. The bastion of $[\mathbf{A}]$ opposite the vertex A_i is the intersection of the hyperplanes a_i, b_i; thus it is true for the $n + 1$ vertices A_i of $[\mathbf{A}]$, and therefore for the remaining $\frac{1}{2}n\,(n + 1)$ vertices, that: *The absolute polar hyperplane of any vertex of the $(n + 2)$-hedroid $[\mathbf{A}]$ contains the opposite bastion.* Thus $[\mathbf{A}]$ is *apolar* to the Absolute. (5·11)

But the cone K_i (3·11) touches the Absolute at every point of its section by the wall a_i of $[\mathbf{A}]$ $(i = 0, 1, ..., n)$. Thus $[\mathbf{A}]$ is apolar to each of the $n + 1$ cones K_i, and therefore: $[\mathbf{A}]$ *is apolar to every quadric of the n-fold linear system $[K]$.* (5·12)

Now the quadrics of an n-fold linear system to which a given $(n + 2)$-hedroid is apolar are the polar quadrics of a certain cubic hypersurface symmetrical q. that $(n + 2)$-hedroid. Thus: *If a simplex be orthocentric, then its absolute Wallacian is the Hessian of a certain cubic hypersurface; and so for its polar Wallacian.* (5·13)

Further, since (3·11) the cone K_j, of which the vertex is B_j, touches the Absolute at every point of its section by a_j, therefore the polar, q. K_j, of B_i contains a_{ij}, and is thus the hyperplane (β_{ij}, say) determined by B_j, a_{ij}; from (2·21) β_{ij} contains B_i'. Similarly β_{ij} is the polar hyperplane of B_j' q. the cone K_i' (4·2) of which the vertex is B_i'.

But if $[A]$ be orthocentric, and W therefore the Hessian of a cubic hypersurface (U, say); then (3·12) K_j is the polar quadric, q. U, of B_j', and β_{ij} is the mixed polar hyperplane, q. U, of the pair of points B_i, B_j'; thus β_{ij} is the polar of B_j', q. that polar cone (K_i'', say), q. U, of which the vertex is B_i'. Regarding B_i' for the moment as fixed, we see that the polar hyperplanes, q. K_i', K_i'', of the n points B_j' $(j \neq i)$ are identical; also K_i', K_i'' have the same vertex B_i'. Thus K_i', K_i'' are identical $(i = 0, 1, ..., n)$, and W, W' are Jacobians of the same $n + 1$ cones K_i', i.e.: *If a simplex be orthocentric, then its absolute and secondary Wallacians are identical.* (5·14)

6. *The Euclidean Case.* 6·1. If (in S_n) the Absolute Q be a quadric in a space of $n - 1$ dimensions (the *Infinite*) which does not contain any vertex of the simplex $[A]$; then all the vertices of the

polar simplex $[B]$ are at infinity, and each of the cones K_i degenerates into the Infinite counted twice (the *doubled Infinite*). The n-fold linear system of quadrics $[K]$ is thus (3·12) determined by the $(n-1)$-fold incentric system and the doubled Infinite. (6·11)

The necessary and sufficient conditions that a point should be on the (absolute) Wallacian are therefore that the point and its isogonal conjugate q. $[A]$ are apolar to the doubled Infinite. Thus: *In the euclidean case the absolute Wallacian of any simplex consists of the Infinite and its isogonal transformation q. that simplex. The correspondent of any finite point of the Wallacian is the point at infinity orthogonal to the pedal hyperplane of the given point.*

(6·12)

6·2. The polar Wallacian (4·1) of $[A]$ is the Jacobian of the $n+1$ point-hyperspheres A_i. In the euclidean case every such hypersphere, and therefore every order-quadric of the n-fold linear determined by them, is a hypersphere orthogonal to the hypersphere which circumscribes $[A]$; in particular (cf. 4·12) the Infinite and each hyperplane which contains the circumcentre of $[A]$ constitute a degenerate quadric of the system. Thus: *In the euclidean case the polar Wallacian of any simplex consists of the hypersphere circumscribing that simplex and the Infinite counted $n-1$ times. The correspondent of each point on the hypersphere is the diametrically opposite point of the hypersphere, and the correspondent of each point of the Infinite is any point of the space of dimensions $n-2$ at infinity orthogonal to the given point.* (6·21)

The second form of the specification (4·11) of $\overline{W}$ of course yields the rectangular property of the hypersphere; while the degenerate form of (4·14) is the theorem that the section by any S_{n-1} of a hypersphere in S_n is a hypersphere in the given S_{n-1}. Wallace's property of the circumcircle appears as a consequence of (4·31).

A focal line method of determining the elastic constants of glass.
By G. F. C. SEARLE, Sc.D., F.R.S., University Lecturer in Experimental Physics.

[*Read* 27 February 1922.]

§ 1. *Introduction.* We suppose that the glass bar is rectangular in form, having a considerable length $2l$, width $2s$ and comparatively small thickness $2t$ cm. Those surfaces which contain the edges $2l$, $2s$ will be called the "faces" of the bar. It is these faces whose curvatures are determined by optical observations on focal lines.

Let the bar be bent by equal and opposite couples applied at the ends of the bar, the magnitude of each couple being G dyne-cm.; the axis of each couple is parallel to the edges $2s$. Let the unstretched longitudinal filament, which lies in the plane bisecting the edges $2s$ at right angles, be bent into an arc of a circle of radius b cm. Then, if E be Young's modulus,

$$E = 3Gb/(4st^3). \qquad \ldots\ldots(1)$$

A transverse filament which intersects at right angles the longitudinal filament just mentioned and, when the bar is unstrained, is parallel to the edges $2s$, will be bent to radius c, where

$$c = -b/\sigma, \qquad \ldots\ldots(2)$$

and σ is Poisson's ratio*. The negative sign indicates that the point of intersection of the two filaments lies *between* the centres of curvature.

When $2t/b$ is small, it will be sufficiently accurate to consider that the principal curvatures of either face of the bar are $1/b$ and $1/c$. The plane of one principal section of a face bisects the edges $2s$ at right angles and the plane of the other principal section cuts the longitudinal filaments at right angles. The two curvatures having opposite signs, the face is anticlastic.

Let the bar be twisted by equal and opposite couples of J dyne-cm. applied at the ends of the bar, the axis of each couple being parallel to the edges $2l$. The particles, which in the unstrained bar lay in the median plane bisecting the edges $2t$, now lie on a helicoid. The lines of curvature of the distorted median surface cut at $\frac{1}{4}\pi$ the median line on this surface to which, before straining, the edges $2l$ were parallel. The principal radii of curvature are equal in length and opposite in sign; let the length of each be h cm. Then, if n be the rigidity, we have to a sufficient approximation, when t/s is small,

$$n = \frac{3Jh}{16st^3(1 - \cdot63t/s)}. \qquad \ldots\ldots(3)$$

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* For the conditions under which results (1) and (2) are good approximations, see Searle, *Experimental Elasticity*, §§ 27–34.

When $2t/h$ is small, either face may be considered as having the same curvatures as the median surface.

Unless the faces of the unstrained bar be, to a reasonable approximation, optically plane, the focal lines will be ill-defined. Bars cut from thin "patent plate" glass are generally satisfactory.

§ 2. *Focal lines due to reflexion at a curved surface.* In place of the general investigation given in books on optics, we use a "stationary time" method, adapting it to the special circumstances of the problem for (1) the bending, (2) the twisting of the bar *.

Take "right-handed" rectangular axes Ox, Oy, Oz, the origin O being on the surface, and let Ox coincide with the normal to the surface through O.

In the first case we suppose that the lines of curvature at O lie in the planes Oxy, Oxz respectively. Then, for points near the origin, the equation to the surface can be written

$$x = \tfrac{1}{2}y^2/b + \tfrac{1}{2}z^2/c. \qquad \qquad \ldots\ldots(4)$$

Here b, c are the radii of curvature of the sections by the planes Oxy, Oxz, and are counted *positive* when the centres of curvature lie on the *positive* part of the axis of x.

Let P (Fig. 1) be a luminous point in the plane Oxy, the line OP lying in the quadrant yOx; let $OP = p$ and let the acute angle POx be θ. Let OQ be a straight line in the same plane such that Ox bisects the angle between OP and OQ. Then the chief incident ray PO becomes, after reflexion, the chief reflected ray OQ. Let $OQ = q$. Let p, q be positive when x is positive for P, Q.
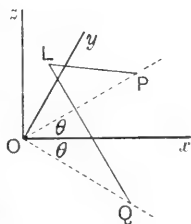
Fig. 1

Take any point L on the reflecting surface near O. Then, if the second and third coordinates of L be y, z, the first is $\tfrac{1}{2}y^2/b + \tfrac{1}{2}z^2/c$. Let $PL = u$, $QL = v$. Then

$$u^2 = (p \cos \theta - \tfrac{1}{2}y^2/b - \tfrac{1}{2}z^2/c)^2 + (p \sin \theta - y)^2 + z^2.$$

Hence, as far as squares of y and z,

$$u^2 = p^2 \{1 - 2y \sin \theta/p - \cos \theta\, (y^2/b + z^2/c)/p + (y^2 + z^2)/p^2\}.$$

Taking the square root, and expanding as far as the squares of y, z, we have

$$u = p \{1 - y \sin \theta/p - \tfrac{1}{2} \cos \theta\, (y^2/b + z^2/c)/p + \tfrac{1}{2} (y^2 + z^2)\, p^2$$
$$- \tfrac{1}{2} y^2 \sin^2 \theta/p^2\}$$
$$= p - y \sin \theta - \tfrac{1}{2} \cos \theta\, (y^2/b + z^2/c) + \tfrac{1}{2} (y^2 \cos^2 \theta + z^2)/p.$$

For v we change the sign of $\sin \theta$ and write q for p. Thus

$$v = q + y \sin \theta - \tfrac{1}{2} \cos \theta\, (y^2/b + z^2/c) + \tfrac{1}{2} (y^2 \cos^2 \theta + z^2)/q.$$

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* Dr G. T. Bennett has kindly verified the work in this section.

For the total distance $PL + LQ = u + v$, we have

$$u + v = p + q - \cos\theta \, (y^2/b + z^2/c) + \tfrac{1}{2}(y^2\cos^2\theta + z^2)(1/p + 1/q).$$
$$\ldots\ldots(5)$$

If $u + v$ have a stationary value, i.e. if $d\,(u + v)/dy = 0$, and $d\,(u + v)/dz = 0$, simultaneously, the ray reflected at L will meet the chief ray OQ in Q. We have $d\,(u + v)/dy = 0$, if either

$$y = 0, \quad \text{or} \quad 1/p + 1/q - 2/(b\cos\theta) \equiv \beta = 0,$$

and $d\,(u + v)/dz = 0$, if either

$$z = 0, \quad \text{or} \quad 1/p + 1/q - 2\cos\theta/c \equiv \gamma = 0.$$

If $y = 0$ and $z = 0$, the reflexion is restricted to a single point on the surface and the positions of P and Q along the lines defined by θ are immaterial. If $\beta = 0$ and $\gamma = 0$, *all* the rays from P which meet the surface near O pass, after reflexion, through the single point Q. This case, however, requires the special conditions $\cos^2\theta = c/b$, and $1/p + 1/q = 2/(bc)^{\frac{1}{2}}$. The points P, Q are then the foci of a quadric surface of revolution, with PQ as axis, the reflecting surface coinciding with the quadric near O. When the surface is anticlastic, c/b is negative and this special case cannot arise.

We are left with the two pairs of conditions, viz. $z = 0$ and $\beta = 0$, and $y = 0$ and $\gamma = 0$. When $z = 0$ and $\beta = 0$, all the rays from P, which meet the surface in the plane $z = 0$, pass after reflexion through the point Q defined by

$$1/p + 1/q = 2/(b\cos\theta). \qquad \ldots\ldots(6)$$

Similarly, all the rays from P, which meet the surface in the plane $y = 0$, pass after reflexion through the point Q' defined by

$$1/p + 1/q' = 2\cos\theta/c. \qquad \ldots\ldots(7)$$

In the experiment, $1/p = 0$, and then

$$b = 2q/\cos\theta, \qquad \ldots\ldots(8)$$
$$c = 2q'\cos\theta. \qquad \ldots\ldots(9)$$

The planes containing (1) Q and Oz, and (2) Q' and Oy are the principal planes of the reflected wave front and contain the focal lines through Q, Q' respectively.

In the second case, we suppose that the lines of curvature at O are inclined at $\tfrac{1}{4}\pi$ to Oy and Oz. Then, for points near O, the equation to the surface may be written

$$x = yz/h. \qquad \ldots\ldots(10)$$

The helicoid is left-handed when h is positive. Writing $2yz/h$ in place of $y^2/b + z^2/c$ in (5), we have

$$u + v = p + q - 2\cos\theta.yz/h + \tfrac{1}{2}(y^2\cos^2\theta + z^2)(1/p + 1/q).$$
$$\ldots\ldots(11)$$

If $d(u + v)/dy = 0$, and $d(u + v)/dz = 0$, simultaneously, the ray reflected at L will meet the chief reflected ray in Q. These conditions give

$$2z/h = y(1/p + 1/q)\cos\theta, \quad 2y/h = z(1/p + 1/q)/\cos\theta. \quad \dots(12)$$

The values of z/y satisfying these equations are given by

$$z/y = \mp\cos\theta.$$

Thus, if Q and Q' correspond to $z = -y\cos\theta$, and to $z = y\cos\theta$ respectively, we have

$$1/p + 1/q = -2/h, \quad 1/p + 1/q' = 2/h. \quad \dots\dots(13)$$

When p is given, the values of q, q' depend only on h and not on θ.

In the experiment $1/p = 0$, and then

$$h = -2q, \quad\quad\quad \dots\dots(14)$$
$$h = 2q'. \quad\quad\quad \dots\dots(15)$$

We will now find the planes which contain the chief reflected ray OQ and the rays reflected at points on the curves in which the surface is cut by the planes $z = -y\cos\theta$, $z = y\cos\theta$ respectively.

On yO produced take a point M (Fig. 2) and draw KMK' parallel to Oz to meet the planes $z = \mp y\cos\theta$ at K, K'. Then $KM = K'M = OM\cos\theta$. Through KM draw a plane KMN to meet OQ at right angles at N. Then KN, MN, $K'N$ are perpendicular to ON, and KNM, $K'NM$ are the angles which the planes OKN, $OK'N$ make with the plane Oxy. Since $NOx = POx = \theta$, and $ONM = \frac{1}{2}\pi$, we have $OMN = \theta$. Hence

$$MN = OM\cos\theta = KM = K'M.$$

Since $KMN = \frac{1}{2}\pi$, we have

$$KNM = K'NM = \frac{1}{4}\pi.$$

Fig. 2

The planes OKN, $OK'N$ contain the rays which meet in Q, Q' respectively and are thus the principal planes of the reflected wave front. The focal lines through Q, Q' lie in the planes $OK'N$, OKN respectively, but their distances from O depend upon h, for $q = -\frac{1}{2}h$, $q' = \frac{1}{2}h$.
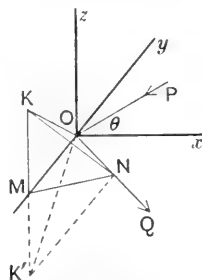
§ 3. *Optical method of measuring curvatures of reflecting surface.* The surface S (Fig. 3) is placed so that the tangent plane at its central point O is vertical. The normal Ox and the axis Oy are horizontal. The surface is placed so that its lines of curvature at O are either vertical and horizontal or are inclined at $\frac{1}{4}\pi$ to Oz. A collimator, consisting of a lens L and cross-wires, which intersect at K in the focal plane of L, is directed to O, the line of collimation being horizontal. The wires are stretched across an opening in a plate and are illuminated by a flame. The rays from K fall on S as

a parallel beam and thus $1/p = 0$ in (6), (7) and (13). The rays reflected at S are received by a converging lens M of focal length f, whose axis passes through O and is horizontal, and are brought to a point-focus or to a focal line on the ground glass screen N, fixed to a carriage sliding on a short optical bench. If S be one surface of a glass plate, the other surface is smeared with vaseline to stop reflexion there.
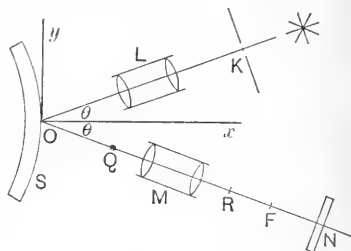
Fig. 3

It is convenient that one focus of M should coincide with O. A plane mirror is substituted for N, and then M is set so that a speck of paper placed on S at O coincides with its own image.

If S be plane, both wires will be sharply focussed when N is in the focal plane through F, the other focus of M.

If S be curved, and θ, the angle of incidence, be finite, it will not be possible to obtain an "image" of either cross-wire, unless the directions of the wires be properly adjusted. It will still be impossible to obtain images of both wires simultaneously, but each can be focussed in turn by adjusting N.

If, instead of cross-wires, there were a fine hole at K, a parallel beam would fall on S and the reflected rays would pass through two focal lines meeting the chief reflected ray in Q, Q'. If a luminous point were placed at Q, it would give rise to two focal lines, one passing through K and one elsewhere. A luminous point at Q' would give rise to two focal lines, one passing through K and one elsewhere. If the cross-wires at K be placed in the directions of those focal lines due to Q and Q' which pass through K, it will be possible to focus each wire in turn by adjusting the screen N. Hence, when the lines of curvature at O are horizontal and vertical, the cross-wires must also be horizontal and vertical. When the lines of curvature are inclined at $\frac{1}{4}\pi$ to the vertical axis Oz, the cross-wires must also be inclined at $\frac{1}{4}\pi$ to Oz.

Let R be the image of Q formed by M, let $FR = \xi$, and let ξ be counted positive when R lies on the same side of F as the focus O. Then, as in § 2, counting OQ or q positive when Q is "real," we have, by Newton's formula, $\xi q = f^2$, or

$$q = f^2/\xi. \qquad \ldots\ldots(16)$$

The distance ξ is easily measured, as it is merely the displacement of the screen N from the position in which both wires are in focus when a good plane surface, such as that of a prism, is used as the reflecting surface.

In the case of (14) and (15), we can at once find h from (16) when f is known. But in the case of (8) and (9), the value of $\cos\theta$

is required. A *convex* spherical surface, e.g. the surface of a lens, of radius a is substituted for S, and is adjusted so that the chief reflected ray has the same direction as before. Then in (8), (9) $b = -a$, $c = -a$, and thus, if η, η' be the values of ξ for (1) the vertical, (2) the horizontal focal line, $f^2 = \eta\left(-\frac{1}{2}a\cos\theta\right)$, and $f^2 = \eta'\left(-\frac{1}{2}a/\cos\theta\right)$. Hence

$$\cos^2\theta = \eta'/\eta. \qquad \ldots\ldots(17)$$

A convex is used in preference to a concave spherical surface, for, unless the curvature of the latter be slight, the images formed by M of the focal lines will not be "real."

In practice f is more easily determined than a, but, if desired, f can be found in terms of a, since $f^4 = \frac{1}{4}a^2\eta\eta'$.

§ 4. *Determination of Young's modulus and of Poisson's ratio.* The glass bar rests with one face against two vertical pillars, whose sections are C, D (Fig. 4), rising from a base H (see Fig. 6); the base is fixed to the table in any convenient manner. The bar is bent by two strings which pass over pulleys U, V and are attached
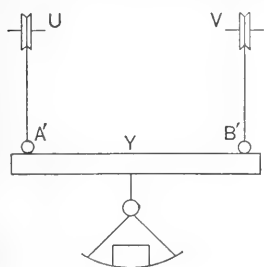
Fig. 5
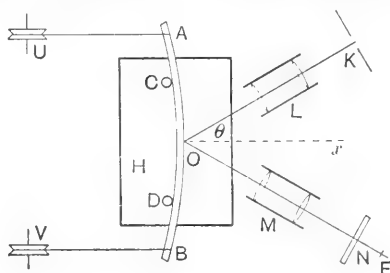
Fig. 4

at A', B' (Fig. 5) to a light yoke Y. A loop at the end of the U-string passes round the bar at A, and similarly for the other string. From the centre of the yoke hangs a light pan in which loads are placed. The screen N is adjusted so that (1) the vertical wire, (2) the horizontal wire is in focus for each value of the load.

Let ξ, ξ' be the values of FN for the vertical and horizontal lines when the *total* load carried by the two strings is m grms. When the bar is bent as in Fig. 4, ON is greater than OF for the vertical focal line and thus ξ is *negative*. Let $\frac{1}{2}(AB - CD) = k$ cm., where CD is measured between the lines of contact. Then, G, the bending moment at any part of the bar between C and D, is constant, if $AC = BD$, and $G = \frac{1}{2}mgk$. Since, by (8), $b = 2q/\cos\theta$, we have, by (1) and (16),

$$E = \frac{3gkf^2}{4st^3\cos\theta} \cdot \left|\frac{m}{\xi}\right|. \qquad \ldots\ldots(18)$$

The value of $\cos\theta$ is given by (17).

By (2), $\sigma = - b/c$, and, by (8), (9), $b/c = q/(q' \cos^2 \theta)$. Hence, by (16) and (17),

$$\sigma = - \xi'/(\xi \cos^2 \theta) = - \xi' \eta / \xi \eta'. \qquad \ldots\ldots(19)$$

A series of loads, increasing by equal steps, is used. It is convenient to adjust the mass of the yoke and pan to form the first of the series of loads. In view of the difficulty of the observations, it is best to plot the values of $- \xi$ against those of m and to draw a straight line to lie as evenly as possible among the plotted points. The ratio of the rate of increase of $- \xi$ to that of m is then found from the straight line, and this ratio is used in place of $- \xi/m$ in (18). The ratio $- \xi'/\xi$ is found by plotting ξ' against $- \xi$.

§ 5. *Determination of rigidity.* The base and pillars used in § 4 are again employed. On the pillars C, D slides a bridge EF (Fig. 6) which can be clamped to C, D. The upper end A of the glass bar
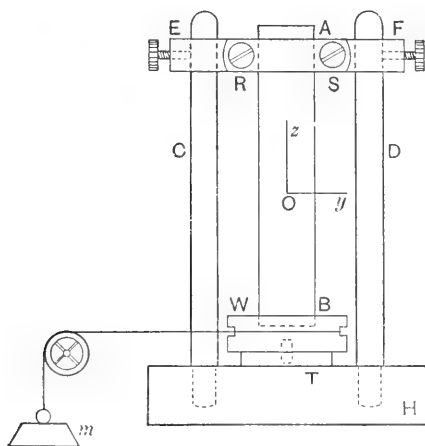
Fig. 6
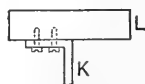
Fig. 7

is placed between the bridge EF and a metal plate RS and is clamped by screws passing through the ends of RS. The lower end B of the bar rests in a groove cut in the wheel W, which turns freely about a vertical pivot rising from a plate T, which is attached to the base H in such a position that the edges of the bar are vertical. A fine string attached to W passes over a pulley and supports a mass m grms. Let the radius of the drum on which the string is wound be r cms., and let J be the couple applied to the glass bar; then $J = mgr$ dyne-cm.

The small table L (Fig. 7), having an angle-piece K, which can be clamped between EF and RS in place of the bar, is used for supporting a prism or a lens; the height is adjusted by sliding EF on the pillars.

We suppose that the axes of y and z are as shown in Fig. 6, and that, in the standard case, the load m turns the wheel W in a

clock-wise direction as seen from the end A of the bar. The bar is thus twisted into a right-handed helicoid, and hence the quantity h in (10) is negative. By § 2, $q = -\frac{1}{2}h$ for that focal line which has a north-east direction, as seen on the ground glass screen N, when N lies between the lens M and the observer, and, by (16), ξq is positive. In the standard case, h is negative and hence ξ is positive for the north-east focal line and ξ' is negative for the north-west line.

If the direction of the couple be reversed, ξ is negative for the north-east line and ξ' is positive for the north-west line.

A series of loads is applied for each direction of the couple; the loads which produce a left-handed helicoid are counted as negative. From the straight lines lying most evenly among the points, when ξ and $-\xi'$ are plotted against m, ξ/m and $-\xi'/m$ are found. The mean of these two values is used in (20) below. By (14), (15) and (16), $h = -2f^2/\xi = 2f^2/\xi'$, and thus, since $J = mgr$, we have for the rigidity,

$$n = \frac{3grf^2}{8st^3(1 - \cdot63t/s)} \cdot \left| \frac{m}{\xi} \right| . \qquad \ldots\ldots(20)$$

§ 6. *Practical example.* Mr C. F. Sharman, of King's College, used a bar cut from a photographic plate of thin "patent plate" glass kindly supplied by the Astronomer Royal at a time when such glass was difficult to obtain. For this bar, $s = 1\cdot25$ cm., $t = \cdot0902$ cm.

The focal length of the lens M (Fig. 3) was found, by the goniometer method, to be $f = 16\cdot13$ cm. When a face of a prism was the reflector, the bench reading of the screen N was $17\cdot72$ cm., corresponding to the point F (Fig. 3). When the convex surface of a lens was the reflector, the bench readings of N for the vertical and horizontal focal lines were $29\cdot12$ and $27\cdot21$ cm. respectively. In each of the latter readings NO was greater than FO, and thus η, η' are negative. Hence $\eta = 17\cdot72 - 29\cdot12 = -11\cdot40$ cm., $\eta' = 17\cdot72 - 27\cdot21 = -9\cdot49$ cm. Then $\cos^2\theta = \eta'/\eta = \cdot83246$; thus $\cos\theta = \cdot91238$ and $\theta = 24° 9' 49''$.

The radius, a, of the convex surface of the lens is not required in this experiment. By § 3, $|a| = 2f^2/(\eta\eta')^{\frac{1}{2}} = 50\cdot03$ cm.

Determination of Young's modulus and Poisson's ratio. The distance AB between the strings was $15\cdot20$ cm., and the distance CD between the lines of contact of the bar with the pillars was $6\cdot98$ cm. Hence $k = \frac{1}{2}(AB - CD) = 4\cdot11$ cm. To allow for any slight initial curvature of the bar, the zero readings of the screen N for the vertical and the horizontal focal lines respectively were obtained, not by aid of the prism, but from the bar itself when the load was zero.

The values found for ξ, ξ' were as follows:

Load = m	ξ	ξ'	Load = m	ξ	ξ'
grms.	cm.	cm.	grms.	cm.	cm.
0	0	0	800	$-1\cdot06$	$\cdot26$
200	$-\cdot23$	$\cdot02$	1000	$-1\cdot33$	$\cdot31$
400	$-\cdot53$	$\cdot15$	1200	$-1\cdot56$	$\cdot38$
600	$-\cdot80$	$\cdot21$	1400	$-1\cdot84$	$\cdot45$

On plotting $-\xi$ against m, and ξ' against $-\xi$, the values

$$-\xi/m = 1{\cdot}3125 \times 10^{-3} \text{ cm. grm.}^{-1}, \qquad \xi'/\xi = -0{\cdot}215,$$

were obtained. Then, by (18), we have for Young's modulus,

$$E = \frac{3gkf^2}{4st^3 \cos\theta \cdot |\xi/m|} = \frac{3 \times 981 \times 4{\cdot}11 \times 16{\cdot}13^2}{4 \times 1{\cdot}25 \times {\cdot}0902^3 \times {\cdot}9124 \times 1{\cdot}3125 \times 10^{-3}}$$

$$= 7{\cdot}16 \times 10^{11} \text{ dyne cm.}^{-2}$$

Also, by (19), we have for Poisson's Ratio,

$$\sigma = -\frac{\xi'\eta}{\xi\eta'} = \frac{{\cdot}215}{{\cdot}8325} = {\cdot}258.$$

Determination of rigidity. The radius of the drum on which the string was wound was (with allowance for thickness of string) $r = 2{\cdot}535$ cm. In the table, couples straining the bar into a left-handed helicoid are indicated by the negative sign prefixed to the loads.

Load m	N.E. image ξ	N.W. image ξ'	Load m	N.E. image ξ	N.W. image ξ'
grms.	cm.	cm.	grms.	cm.	cm.
200	·15	−·17	−200	−·17	·22
400	·29	−·35	−400	−·40	·42
600	·53	−·59	−600	−·58	·57
800	·72	−·72	−800	−·77	·81

On plotting ξ and $-\xi'$ against m, the values $\xi/m = 9{\cdot}30 \times 10^{-4}$, and $-\xi'/m = 9{\cdot}60 \times 10^{-4}$ were obtained. The mean is $9{\cdot}45 \times 10^{-4}$ cm. grm.$^{-1}$ Then, by (20), we have for the rigidity,

$$n = \frac{3grf^2}{8st^3(1 - {\cdot}63t/s)} \cdot \left|\frac{m}{\xi}\right| = \frac{3 \times 981 \times 2{\cdot}535 \times 16{\cdot}13^2}{8 \times 1{\cdot}25 \times {\cdot}0902^3 \times {\cdot}9545} \cdot \frac{1}{9{\cdot}45 \times 10^{-4}}$$

$$= 2{\cdot}93 \times 10^{11} \text{ dyne cm.}^{-2}$$

*The absorption spectra of some organic and inorganic salts **of** didymium.* By J. E. PURVIS, M.A.

(Plates VII and VIII.)

[*Read* 7 May 1923.]

Ostwald's well-known experiments on the absorption spectra of a number of permanganates[*] endeavoured to prove that there was no difference in the character and appearance of these spectra. It was considered to be conclusive that the spectrum common to all the salts of the same metal is due to the metallic ions.

Merton (*Trans. Chem. Soc.* 1911, 99. 637), from a study of the absorption spectra of permanganates in certain solvents, concluded that the general nature of the absorption is due to the atomic grouping MnO_4, and that it is influenced very little, if at all, by the nature of the kation or the extent of the dissociation, the position of the points of maximum absorption being determined by the nature of the medium in which the salt is dissolved.

It has been suggested that any differences in the appearance of such absorption bands would be small, and instruments of considerable dispersive power would be required. The author has shown, for example[†], that in the third order of the dispersion by a Rowland grating of 21 feet radius, a band at $\lambda 427$ (described as a sharp band normally), in the spectrum of didymium, divides into two. It is certain that, with larger dispersions than have hitherto been employed, bands would split up which have been considered indivisible.

Dr Liveing[‡] describes the effects of dilution, temperature, and other circumstances on the absorption spectra of solutions of didymium and erbium salts. He found (1) that with regard to temperature the spectrum afforded no confirmation of the supposition that the absorptions are due to an increase in the number of ions, (2) that the absence of any diminution of intensity in the bands by the addition of acid, taken in conjunction with the fact that rise of temperature does not increase their intensity, goes a long way to negative the supposition that these bands are produced by the metallic ions. Dr Liveing concluded from the whole series of observations that the characteristic absorptions of didymium (and erbium) compounds, namely those which are common to dilute aqueous solutions, and are only modified by concentration, by heat, and by variations of the solvent, are due to molecules which are identical in all cases, though their vibrations are modified by their relations to other molecules surrounding them.

[*] *Zeit. Phys. Chem.* 1892, 9. 579.

[†] *Proc. Camb. Phil. Soc.* 1903, 11. 206.

[‡] *Trans. Camb. Phil. Soc.* 1899, 18. 298 sqq.

Hartley*, in an examination of the absorption spectra of metallic nitrates, considers that the absorption bands of the different salts are not identical, but only very similar. He argues that there is no separation of the compound into ions, but only a dissociation of such a character that the molecule is shown to consist of two parts, the movements of the one being influenced by those of the other, so that the molecule is not resolved into ions, but is in a condition of molecular tension.

The author† has described the influence on the absorption of much greater dilutions of the chlorides and nitrates of didymium and erbium chlorides than were employed by Dr Liveing, and found similar results. The bands in the concentrated solutions of the chlorides of didymium and erbium were not altered on dilution, the bands of very concentrated solutions of the nitrates of didymium and erbium were less diffuse when these solutions were diluted, and the bands of concentrated and diluted solutions of the chlorides of didymium and erbium are precisely similar to those in the diluted solutions of the nitrates. In the ultra-violet regions, the only point of difference in the strong and diluted solutions was an extension of the general absorption of the more refrangible end of the spectrum in the concentrated didymium and erbium chlorides, and this was comparable with that of the concentrated didymium and erbium nitrates.

None of these inorganic salts of didymium possesses any bands in the ultra-violet regions; and, so far as the author knows, no organic salts have been studied. He has investigated the absorption of didymium-phenyl-acetate, which gives a large band in the ultra-violet regions, and compared it with the absorption of equivalent amounts of the acetate, the chloride, and the nitrate of didymium. Acetic acid has no absorption band in the ultra-violet regions; but phenyl acetic acid in considerable thickness exhibits a well-marked band, and also the weak remnants of three toluene bands, when the light passes through a few millimetres of the solution‡. Didymium-phenyl-acetate is not very soluble, but it is sufficiently soluble to produce a solution of about $M/1000$ strength. Equivalent amounts of the four salts, i.e., $0·248$ gm. Di_2O_3 in 1000 c.c., acetate, phenyl-acetate, chloride and nitrate of didymium and also of phenyl-acetic acid itself, were investigated.

For surveying the visible regions of the spectrum the method of comparison was similar to that used by Dr Liveing and the author§. The tubes containing the liquids were 610 mm. and 300 mm. long respectively, and 20 mm. inside diameter. For

* *Trans. Chem. Soc.* 1903, 83. 221.
† *Proc. Camb. Phil. Soc.* 1903, 12. 206.
‡ Purvis, *Trans. Chem. Soc.* 1915, 107. 966.
§ *Loc. cit.*

the ultra-violet regions the tube was similar to that introduced by Baly, with a movable inside cylinder to produce varying thickness of the liquid. A Welsbach burner, without the cover, was the source of radiation for the visible spectrum; and a condensed cadmium spark for the ultra-violet regions. The times of exposure for the visible regions varied from 5 to 15 seconds, determined by a stop-watch, and spectrum plates were employed; for the ultra-violet regions two minutes' exposure was given, and the thickness of the liquid varied between 2 mm. and 30 mm. The following bands were examined. Other bands of didymium were too weak for a close examination.

$\lambda 623$ a weak band.
$\lambda 596$ a somewhat weak band.
$\lambda 590$–$\lambda 570$ a group of bands overlapping one another.
$\lambda 531$ a somewhat weak band.
$\lambda 528$–$\lambda 520$ a strong group of overlapping bands.
$\lambda 510$ a strong group of two diffuse bands.
$\left.\begin{array}{l}\lambda 483 \\ \lambda 476 \\ \lambda 469\end{array}\right\}$ a well-marked series of bands of which $\lambda 476$ is weaker than the other two.
$\lambda 444$ a strong broad band.

Visible Spectrum. In the visible regions didymium acetate shows distinct differences from the didymium-phenyl-acetate, and also from the chloride and the nitrate. If didymium acetate is compared with didymium-phenyl-acetate (Fig. 1) the more marked differences are the following:

$\lambda 623$ is weaker in the didymium acetate and nearly disappears.
$\lambda 596$ is also weaker and more diffuse and widens into the group $\lambda 590$–$\lambda 570$.
$\lambda 590$–$\lambda 570$ this also widens, but there is no difference in the intensity.
$\lambda 531$ a shade weaker and more diffuse.
$\lambda 528$–$\lambda 520$ a group of bands which is perhaps a shade stronger.
$\lambda 510$ a group which is a shade stronger.
Of the triplet $\lambda 483$, $\lambda 476$, $\lambda 469$ the band at $\lambda 476$ is weaker, the other two show no difference.
$\lambda 444$ a band which is a shade stronger.

The most definitely marked differences in the bands of these two salts are at $\lambda 596$, $\lambda 590$–$\lambda 570$, $\lambda 476$ and $\lambda 444$.

On comparing didymium acetate with the nitrate and chloride of didymium (Fig. 2) there are some slight differences, but they are not so well marked as those in didymium acetate and didymium-phenyl-acetate described above. For example, the band at $\lambda 623$ is weaker in didymium acetate and at $\lambda 596$ is a shade more

diffuse than in the nitrate and chloride; the group $\lambda590$–$\lambda570$ is a shade more diffuse on the less refrangible edge than the corresponding group of the nitrate and chloride, the band at $\lambda531$ is a shade weaker in didymium acetate; and the narrow band at $\lambda476$ is weaker in didymium acetate. The nitrate and chloride of didymium in thin dilute solutions show no difference, as Dr Liveing has shown before[*].

These results, therefore, indicate that, in the visible regions, various bands of didymium show slight differences according to the nature of the acid radicle.

Ultra-violet regions. The absorption of phenyl-acetic acid has been described by the author[†]. In solutions of 2 and 4 mm. thickness the three bands of toluene appeared but much weaker and wider than in toluene itself. In thicker solutions these widened out into one large band. The author has shown[‡] that this band almost disappears in triphenyl-acetic acid.

Now, through thicknesses of solution of 10 mm. (Fig. 3) the chloride and the acetate of didymium transmit the radiations to $\lambda2100$, *i.e.* as far as the photographic plate is sensitive. There is no difference either in the intensities or the position of the transmitted rays. In didymium-phenyl-acetate there is a large band between $\lambda2830$–$\lambda2450$ and the rays are then transmitted to about $\lambda2300$. This band is more diffuse on the more refrangible side than on the less refrangible. In didymium nitrate the rays are transmitted to about $\lambda2320$. So that in both these salts there is more general absorption than in the acetate and chloride.

On comparing the absorption of phenyl-acetic acid and didymium-phenyl-acetate through a thickness of 6 mm., the width of the band of the former is 160 Å. units ($\lambda2780$–$\lambda2620$), and that of the latter is 170 Å. units ($\lambda2790$–$\lambda2620$). And, through the same thickness of 6 mm., the line of general absorption is at $\lambda2290$ in phenyl-acetic acid and at $\lambda2310$ in didymium-phenyl-acetate; through 20 mm. thickness the line of general absorption stops at $\lambda2820$ in phenyl-acetic acid, and at $\lambda2835$ in didymium-phenyl-acetate. Through a thickness of 2 mm. the two weak "toluene" bands at about $\lambda2730$ and $\lambda2660$, in both of these substances, do not show any well-defined differences. These photographs have not been reproduced. Figs. 1, 2, and 3 show the differences which are described above, fairly well, but the original photographs are, of course, much clearer.

* *Loc. cit.*
† *Loc. cit.*
‡ *Trans. Chem. Soc.* 1914, 105. 1372.

78c
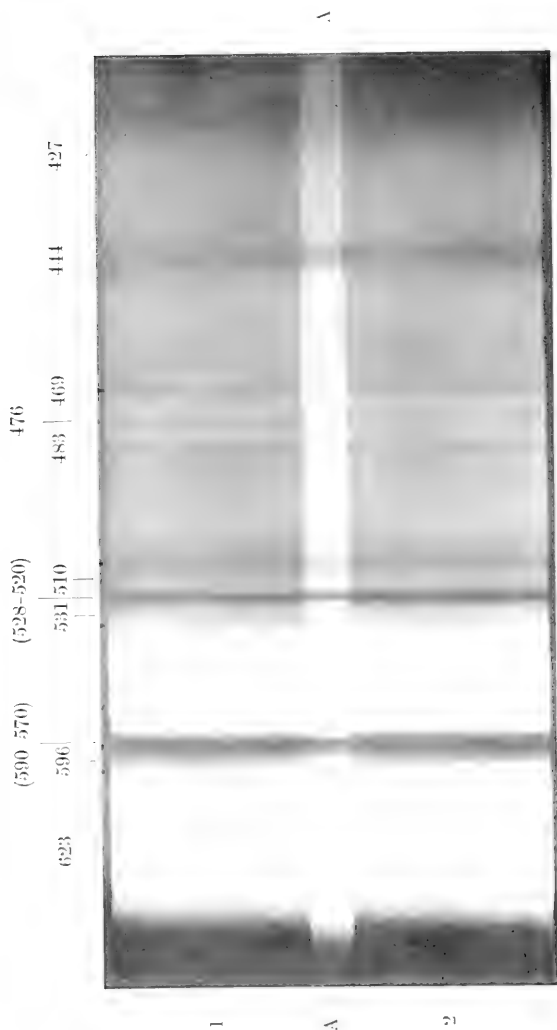
Fig. 1

1. Didymium phenyl acetate.
2. Didymium acetate.

The narrow band A. A. is produced by the overlapping of 1 and 2.

(590–570) (528–520)   476

623 596   531 510   483 469   444   427

Fig. 2

1. Didymium nitrate.   2. Didymium chloride.   3. Didymium acetate.
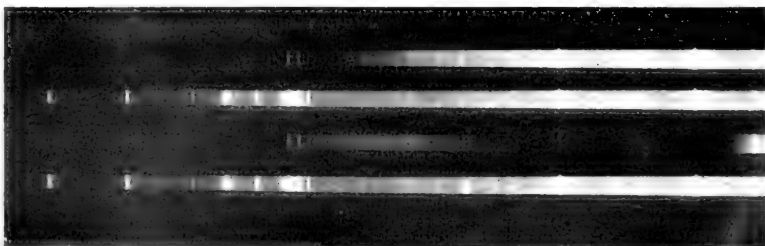
Cd. 2146 Cd. 2196   Cd. 2329   Cd. 2572   Cd. 2717

Fig. 3

1. Didymium nitrate.   2. Didymium chloride.   3. Didymium phenyl acetate.
                        4. Didymium acetate.

The transmitted rays in 2 and 4 are well seen in this reproduction, and the large
band in 3 between λ2830 λ2450 is quite clear.

RESULTS.

The results show (1) that in the visible regions of the spectrum there are some slight differences in the width and intensity of the bands of didymium chloride, nitrate, acetate, and phenyl-acetate; the differences are more apparent in the two latter substances, particularly at $\lambda596$, $\lambda590$–$\lambda570$, and $\lambda476$: (2) that in the ultra-violet regions the differences in the absorption of the rays are much more marked, as for example, in didymium chloride, didymium nitrate and didymium acetate; and (3) the large band of didymium-phenyl-acetate is a little wider than that of phenyl-acetic acid and the line of general absorption is shifted a little more towards the red end.

If the basic and acidic constituents of these salts were completely separated from each other in solution, the absorption bands should be similar in the visible regions, and there would be well-marked differences in the ultra-violet regions, depending on the chemical nature of the acid radicle. These experiments prove that there are some differences in the bands of various salts both in the visible and the ultra-violet regions. Each of the constituent parts of the salts exercises its own powers to a considerable extent; but they are not free from each other's influence. The anions and kations of the molecule do not vibrate independently of each other.

There is also the influence exerted by each molecule acting on and being acted on by other molecules in their mutual encounters. Didymium is not a single element. It is composed of several constituents two of which we know as neodymium and praseodymium, the atomic weights of which are 144·3 and 140·6 respectively. The various salts of these elements would differ, to some extent, in their absorptive capacities, and what we observe is the result of a complex series of vibrations of the electrons, atoms and molecules of closely related elements.

The absorption spectra of solutions of benzene and some of its derivatives at various temperatures. By J. E. PURVIS, M.A.

(Plates IX and X.)

[*Read* 7 May 1923.]

The absorption spectra of the vapours of benzene and its homologues at different temperatures and pressures and of solutions of benzene have been described by Hartley*. He found that at increasingly higher temperatures, the general absorption is broadened and extended towards the less refrangible rays, and that the narrow vapour bands were unaltered in position by variations in temperature and pressure. He explains the general absorption as due to the translatory kinetic energy of the molecules, and the selective absorption to the vibrations of the atoms or atom-complexes within the molecules, so that they are not affected by rise of temperature in the same manner as the translatory molecular movements; they are not displaced. When a large number of molecules pass the slit of the spectroscope, a greater number of the intra-molecular vibrations is brought into the field and the feeble bands are intensified.

So far as the author knows, there are no recorded results of the effect of temperature on the spectra of *solutions* of organic substances†. He has investigated the vapours of a number of organic substances, and he noticed that narrow bands widened into each other at the higher temperatures and pressures. They gradually became comparable with the wide and diffuse solution bands. The results are published in various volumes of the *Trans. Chem. Soc.* and references to these earlier investigations of the substances are given at the end of this paper. From among these substances he has selected the following for an examination of the effect of changes of temperature on their solution spectra, viz. benzene, toluene, monochlorobenzene, benzonitrile, phenol and aniline.

Apparatus. The tube containing the alcoholic solutions was 40 mm. long and 15 mm. internal diameter; polished quartz plates were screwed on each end, and there was a side tube for introducing the solutions. The absorbing tube was placed in a brass cell with an opening at each end, and a polished quartz plate screwed over

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* *Trans. Roy. Soc.* A, 1907, 208. 475 et seq.

† Hartridge (*Proc. Phys. Soc.* 1920–21, 54. 128) found that, at the temperature of liquid air, the absorption bands of oxy-hæmoglobin were sharper to the eye than those at laboratory temperatures, and were shifted towards the violet approximately 41 Å.U.

each opening. The cell was covered with thick asbestos board and filled with water—the absorbing tube being completely immersed. The cell and its contents were mounted in front of the slit of the spectrometer, and a condensed cadmium spark was the source of radiation. The times of exposure of the photographic plate varied from two to five minutes. Alcoholic solutions of various strengths were used—*that* solution being selected which showed clear and well-defined bands of each substance. The temperatures ranged from that of the unheated cell, and its contents, to $30°$ C., $45°$ C., and $60°$ C.

Benzene. Alcoholic solutions of benzene of $M/500$, $M/750$ and $M/1000$ strengths were investigated. Three of the seven bands were selected as they are strong and well defined; these are at $\lambda 2618$, $\lambda 2554$, $\lambda 2497$. It will be noticed (Fig. 1) that at the higher temperatures they are a little wider, more diffuse, and with a slight shift towards the red end. They are also slightly weaker, but this does not appear very clearly in the reproduction. The reproductions are twice as large as the original photographs.

Toluene. Fig. 2 is a reproduction of the absorption by a solution in alcohol of $M/500$ strength. As the temperature increases the three bands at $\lambda 273$, $\lambda 266$, and $\lambda 263$ gradually widen, become a little more diffuse and are shifted towards the red end. The band at $\lambda 263$ nearly disappears at $60°$ C.

Monochlorobenzene. The absorption by an alcoholic solution of $M/400$ strength is reproduced in Fig. 3. This substance has seven bands of varying degrees of width and intensity. At the higher temperatures the band at $\lambda 2714$ widens, is more diffuse and is shifted towards the red end. The bands at $\lambda 2645$, $\lambda 2622$, $\lambda 2577$ and $\lambda 2549$ almost disappear. The bands at $\lambda 2510$ and $\lambda 2450$ are too weak to show anything in this strength of solution.

Phenol. An alcoholic solution of $M/5000$ strength was investigated. This substance has one large diffuse band the centre of which is at about $\lambda 2750$. At the higher temperatures the edges of this band become more diffuse, but the differences do not come out very well in the enlarged reproduction (Fig. 3); the darkening at $60°$ on the more refrangible side of Cd. 2836 is well seen.

Aniline. The absorption by an alcoholic solution of $M/2500$ strength was investigated. This substance has one large band the centre of which is at about $\lambda 2850$. At the higher temperature of $60°$ C. the edges of the band appear to be a little more diffuse. Photographs have not been reproduced.

Benzonitrile. In an alcoholic solution of $M/1500$ strength at the higher temperature of $60°$ C. the band at $\lambda 2785$ is wider, more diffuse and shifted a little towards the red end (Fig. 4). The bands at $\lambda 2710$ and $\lambda 2640$ are also more diffuse, but this is not so definite

as in the band λ2785. In a solution of $M/1000$ strength the widening of the band at λ2785 is better seen.

The results show that the effect of an increased temperature on solutions of these six substances is a widening of the bands; the edges become more diffuse; there is a slight shift towards the red end; and they are a shade weaker.

Dr Liveing[*] concludes from an investigation of various salts of didymium and erbium, whether in dilute or strong solutions, that increasing the temperature makes the bands more diffuse, spreads them out, makes their limits less definite, and, in the case of weak bands, makes them appear weaker. He argues that the effects of heat on the spectra afford no confirmation of the supposition that the absorptions are due to an increase in the number of ions.

The author's observations of the action of heat on dilute solutions of organic substances are similar to Dr Liveing's results with solutions of inorganic substances. There is no suggestion that benzene or toluene or aniline are ionised in solution, but the effects of heat on the absorption bands are similar to the effects of heat on solutions of inorganic salts which are supposed to be dissociated. The two sets of phenomena are similar. This may be explained by assuming that the selective absorption of all these substances, inorganic and organic, is caused by vibrations within the molecule, *i.e.* they owe their origin to the acting and reacting atoms and their electrons. The fundamental vibrations of the simpler substances are modified by the type of atom or group of atoms which may be introduced. In regard to benzene, for example, the fundamental vibrations which produce the seven solution bands in the ultra-violet regions are modified in every one of its compounds. The bands may be shifted towards the red end, or be diminished in number, or may close up to produce one large band, or they may disappear altogether. Neither dilution nor heat has any such influence. Within the limits of the temperatures to which they were submitted there is no difference in the specific type of the absorption peculiar to each substance. The type remains constant and peculiar to each molecule; and, according to this investigation, at the higher temperatures it is only slightly widened and shifted towards the red, it becomes a little more diffuse and a little weaker.

* *Trans. Camb. Phil. Soc.* 1899, 18. 298 et seq.

Plate IX

Cd. 2572
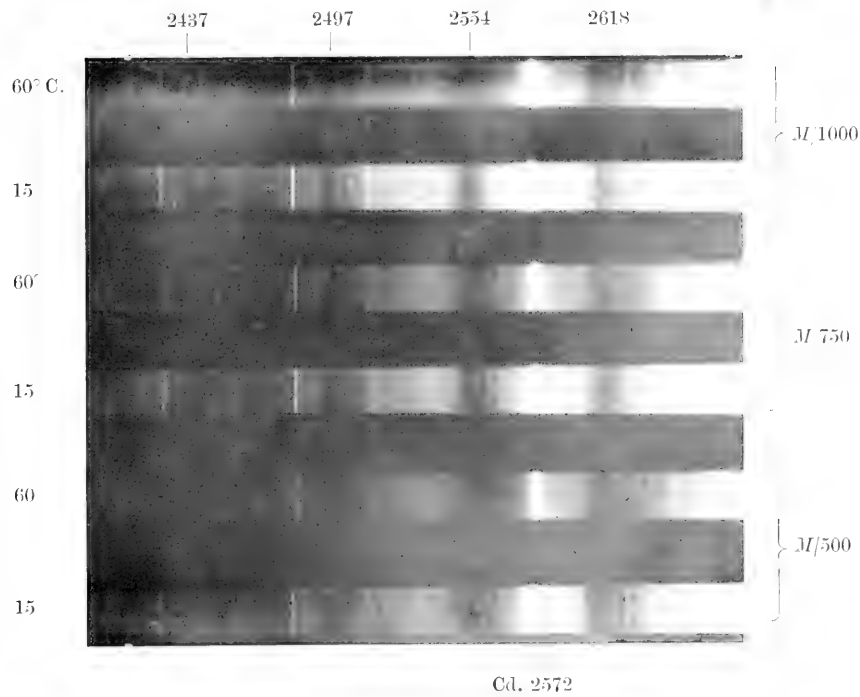
Fig. 1. Benzene of various concentrations in alcohol.

Cd. 2572
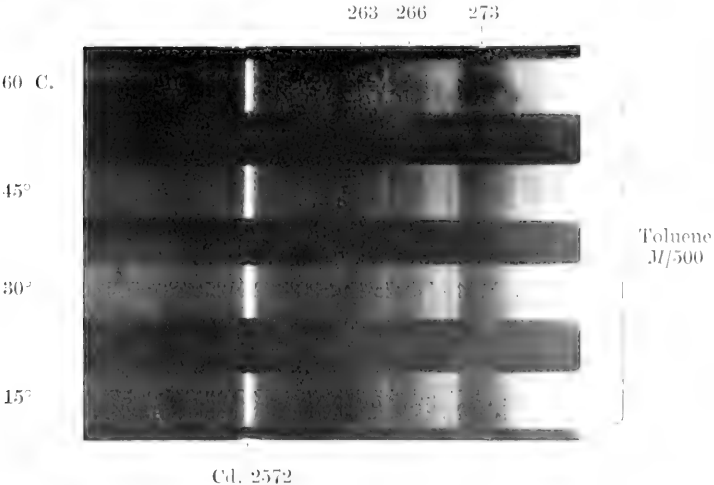
Fig. 2. Toluene, $M/500$ strength in alcohol.
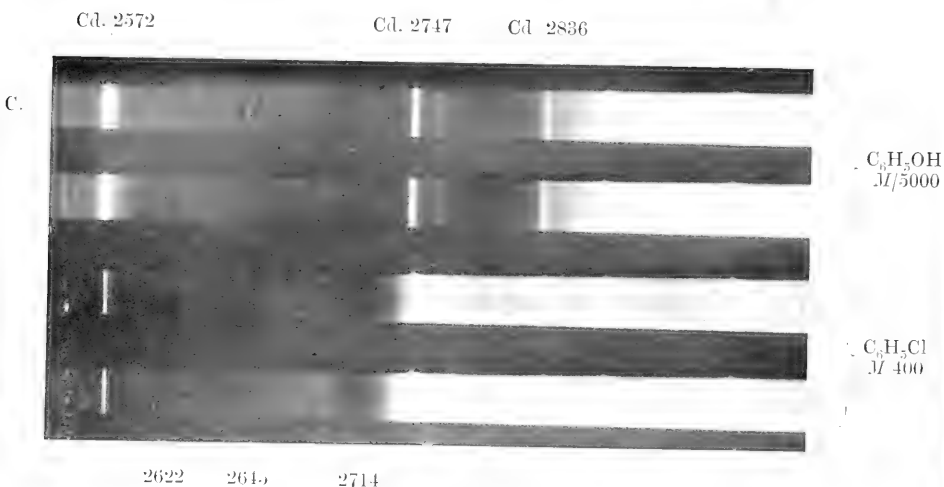
Fig. 3
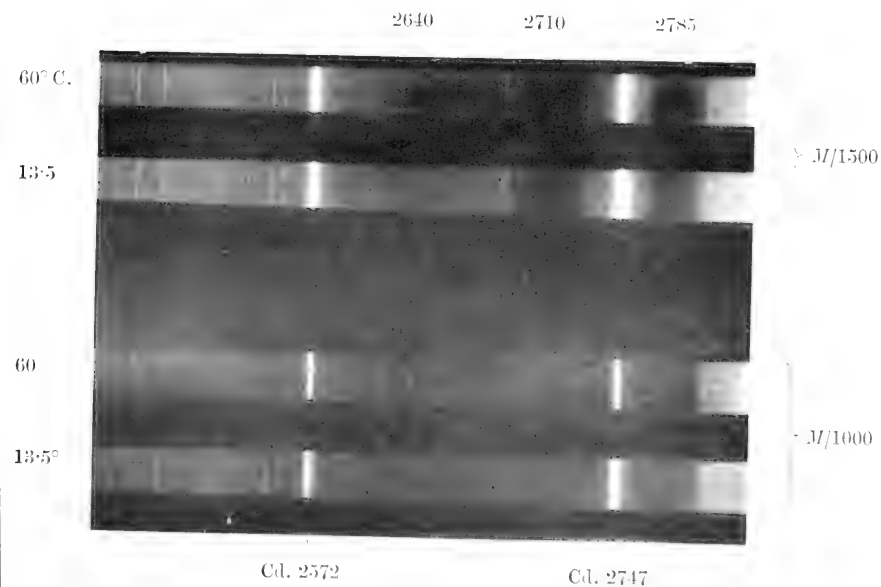
Fig. 4. Benzonitrile of two different strengths in alcohol.

Further references to investigations on the absorption spectra of the substances described in this paper:

Benzene:—

For solution: Hartley and Huntingdon, *Phil. Trans.* 1879, 170. i. 257.
Hartley and Dobbie, *Trans. Chem. Soc.* 1898, 73. 695.
Baly and Collie, *Trans. Chem. Soc.* 1905, 87. 1332.

For vapour: Hartley, *Phil. Trans.* A. 1907, 208. 475.

Monochlorobenzene:—

For solution: Baly, *Trans. Chem. Soc.* 1905, 87. 1332, and *Proc. Chem. Soc.* 1911, 27. 72.
Purvis, *Trans. Chem. Soc.* 1911, 99. 824.

For vapour: Purvis, *Trans. Chem. Soc.* 1911, 99. 812.

Benzonitrile:—

For solution and vapour: Purvis, *Trans. Chem. Soc.* 1915, 107. 501.

Phenol:—

For solution: Hartley, Dobbie and Lander, *Trans. Chem. Soc.* 1902, 81. 929.
Baly and Ewbank, *Trans. Chem. Soc.* 1905, 87. 1347.

For vapour: Purvis and McCleland, *Trans. Chem. Soc.* 1913, 103. 1089.

Aniline:—

For solution: Hartley and Huntingdon (*loc. cit.* above).
Baly and Collie (*loc. cit.* above).

For vapour: Purvis, *Trans. Chem. Soc.* 1910, 97. 1547.

PROCEEDINGS AT THE MEETINGS HELD DURING THE SESSION 1922—1923.

ANNUAL GENERAL MEETING.

October 30, 1922.

In the Cavendish Laboratory.

PROFESSOR SEWARD, PRESIDENT, IN THE CHAIR.

The following were elected Officers for the ensuing year:

President:

Mr C. T. Heycock.

Vice-Presidents:

Prof. Newall.
Prof. Seward.
Dr H. Lamb.

Treasurer:

Mr F. A. Potts.

Secretaries:

Prof. Baker.
Mr F. W. Aston.
Mr J. Gray.

Other Members of Council:

Prof. Hopkins.	Mr T. C. Nicholas.
Dr Bennett.	Dr E. H. Griffiths.
Dr Hartridge.	Mr C. T. R. Wilson.
Mr H. Hamshaw Thomas.	Mr J. M. Wordie.
Mr R. H. Fowler.	Mr G. I. Taylor.
Mr E. Cunningham.	Mr H. McCombie.

The following were elected Associates of the Society:

Nazir Ahmid, Peterhouse.
L. F. Bates, Trinity College.
F. Bath, King's College.
A. H. Bebb, Downing College.
L. F. Curtiss.
F. Goldby, Gonville and Caius College.
W. G. S. Hopkirk, Gonville and Caius College.
J. Hyslop, St John's College.
J. E. Jones, Trinity College.
Hem Singh Pruthi, Fitzwilliam Hall.
F. I. G. Rawlins, Trinity College.
J. S. Rogers, Trinity College.
Miss H. G. Telling, Newnham College.

The following Communications were made to the Society:

1. Determinations of the velocity with which Carbon Monoxide displaces Oxygen from its combination with the blood pigment Haemoglobin.

    (*a*) A spectroscopic method of estimating the relative proportions of Oxygen and Carbon Monoxide in combination with Haemoglobin.

    (*b*) Factors which affect the accuracy of the Spectroscopic method.

    (*c*) Basic principles of the two methods of estimating the velocity of the Reaction.

    (*d*) Description of the actual technique employed.

    (*e*) The results obtained and conclusions.

By Dr HARTRIDGE and F. J. W. ROUGHTON, B.A., Trinity College.

2. Some problems of Diophantine approximation. By Prof. G. H. HARDY and J. E. LITTLEWOOD, M.A., Trinity College.

3. A preliminary investigation of the Intensity Distribution in the β-ray Spectra of Radium B and C. By Dr CHADWICK and C. D. ELLIS, M.A., Trinity College.

4. Partition Functions for temperature radiation and the internal energy of a crystalline solid. By C. G. DARWIN, M.A., Christ's College, and R. H. FOWLER, M.A., Trinity College.

5. On an integral equation. By J. E. LITTLEWOOD, M.A., Trinity College, and E. A. MILNE, M.A., Trinity College.

6. The automatic synchronization of Triode Oscillators. By E. V. APPLETON, M.A., St John's College.

7. Note on the curved tracks of β Particles. By P. L. KAPITZA. (Communicated by Mr C. G. Darwin.)

8. Meteorology and the non-flapping flight of tropical birds. By Dr G. T. WALKER.

9. The algebra of symmetric functions. By Major P. A. MACMAHON.

November 13, 1922.

In the Botany School.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following was elected an Associate of the Society:

R. Schlapp, St John's College.

Dr Arthur Smith Woodward delivered a Lecture entitled "The Skulls of Palaeolithic Men," which was illustrated with lantern slides.

The following Communications were made to the Society:

1. On a system of differential equations which appear in the theory of Saturn's rings. By W. M. H. GREAVES, M.A., St John's College.

2. Fluctuations in an assembly in statistical equilibrium. By C. G. DARWIN, M.A., Christ's College, and R. H. FOWLER, M.A., Trinity College.

November 27, 1922.

In the Cavendish Laboratory.

Mr C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

J. C. Burkill, B.A., Trinity College.
J. A. Carroll, B.A., Sidney Sussex College.
R. A. Fisher, M.A., Gonville and Caius College.
J. P. Gabbatt, M.A., Peterhouse.
A. Hopkinson, M.B., Emmanuel College.
A. E. Ingham, B.A., Trinity College.
C. G. F. James, B.A., Trinity College.
F. J. W. Roughton, B.A., Trinity College.
M. Thomas, B.A., Trinity Hall.
C. E. Tilley, Ph.D., Emmanuel College.
Hugh Watson, M.A., Trinity College.

The following were elected Associates of the Society:

E. A. Guggenheim, Gonville and Caius College.
O. H. Malik, Fitzwilliam Hall.

The following Communications were made to the Society:

1. On some α-ray tracks. By C. T. R. WILSON, M.A., Sidney Sussex College.

2. The interpretation of the Pelvic Region and Thigh of Monotremata. By A. B. APPLETON, M.D., Downing College.

3. Observations on the innervation of the pubi-tibialis (sartorius) muscle. By A. B. APPLETON, M.D., Downing College, and F. GOLDBY, Gonville and Caius College.

4. The axioms of elliptic geometry. By Dr W. BURNSIDE.

5. The periodic solutions of the differential equation for the triode oscillator. By W. M. H. GREAVES, M.A., St John's College.

6. Complexes of cubics in ordinary space. By C. G. F. JAMES, B.A., Trinity College.

January 22, 1923.

In the Comparative Anatomy Lecture Room.

Mr C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

L. E. Bayliss, B.A., Trinity College.
F. Debenham, M.A., Gonville and Caius College.
Sir G. P. Lenox-Conyngham, M.A., Trinity College.
R. E. Priestley, B.A., Clare College.

The following were elected Associates of the Society:

H. J. Brennan, Trinity Hall.
H. Davis, Trinity College.
G. P. Wells, Trinity College.
Miss G. L. Elles, Newnham College.
Miss D. L. Foster, Newnham College.
Miss D. M. Wrinch, Girton College.

The following Communications were made to the Society:

1. (1) The stellate appendages of telescopic and entoptic diffraction.
   (2) Can gravitation really be absorbed into the frame of space and time?

By Prof. Sir JOSEPH LARMOR.

2. The representation of a cubic surface upon a quadric surface. By Prof. H. F. BAKER.

3. Measurements of the rate of oxidation and reduction of Haemoglobin. By Dr HARTRIDGE and F. J. W. ROUGHTON, B.A., Trinity College.

4. A method of measuring the Carbon Dioxide output of aquatic animals. By J. T. SAUNDERS, M.A., Christ's College.

5. Changes in the specific gravity of *Daphnia pulex* L. By Miss D. EYDEN. (Communicated by Mr J. T. Saunders.)

February 5, 1923.

In the Cavendish Laboratory.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

E. G. Dymond, B.A., St John's College.
W. B. R. King, M.A., Jesus and Magdalene Colleges.
E. G. Williams, B.A., Trinity Hall.
A. F. R. Wollaston, B.Ch., King's College.

The following were elected Associates of the Society:

Miss A. Bishop, Girton College.
Miss F. M. Hamer, Girton College.
C. C. Hurst, Trinity College.
Miss D. M. Moyle, Girton College.
Miss M. B. Thomas, Girton College.
Miss B. Trevelyan, Girton College.

The following Communications were made to the Society:

1. The escape of molecules from an atmosphere, with special reference to the boundary of a gaseous star. By E. A. MILNE, M.A., Trinity College.

2. Free paths in a non-uniform rarefied gas with an application to the escape of molecules from an isothermal atmosphere. By J. E. JONES. (Communicated by Mr R. H. Fowler.)

3. L Series of Tungsten and Platinum. By J. S. ROGERS. (Communicated by Prof. Sir E. Rutherford.)

4. Contributions to the theory of α-particle phenomena. Part I. Stopping Powers. Part II. Ionization. By R. H. FOWLER, M.A., Trinity College.

5. The representation of varieties in space of three and four dimensions. By C. G. F. JAMES, B.A., Trinity College.

6. On the fifth book of Euclid's elements. By Dr M. J. M. HILL.

7. A chapter from the note-book of Mr Ramanujan. By G. H. HARDY, M.A., Trinity College.

8. Hankel Transforms. By E. C. TITCHMARSH. (Communicated by Mr G. H. Hardy.)

9. A generalisation of Feuerbach's theorem. By J. P. GABBATT, M.A., Peterhouse.

February 19, 1923.

In the Anatomy School.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

Dr Richard Goldschmidt (Kaiser Wilhelm-Institut, Berlin) delivered a Lecture entitled " Intersexuality and the problem of sex determination."

March 5, 1923.

In the Cavendish Laboratory.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

A. W. Barton, B.A., Trinity College.
L. C. G. Clarke, M.A., Trinity Hall.
Prof. H. R. Dean, M.D., Trinity Hall.
H. M. Fox, M.A., Gonville and Caius College.
E. C. Francis, M.A., Peterhouse.
W. A. C. Gardiner, B.A., Gonville and Caius College.
A. K. Goard, B.A., Trinity College.
J. B. S. Haldane, M.A., Trinity College.
D. R. Hartree, B.A., St John's College.
A. D. Hobson, B.A., Christ's College.
L. H. Matthews, B.A., King's College.
G. Merton, M.A., Trinity College.
M. G. L. Perkins, B.A., Trinity College.
H. W. B. Skinner, B.A., Trinity College.

The following were elected Associates of the Society:

A. T. Akers, Gonville and Caius College.
O. R. Howell, Emmanuel College.
T. Moran.
J. Piqué.

The following Communications were made to the Society:

1. Some observations on α-particle tracks in a magnetic field. By P. KAPITZA. (Communicated by Prof. Sir Ernest Rutherford.)

2. The capture and loss of electrons by α-particles. By Prof. Sir ERNEST RUTHERFORD.

3. The magnetic field of a helix. By Dr H. LAMB.

4. (1) The theory of errors of observation.
   (2) The solution of a certain partial difference equation.
By Dr W. BURNSIDE.

5. A note on the natural curvature of alpha Ray Tracks. By P. M. S. BLACKETT, B.A., Magdalene College.

March 12, 1923.

In the Chemical Laboratory.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

Professor Sir William Pope gave a Demonstration on **Doubly Re-**
fracting Liquids.

May 7, 1923.

In the Cavendish Laboratory.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following Communications were made to the Society:

1. The recuperation of energy in the Universe. By Dr G. D. LIVEING.

2. (1) Infra-red spectra.
   (2) The absorption spectra of some organic and inorganic salts of didymium.
   (3) The absorption spectra of solutions of benzene and some of its derivates at various temperatures.
   (4) The absorption of the ultra-violet rays by phosphorus and some of its compounds.
By J. E. PURVIS, M.A., Corpus Christi College.

3. A note on the electromagnetic mass of the electron. By E. C. STONER, B.A., Emmanuel College.

4. Chemical constants of diatomic molecules. By R. R. S. COX, B.A., Christ's College. (Communicated by Mr R. H. Fowler.)

May 21, 1923.

In the Cavendish Laboratory.

MR C. T. HEYCOCK, PRESIDENT, IN THE CHAIR.

The following were elected Fellows of the Society:

Sir Robert Waley Cohen, M.A., Emmanuel College.
V. C. Pennell, M.A., Pembroke College.

The following was elected an Associate of the Society:

J. S. Bentwich, Trinity College.

The following Communications were made to the Society:

1. (1) The partitions of Infinity.
   (2) The prime numbers of measurement.
By Major P. A. MacMahon.

2. On Approximate Continuity. By M. H. A. Newman, B.A., St John's College. (Communicated by Prof. H. F. Baker.)

3. The pedal locus in hyperspace. By J. P. Gabbatt, M.A., Peterhouse.

4. On some approximate numerical applications of Bohr's theory of spectra. By D. R. Hartree, B.A., St John's College.

5. Some statistical aspects of Geographical Distribution. By A. G. Thacker. (Communicated by Mr J. Gray.)

6. On the structure of a middle Cambrian Alga from British Columbia (*Marpolia spissa*, Walcott). By J. Walton, M.A., St John's College.

7. On the invasion of woody tissues by wound parasites. By F. T. Brooks, M.A., Emmanuel College, and W. C. Moore, B.A., Trinity College.

June 12, 1923.

In the Cavendish Laboratory.

Mr C. T. Heycock, President, in the Chair.

Niels Bohr (Hon. Sc.D., Cambridge), Professor of Theoretical Physics in the University of Copenhagen, was elected an Honorary Member of the Society, on the ground of his valuable contributions to Mathematical Physics, and especially to the Theory of the Constitution of the Atom.

July 16, 1923.

In the Cavendish Laboratory.

Mr C. T. Heycock, President, in the Chair.

The following were elected Fellows of the Society:

W. W. Rouse Ball, M.A., Trinity College.
Rev. F. D. Bateman, M.A., Sidney Sussex College.
John Brill, M.A., St John's College.
S. Brodetsky, M.A., Trinity College.
G. S. Carter, B.A., Gonville and Caius College.
T. M. Cherry, B.A., Trinity College.
E. F. Collingwood, B.A., Trinity College.
C. E. Cullis, M.A., Gonville and Caius College.
H. E. J. Curzon, M.A., Downing College.
J. B. Dale, M.A., St John's College.
B. Bentham Dickinson, M.A., Emmanuel College.

W. C. Fletcher, M.A., St John's College.
C. Fox, M.A., Sidney Sussex College.
Peter Fraser, M.A., Queens' College.
R. A. P. Gray, B.A., St John's College.
H. Ronald Hassé, M.A., St John's College.
L. Isserlis, B.A., Christ's College.
T. R. Lee, B.A., Pembroke College.
G. H. Livens, M.A., Jesus College.
F. S. Macaulay, M.A., St John's College.
H. R. Mehra, Ph.D., Fitzwilliam Hall.
W. P. Milne, M.A., Clare College.
M. H. A. Newman, B.A., St John's College.
T. R. Parsons, M.A., Sidney Sussex College.
G. Prasad, M.A., Christ's College.
J. Proudman, M.A., Trinity College.
Balak Ram, M.A., St John's College.
Colin M. Ross, M.A., King's College.
R. A. Sampson, M.A., St John's College.
E. H. Smart, M.A., Queens' College.
R. O. Street, M.A., St John's College.
J. F. Tinto, M.A., Trinity College.
H. W. Turnbull, M.A., Trinity College.
C. Walmsley, M.A., King's College.
E. L. Watkin, M.A., St John's College.

The following was elected an Associate of the Society:

T. Alty, Trinity College.

The following Communications were made to the Society:

1. The possible mechanics of the hydrogen atom. By W. M. H. GREAVES, M.A., St John's College.

2. The motion of a neutral ionised stream in the earth's magnetic field. By S. CHAPMAN, M.A., Trinity College.

3. Analytical theory of crystals. By J. D. BERNAL, B.A., Emmanuel College. (Communicated by Prof. H. F. Baker.)

4. Two geometrical notes: (1) Theory of confocal quadrics and Poncelet's porism of inscribed triangles. (2) A self-reciprocal figure, and the associated cubic surfaces. By Prof. H. F. BAKER.

5. Sur la représentation analytique des congruences de coniques. By Monsieur L. GODEAUX. (Communicated by Prof. H. F. Baker.)

6. Dougall's theorem on hypergeometric functions. By C. T. PREECE. (Communicated by Dr G. N. Watson.)

7. A quintic locus defined by five points in a plane. By W. L. MARR. (Communicated by Mr W. P. Milne.)

8. On the problem of three bodies. By J. BRILL, M.A., St John's College.

9. Extensions of a theorem of Segre's, with their natural position in space of seven dimensions. By C. G. F. JAMES, B.A., Trinity College.

10. The form of the solution of the equations of dynamics. By T. M. CHERRY, B.A., Trinity College.

11. Note on Dr Burnside's recent paper on errors of observation. By R. A. FISHER, M.A., Gonville and Caius College.

12. Further examples of partition functions. By C. G. DARWIN, M.A., Christ's College, and R. H. FOWLER, M.A., Trinity College.

13. Real twisted cubics which are geodesics on quadric surfaces. By H. W. RICHMOND, M.A., King's College.

[*Note*. The Membership of the Society on 30 September, 1923, consists of:

Honorary members, 32; Life Fellows, 215; Life Fellows paying a voluntary annual subscription of one or two guineas, 20; Fellows paying an annual subscription of two guineas, 180; Associates, paying one guinea annually, 52; Readers in the Library, paying ten shillings annually or five shillings for a single term, 56.

A revised list of the members of the Society will be issued as soon as the expense of printing this seems justified.]

MEETINGS OF THE SOCIETY 1923—1924

Mondays Oct. 29, Nov. 12, Nov. 26, Dec. 3
Jan. 21, Feb. 4, Feb. 18, Mar. 3.
May 5, May 19
July 14

at 4.30 p.m., except on Nov. 12 and Feb. 18, when the Meetings will be at 8.45 p.m.

STATEMENT OF ACCOUNTS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, FROM JANUARY 1, 1922, TO DECEMBER 31, 1922.

Receipts.

	£ s. d.	£ s. d.
Balance due to the Society, Jan. 1, 1922		
With Treasurer ...	7 4	
Special Donation Fund ...	25 0 0	25 7 4
Arrears of Subscriptions ...		19 0 0
Subscriptions for 1922 ...		244 13 0
Entrance Fees ...		4 4 0
Subscription paid in advance ...		1 1 0
Associates' Subscriptions for 1922 ...		24 13 6
Composition Fees ...		246 15 0
Special Donations ...		390 1 0
Dividends on Liverpool Corporation Stock ...		7 15 8
Publishing Account: University Press (1921) ...		41 0 8
,, ,, Deighton Bell ...		11 5
Interest on Deposit Account ...		6 5 0
Anthropometric Committee ...		4 6 2
Botany School: Sale of Proceedings ...		8 4
Postage refunded ...		1 0
		£1016 3 1

Expenditure.

	£ s. d.	£ s. d.
Balance due from the Society, Jan. 1, 1922		
University Press ...		257 10 2
Expenses connected with the Philosophical		263 18 3
Library:		
Cambridge Bookbinders' Guild ...	72 5 9	
Wilson, Bookbinder ...	30 0 0	102 5 9
Stationery (Spalding) ...		3 3 1
Heffer ...		2 16 9
Purchase of £332 4/3 Conversion Loan at 3½ % ...		250 0 0
Carriage of books from Germany ...		1 15 6
Carriage of parcels and postage ...		16 1 4
		897 10 10
Balance due to the Society, Dec. 31, 1922		
Special Donation Fund ...	93 1 9	
With Treasurer ...	25 0 0	
	10 6	118 12 3
		£1016 3 1

HOPKINS PRIZE FUND ACCOUNT.

Receipts.		£ s. d.
Balance due to the Fund, Jan. 1, 1922		235 3 8
Dividends on £750 G.E.R. 4 per cent. Debentures:		
January, 1922		10 10 0
July, 1922		10 17 6
Income Tax refunded (1920–1922)		27 0 0
		£283 11 2

Expenditure.		£ s. d.
Balance due to the Fund, Dec. 31, 1922		283 11 2
		£283 11 2

STATEMENT OF BALANCES IN THE BANK.

		£ s. d.	£ s. d.
Balance at Messrs Barclay's, Dec. 31, 1922 ...		179 17 11	
On Deposit		506 5 0	
		686 2 11	
Less 1922 accounts paid subsequently ...		284 10 0	
			401 12 11
Balance due to Hopkins Prize Fund Account		283 11 2	
Balance due to the Society		93 1 9	
Special Donation to Library		25 0 0	
			401 12 11

CAPITAL ACCOUNT (RESERVE FUND).

			£ s. d.
Liverpool Corporation 3½ per cent. Stock			307 1 10
£332 ⅓ Conversion Loan 3½ per cent.			250 0 0

F. A. POTTS, *Treasurer.*

REPORT OF THE AUDITORS.

We have examined these accounts and find them correct. There is a balance due to the Society of £93. 1s. 9d.; and a balance with the Treasurer of 10s. 6d. due to the Society; and a balance due to the Hopkins Fund of £283. 11s. 2d., together with a balance of £25 due to the Special Donation Fund.

13 *February*, 1923.

F. H. A. MARSHALL,
F. J. M. STRATTON } *Auditors.*

INDEX* TO THE PROCEEDINGS VOL. XXI

with references to the Transactions